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Transmuted Mukherjee-Islam Distribution: A Generalization of Mukherjee-Islam Distribution

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Abstract

A new continuous distribution is proposed in this paper. This distribution is a generalization of Mukherjee-Islam distribution using the quadratic rank transmutation map. It is called transmuted Mukherjee-Islam distribution (TMID). We have studied many properties of the new distribution: Reliability and hazard rate functions. The descriptive statistics: mean, variance, skewness, kurtosis are also studied. Maximum likelihood method is used to estimate the distribution parameters. Order statistics and Renyi and Tsallis entropies were also calculated. Furthermore, the quantile function and the median are calculated.

Keywords: Transmuted Mukherjee-Islam distribution, Moments, Entropy, Order statistics, quantile function

1. Introduction

Shaw and Buckley (2007) have proposed to transmutation maps, the sample and rank transmutations. The simplest rank transmutation map is the quadratic rank transmutation map. The quadratic rank transmutation map will be used through this paper to derive a generalization of the Mukherjee-Islam distribution with some of its properties. This generalization is called the transmuted Mukherjee-Islam (TMI) distribution. Al-Omari et al. (2017) proposed the transmuted janadran distribution as a generalization of the Janadran distribution. Aryal and Tsokos (2011) worked out a generalization of the weibull probability distribution (transmuted weibull distribution). Merovci (2013a) used the quadratic rank transmutation map to develop a new distribution called the Transmuted Lindley Distribution. Merovci (2013b) used this map to develop a Transmuted Rayleigh Distribution. An extension of the exponentiated generalized G class of distributions (Cordeiro et al., 2013) called the transmuted exponentiated generalized G family. A simple representation for the transmuted G-family density function as a linear mixture of the G and the exponentiated-G densities was derived by Bourguignon et al. (2016). Many authors worked out generalizations to some distributions using the quadratic rank transmutation map. For example Merovci and Elbatal (2014) introduced the Transmuted Lindley-Geometric distribution, whereas, Vardhan and Balaswamy (2016) proposed a transmuted new modified Weibull distribution. A Transmuted Lomax distribution (Ashour and Eltehiwy, 2013), a Transmuted Log-Logistic Distribution (Aryal and Tsokos, 2013), Transmuted Burr Type XII Distribution (Khazaleh, 2016). El- batal et al. (2014) studied some general properties of the transmuted exponentiated Frêchet distribution. Based on new modified weibull distribution, Vardhan and Balaswamy (2016) produced a transmuted distribution using the quadratic rank transmutation map, named transmuted new modified weibull distribution. A transmuted modified weibull distribution is introduced by Khan and King (2013).

We organized the rest of this paper as follows: In Section 2 the pdf and CDF of the TMI distribution are demonstrated. In Section 3, the reliability and hazard rate functions of our model are computed. We summarized the distributions of order statistics in Section 4. Some properties, like the r^{th} moment, mean, variance, skewness, kurtosis, coefficient of variation and the moment generating function of the TMI distribution are derived in Section 5. In Section 6 the maximum likelihood estimates of the distribution parameters are demonstrated. The Renyi and Tsallis entropies are calculated in Section 7. The quantile function is derived in Section 8. Finally, in Section 9 we will draw conclusions.

2. Transmuted Mukherjee-Islam Distribution

A random variable, X, is said to have a Mukherjee-Islam distribution (Mukheerji and Islam, 1983) with parameters θ and p if it has a cumulative distribution function (CDF)

$$W(x) = \begin{cases} 0, & x < 0 \\ \frac{x^p}{\theta^p}, & 0 < x \leq \theta \\ 1, & x > \theta \end{cases} \tag{1}$$

with a corresponding probability density function (pdf) given by:

$$w(x) = \begin{cases} \frac{px^{p-1}}{\theta^p}, & 0 < x \leq \theta,\ \theta, p > 0 \\ 0, & other\,wise \end{cases} \tag{2}$$

Definition 2.1 A random variable X is said to have a transmuted distribution if its CDF is given by

$$\Psi(x) = (1+\lambda)W(x) - \lambda[W(x)]^2, \quad |\lambda| \leq 1 \tag{3}$$

where $W(x)$ is the CDF of the base distribution. The pdf of the transmuted random variable is given by

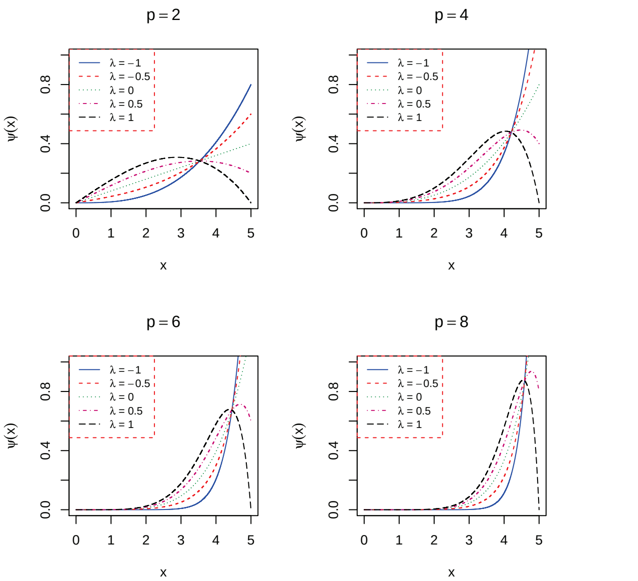
$$\psi(x) = w(x)(1+\lambda-2\lambda W(x)) \tag{4}$$

The CDF of this random variable is, hence, defined using Equations (1) and (3) as:

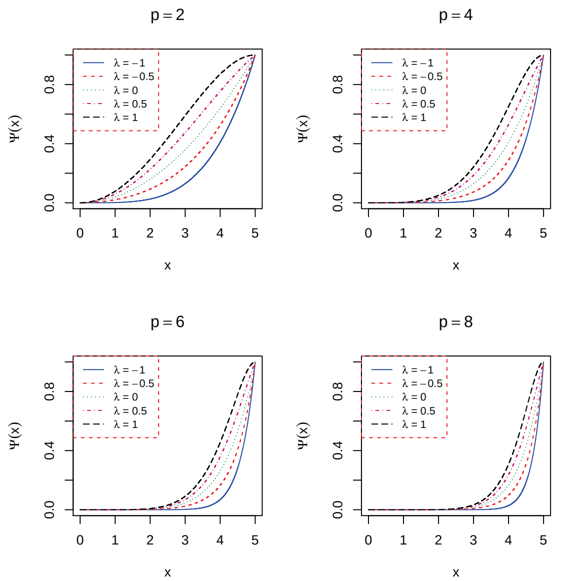
$$\Psi(x) = (1+\lambda)\frac{x^p}{\theta^p} - \lambda\frac{x^{2p}}{\theta^{2p}}, 0 < x \leq \theta \tag{5}$$

Therefore, the pdf of the transmuted Mukherjee-Islam random variable, X, is defined using Equations (1), (2) and (4) as:

$$\psi(x) = \frac{(1+\lambda)p}{\theta^p}x^{p-1} - \frac{2\lambda p}{\theta^{2p}}x^{2p-1}, 0 < x \leq \theta \tag{6}$$



(a) The pdf of the TMI distribution with different values of p, λ when $\theta = 5$



(b)The CDF of the TMI distribution with different values of p, λ when $\theta = 5$

Figure (1a) shows the pdf of the TMI for $\theta = 5$ and different values of p, 2, 4, 6 and 8. We varied the value of λ from -1 to 1 with a step of 0.5. The figure shows that the TMI random variable has a left skewed distribution. The tail of the distribution gets heavier as the value of λ gets smaller. Figure (1b) shows the plot of the CDF of the TMI random variable for θ=5 with p equals to 2, 4, 6 and 8 and λ = -1, -0.5, 0, 0.5, and 1.

3. Reliability Analysis

The reliability and hazard rate functions are defined by:

$$R(t) = 1 - \Psi(t) \tag{7}$$

$$H(t) = \frac{\psi(t)}{1-\Psi(t)} \tag{8}$$

Theorem 3.1 The reliability and hazard rate functions of the TMI distribution, respectively are

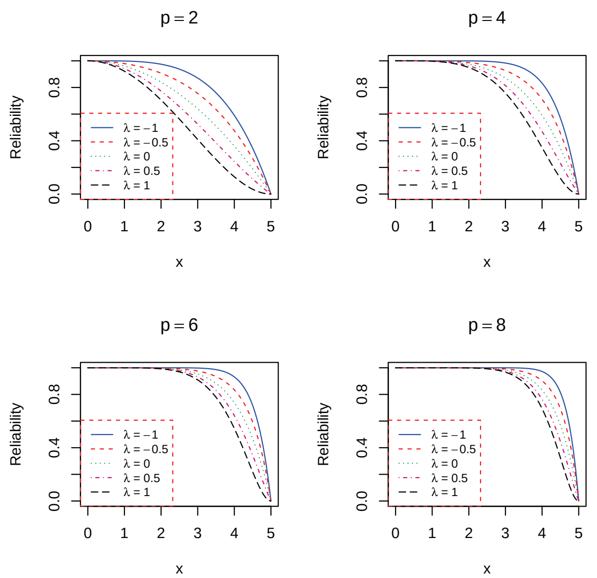
$$R(t) = 1 - \frac{t^p}{\theta^p}\left[1 + \lambda - \lambda\frac{t^p}{\theta^p}\right]$$

$$H(t) = \frac{pt^{p-1}\left[(1+\lambda)\theta^p - 2\lambda t^p\right]}{\theta^{2p} - t^p\left[(1+\lambda)\theta^p - \lambda t^p\right]}$$

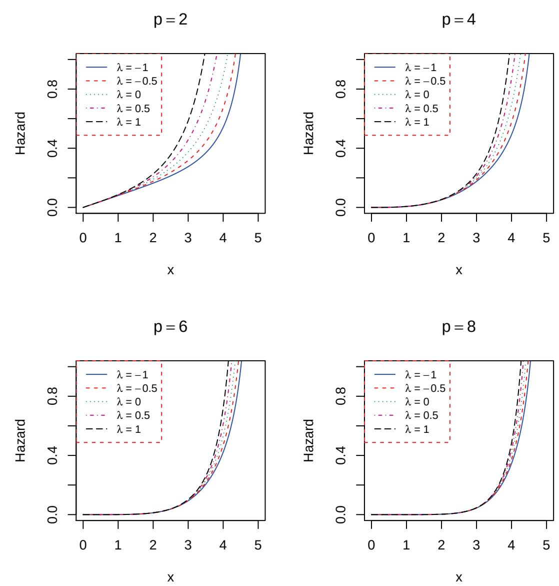
Proof. The proof of the reliability is straightforward, by substituting the CDF of the TMI distribution Equation (5) in Equation (7). Now, for the hazard rate function, substituting Equations (5) and (6) in Equation (8), we get

$$\begin{aligned} H(t) &= \frac{\frac{(1+\lambda)p}{\theta^p}t^{p-1} - \frac{2\lambda p}{\theta^{2p}}t^{2p-1}}{1 - \frac{t^p}{\theta^p}\left[1 + \lambda - \lambda\frac{t^p}{\theta^p}\right]} \\ &= \frac{pt^{p-1}\left[(1+\lambda)\theta^p - 2\lambda t^p\right]}{\theta^{2p} - t^p\left[(1+\lambda)\theta^p - \lambda t^p\right]} \end{aligned}$$

□



(a) Reliability of the TMID with different values of p, λ when $\theta = 5$



(b) Hazard rate function of the TMID with different values of p, λ when $\theta = 5$

4. Order Statistics

Let $X_1, X_2, \ldots X_n$ be a random sample with pdf $\psi(x)$ and CDF $\Psi(x)$. If $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ is the order statistic of this sample, where $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$. Then the pdf of the j^{th} order statistics, $X_{(j)}$ is given by:

$$\psi_{(j)}(x) = j\binom{n}{j}\psi(x)[\Psi(x)]^{j-1}[1-\Psi(x)]^{n-j}, \tag{9}$$

Substituting $j = 1$ in Equation (9), we get the pdf of first order statistics $X_{(1)} = min(X_1, X_2, \ldots X_n)$.

$$\psi_{(1)}(x) = np\frac{x^p}{\theta^p}\left(1 + \lambda - 2\lambda\frac{x^p}{\theta^p}\right)\left(1 - (1+\lambda)\frac{x^p}{\theta^p} + \lambda\frac{x^{2p}}{\theta^{2p}}\right)^{(n-1)}, \quad x \in (0, \theta) \tag{10}$$

The pdf of the n^{th} order statistic $X_{(n)} = max(X_1, X_2, ... X_n)$, is defined as:

$$\psi_{(n)}(x) = np\frac{x^p}{\theta^p}\Big(1+\lambda-2\lambda\frac{x^p}{\theta^p}\Big)\left((1+\lambda)\frac{x^p}{\theta^p}-\lambda\frac{x^{2p}}{\theta^{2p}}\right)^{(n-1)} \quad (11)$$

Furthermore, for any value of j the common form of $\psi_{(j)}(x)$ can be obtained as

$$\begin{aligned}\psi_{(j)}(x) &= j\binom{n}{j}\left[\frac{(1+\lambda)p}{\theta^p}x^{p-1}-\frac{2\lambda p}{\theta^{2p}}x^{2p-1}\right]\left[(1+\lambda)\frac{x^p}{\theta^p}-\lambda\frac{x^{2p}}{\theta^{2p}}\right]^{j-1}\\ &\times \left[1-(1+\lambda)\frac{x^p}{\theta^p}+\lambda\frac{x^{2p}}{\theta^{2p}}\right]^{n-j}\\ &= j\binom{n}{j}\frac{x^{pj-1}}{\theta^{pj}}\left[(1+\lambda)p-2\lambda p\frac{x^p}{\theta^p}\right]\left[1+\lambda-\lambda\frac{x^p}{\theta^p}\right]^{j-1}\left[1-(1+\lambda)\frac{x^p}{\theta^p}+\lambda\frac{x^{2p}}{\theta^{2p}}\right]^{n-j} \quad (12)\end{aligned}$$

5. Moments

5.1 r^{th} Moment

Theorem 5.1 The r^{th} moment of the TMI random variable is defined as:

$$E(X^r) = \left(\frac{2p^2+rp-rp\lambda}{(r+p)(r+2p)}\right)\theta^r \quad (13)$$

Proof.

$$\begin{aligned}E(X^r) &= \int_0^\theta x^r\psi(x)dx\\ &= \int_0^\theta x^r\left(\frac{(1+\lambda)p}{\theta^p}x^{p-1}-\frac{2\lambda p}{\theta^{2p}}x^{2p-1}\right)dx\\ &= \frac{p(1+\lambda)}{\theta^p}\int_0^\theta x^{r+p-1}dx-\frac{2p\lambda}{\theta^{2p}}\int_0^\theta x^{r+2p-1}dx\end{aligned}$$

$$\begin{aligned}&= \frac{p(1+\lambda)}{\theta^p}\left(\frac{\theta^{r+p}}{r+p}\right)-\frac{2p\lambda}{\theta^{2p}}\left(\frac{\theta^{r+2p}}{r+2p}\right)\\ &= \left(\frac{2p^2+rp-rp\lambda}{(r+p)(r+2p)}\right)\theta^r\end{aligned}$$

□

5.1.1 Mean, Variance, Skewness, Kurtosis and Coefficient of Variation

The first and the second moments can be computed by replacing r by 1 and 2; respectively in (13) as follows:

$$\begin{aligned}\mu = E(X) &= \left(\frac{2p^2+p-p\lambda}{(1+p)(1+2p)}\right)\theta\\ E(X^2) &= \left(\frac{p^2+p-p\lambda}{(2+p)(1+p)}\right)\theta^2\end{aligned}$$

But the variance of a random variable is defined as $var(X) = E(X^2) - (E(X))^2$, therefore

$$\begin{aligned}var(X) &= \left(\frac{p^2+p-p\lambda}{(2+p)(1+p)}\right)\theta^2-\left(\frac{2p^2+p-p\lambda}{(1+p)(1+2p)}\right)^2\theta^2\\ &= \frac{-p(p^2\lambda^2+2p\lambda^2-2p^2\lambda+p\lambda+\lambda-4p^2-4p-1)}{(1+p)^2(2+p)(1+2p)^2}\theta^2\end{aligned}$$

The coefficient of variation (cv) is defined to be the ratio of standard deviation of the random variable to it expected value, that is $cv = \frac{\sqrt{var(X)}}{E(X)}$. Therefore,

$$cv = \frac{\sqrt{-p(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1)}}{(2p^2 + p - p\lambda)\sqrt{2+p}} \tag{14}$$

The third and fourth moments of the random variable X can be determined by replacing r by 3 and 4; respectively in Equation (13). Thus, they are given by:

$$E(X^3) = \left(\frac{2p^2 + 3p - 3p\lambda}{(3+p)(3+2p)}\right)\theta^3 \tag{15}$$

$$E(X^4) = \left(\frac{2p^2 + 4p - 4p\lambda}{(4+p)(4+2p)}\right)\theta^4 \tag{16}$$

The skewness and the kurtosis of a random variable are defined as:

$$sk(X) = \frac{E(X^3) - 3E(X)var(X) - (E(X))^3}{(var(X))^{\frac{3}{2}}} \tag{17}$$

$$kur(X) = \frac{E(X^4) - 4(E(X))(E(X^3)) + 6(E(X))^2 var(X) + 3(E(X))^4}{(var(X))^2} \tag{18}$$

Based on the first four moments, the skewness and the kurtosis of the TMI random variable, X are given by:

$$\begin{aligned} Sk(X) &= \frac{(2p^2 + 3p - 3p\lambda)((1+p)^3(1+2p)^3(2+p)^{\frac{3}{2}})}{(3+p)(3+2p)(-p(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1))^{\frac{3}{2}}} \\ &- \frac{(2p^2 + p - p\lambda)^3(2+p)^{\frac{3}{2}}}{(-p(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1))^{\frac{3}{2}}} \\ &- \frac{3(2p^2 + p - p\lambda)\sqrt{(2+p)}}{\sqrt{(-p(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1))}} \end{aligned}$$

$$\begin{aligned} kur(X) &= \frac{(p^2 + 2p - 2p\lambda)(1+p)^4(2+p)(1+2p)^4}{(2+p)(-p(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1))} \\ &- \frac{4(1+p)^3(2+p)^2(1+2p)^3(2p+1-\lambda)(2p+3-3\lambda)}{(3+p)(3+2p)(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1)} \\ &+ \frac{[(1+p)(2+p)(1+2p)(2p+1-\lambda)]^2}{p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1} \\ &+ \frac{4p^2(2p+1-\lambda)(2+p)^2}{(p^2\lambda^2 + 2p\lambda^2 - 2p^2\lambda + p\lambda + \lambda - 4p^2 - 4p - 1)^2} \end{aligned}$$

Table 1. The mean, standard deviation, skewness, kurtosis and the coefficient of variation of the TMI distribution for different values of λ when p=2 and θ=5

λ	$\mu = E(X_{TMI})$	σ_x	$Sk(X)$	$Kur(X)$	$CV(\%)$
-1.0	4.000	0.816	-1.050	3.696	20.412
-0.9	3.933	0.883	-1.103	3.883	22.438
-0.8	3.867	0.939	-1.088	3.776	24.291
-0.7	3.800	0.988	-1.042	3.574	26.007
-0.6	3.733	1.031	-0.981	3.352	27.606
-0.5	3.667	1.067	-0.913	3.138	29.105
-0.4	3.600	1.098	-0.842	2.944	30.513
-0.3	3.533	1.125	-0.771	2.774	31.839
-0.2	3.467	1.147	-0.701	2.627	33.086
-0.1	3.400	1.165	-0.632	2.503	34.258
0.0	3.333	1.179	-0.566	2.400	35.356
0.1	3.267	1.188	-0.501	2.317	36.378
0.2	3.200	1.194	-0.440	2.252	37.326
0.3	3.133	1.197	-0.381	2.204	38.194
0.4	3.067	1.195	-0.326	2.172	38.979
0.5	3.000	1.190	-0.275	2.153	39.675
0.6	2.933	1.181	-0.230	2.148	40.276
0.7	2.867	1.169	-0.189	2.150	40.764
0.8	2.800	1.152	-0.156	2.161	41.136
0.9	2.733	1.131	-0.134	2.173	41.374
1.0	2.667	1.106	-0.125	2.180	41.457

Table 1 shows the values of the mean, standard deviation, skewness, kurtosis and the coefficient of variation (CV) of the TMI random variable for different values of λ when $p = 2$ and θ=5. The table shows that as λ increases the mean decreases. The kurtosis decreases as well. It, also tells us that the shape of the distribution is always skewed to the left regardless the value of λ. The shape of the distribution has sharper peak as λ decreases. The table, as well shows that the mean of the MI distribution, which equals to 3.333, is not far from the mean of the TMI distribution for $-0.5 \leq \lambda \leq 0.5$. Both means are equal when $\lambda = 0$.

5.2 Moment Generating Function

Theorem 5.2 The moment generating function (MGF) of the TMI random variable is given by

$$M_x(t) = \sum_{k=0}^{\infty} \frac{p(2p + k - \lambda k)}{(k + p)(k + 2p)} \frac{(t\theta)^k}{k!} \tag{19}$$

Proof.

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \int_0^\theta e^{tx}\psi(x)dx \\
&= \int_0^\theta \left(\frac{(1+\lambda)p}{\theta^p}x^{p-1} - \frac{2\lambda p}{\theta^{2p}}x^{2p-1}\right)e^{tx}dx \\
&= \frac{(1+\lambda)p}{\theta^p}\int_{k=0}^\theta x^{p-1}e^{tx}dx - \frac{2\lambda p}{\theta^{2p}}\int_{k=0}^\theta x^{2p-1}e^{tx}dx \\
&= \frac{(1+\lambda)p}{\theta^p}\int_{k=0}^\theta x^{p-1}\sum_{k=0}^{\infty}\frac{t^kx^k}{k!}dx - \frac{2\lambda p}{\theta^{2p}}\int_{k=0}^\theta x^{2p-1}\sum_{k=0}^{\infty}\frac{t^kx^k}{k!}dx \\
&= \frac{(1+\lambda)p}{\theta^p}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{k=0}^\theta x^{k+p-1}dx - \frac{2\lambda p}{\theta^{2p}}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{k=0}^\theta x^{k+2p-1}dx \\
&= \frac{(1+\lambda)p}{\theta^p}\sum_{k=0}^{\infty}\frac{t^k}{k!(k+p)}\theta^{k+p}dx - \frac{2\lambda p}{\theta^{2p}}\sum_{k=0}^{\infty}\frac{t^k}{k!(k+2p)}\theta^{k+2p}dx \\
&= (1+\lambda)p\sum_{k=0}^{\infty}\frac{t^k}{k!(k+p)}\theta^k dx - 2\lambda p\sum_{k=0}^{\infty}\frac{t^k}{k!(k+2p)}\theta^k dx \\
&= \sum_{k=0}^{\infty}\left[\frac{(1+\lambda)p}{k+p} - \frac{2\lambda p}{k+2p}\right]\frac{(t\theta)^k}{k!} \\
&= \sum_{k=0}^{\infty}\left[\frac{(1+\lambda)(k+2p)p - 2\lambda p(k+p)}{(k+p)(k+2p)}\right]\frac{(t\theta)^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{p(2p+k-\lambda k)}{(k+p)(k+2p)}\frac{(t\theta)^k}{k!}
\end{aligned}$$

□

6. Maximum Likelihood Estimates

Definition 6.1 Let $X_1, X_2, ..., X_n$ be a random sample size n with a $pdf\ \psi(x)$. The likelihood function is defined as the joint density of the random sample, which is defined as

$$\ell = L(\lambda, \theta, p|x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \psi(x_i|\lambda, \theta, p)$$

Hence, the likelihood function is given by

$$\begin{aligned} \ell &= \prod_{i=1}^{n}\left(\frac{(1+\lambda)p}{\theta^p}x_i^{p-1} - \frac{2\lambda p}{\theta^{2p}}x_i^{2p-1}\right) \\ &= \prod_{i=1}^{n} p\frac{x_i^{p-1}}{\theta^p}\left((1+\lambda) - \frac{2\lambda}{\theta^p}x_i^p\right) \end{aligned}$$

Therefore, the log-likelihood function is given by

$$\begin{aligned} ln\ell &= ln\left(\prod_{i=1}^{n} p\frac{x_i^{p-1}}{\theta^p}\left((1+\lambda) - \frac{2\lambda}{\theta^p}x_i^p\right)\right) \\ &= nlnp + (p-1)\sum_{i=1}^{n} lnx_i - npln\theta + \sum_{i=1}^{n} ln\left(1 + \lambda - \lambda\frac{x_i^p}{\theta^p}\right) \end{aligned} \tag{20}$$

Deriving Equation (20) withe respect to the parameters we get:

$$\begin{aligned} \frac{\partial\ell}{\partial\theta} &= \frac{1}{\theta}\left(-np + \sum_{i=1}^{n} \frac{\lambda x_i^p}{(1+\lambda)\theta^p - \lambda x_i^p}\right) \\ \frac{\partial\ell}{\partial p} &= \frac{n}{p} + \sum_{i=1}^{n} ln(x_i) - nln(\theta) - \sum_{i=1}^{n} \frac{x_i^p ln\left(\frac{x_i}{\theta}\right)}{(1+\lambda)\theta^p - \lambda x_i^p} \\ \frac{\partial\ell}{\partial\lambda} &= \sum_{i=1}^{n} \frac{\theta^p - x_i^p}{\theta^p - \lambda(\theta^p - x_i^p)} \end{aligned} \tag{21}$$

Equating the system of derivatives in Equation (21) to zero, we get the following system of equations

$$\begin{aligned} \sum_{i=1}^{n} \frac{\lambda x_i^p}{(1+\lambda)\theta^p - \lambda x_i^p} &= np \\ \frac{n}{p} - \sum_{i=1}^{n} \frac{x_i^p ln\left(\frac{x_i}{\theta}\right)}{(1+\lambda)\theta^p - \lambda x_i^p} &= nln(\theta) - \sum_{i=1}^{n} ln(x_i) \\ \sum_{i=1}^{n} \frac{\theta^p - x_i^p}{\theta^p - \lambda(\theta^p - x_i^p)} &= 0 \end{aligned}$$

There is no exact solution for this system of equations. So, to get the maximum likelihood estimates for the distribution parameters, we have to solve this system numerically.

7. Entropy

7.1 Renyi Entropy

Theorem 7.1 The Renyi entropy of order $\beta \geq 0$ is defined is

$$E_\beta = \frac{1}{1-\beta} log \sum_{i=0}^{n} \binom{\beta}{i} (1+\lambda)^i (-2\lambda)^{\beta - i} \frac{p^\beta \theta^{(1-\beta)}}{2p\beta - pi - \beta + 1}$$

Proof.

$$
\begin{aligned}
E_\beta &= \frac{1}{1-\beta} log \int_0^\theta (\psi(x))^\beta dx \\
&= \frac{1}{1-\beta} log \int_0^\theta \left(\frac{(1+\lambda)p}{\theta^p} x^{p-1} - \frac{2\lambda p}{\theta^{2p}} x^{2p-1} \right)^\beta dx \\
&= \frac{1}{1-\beta} log \int_0^\theta p^\beta \left(\frac{(1+\lambda)x^{p-1}}{\theta^p} - \frac{2\lambda x^{2p-1}}{\theta^{2p}} \right)^\beta dx \\
&= \frac{1}{1-\beta} log \int_0^\theta \frac{p^\beta x^{\beta(p-1)}}{\theta^{\beta p}} \left((1+\lambda) - \frac{2\lambda x^p}{\theta^p} \right)^\beta dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1-\beta} log \int_0^\theta \frac{p^\beta x^{\beta(p-1)}}{\theta^{\beta p}} \left(\sum_{i=0}^n \binom{\beta}{i} (1+\lambda)^i (-2\lambda)^{\beta-i} \left(\frac{x^p}{\theta^p} \right)^{\beta-i} \right) dx \\
&= \frac{1}{1-\beta} log \sum_{i=0}^n \binom{\beta}{i} \frac{(1+\lambda)^i (-2\lambda)^{\beta-i} p^\beta}{\theta^{p(2\beta-i)}} \int_0^\theta x^{(2p\beta-\beta-pi)} dx \\
&= \frac{1}{1-\beta} log \sum_{i=0}^n \binom{\beta}{i} \frac{(1+\lambda)^i (-2\lambda)^{\beta-i} p^\beta}{\theta^{p(2\beta-i)}} \frac{\theta^{(2p\beta-\beta-pi+1)}}{2p\beta-\beta-pi+1} \\
&= \frac{1}{1-\beta} log \sum_{i=0}^n \binom{\beta}{i} (1+\lambda)^i (-2\lambda)^{\beta-i} \frac{p^\beta \theta^{(1-\beta)}}{2p\beta - pi - \beta + 1}
\end{aligned}
$$

□

7.2 Tsallis Entropy

Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows:

$$
\begin{aligned}
E_T(X) &= \frac{1}{q-1} \Big(1 - \int_0^\theta (\psi(x))^q dx \Big), \quad x \geq 0 \\
&= \frac{1}{q-1} \left[1 - \sum_{i=0}^n \binom{q}{i} (1+\lambda)^i (-2\lambda)^{q-i} \frac{p^q \theta^{(1-q)}}{(2pq - pi - q + 1)} \right]
\end{aligned}
$$

8. Quantile Function

The quantile value is a value, say x, of the random variable, X, with CDF $\Psi(x)$ such that $\Psi(x) = p(X \leq x) = q$, where $0 < q < 1$. Therefore, the quantile of the TMI distribution is given by

$$
x = \theta \sqrt[p]{\frac{1 + \lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}}, \quad \lambda \neq 0 \tag{22}
$$

Proof.

$$
\begin{aligned}
\Psi(x) &= q \\
(1+\lambda)\frac{x^p}{\theta^p} - \lambda \frac{x^{2p}}{\theta^{2p}} &= q
\end{aligned}
$$

Assume $y = \frac{x^p}{\theta^p}$, then

$$
\begin{aligned}
(1+\lambda)y - \lambda y^2 &= q \\
\lambda y^2 - (1+\lambda)y + q &= 0
\end{aligned}
$$

Using the general formula for quadratic equations, we get

$$
y = \frac{1 + \lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}
$$

replacing y by its value $\frac{x^p}{\theta^p}$, we have

$$\begin{aligned}
\frac{x^p}{\theta^p} &= \frac{1+\lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda} \\
x^p &= \left(\frac{1+\lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right)\theta^p \\
\therefore x &= \theta \sqrt[p]{\frac{1+\lambda \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}}
\end{aligned}$$

□

The median of a continuous random variable X is defined to be the value m such that $\Psi(m) = \frac{1}{2}$. Hence, the median is the quantile value when $q = \frac{1}{2}$. Therefore,

$$\begin{aligned}
m &= \theta \sqrt[p]{\frac{1+\lambda \pm \sqrt{(1+\lambda)^2 - 2\lambda}}{2\lambda}} \\
&= \theta \sqrt[p]{\frac{1+\lambda \pm \sqrt{1+\lambda^2}}{2\lambda}}
\end{aligned}$$

9. Conclusion

In this paper, a generalization of Mukherjee-Islam distribution of failure time is introduced. It is called the transmuted Mukherjee-Islam distribution. We have studied some properties of this distribution, such as: moments, mean, variance, order statistics, maximum likelihood estimates of the distribution parameters. we, also have found the reliability and hazard rate functions, Renyi and Tsallis entropies and the quantile function as well as the median. The mean and the kurtosis decrease as the value of λ increases. The shape of the distribution is left skewed always regardless the value of p and λ.

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