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The Convergence of Calderón Reproducing Formulae of Two Parameters in L^p, in $\mathscr{S}$ and in $\mathscr{S}'$

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Abstract

The Calderón reproducing formula is the most important in the study of harmonic analysis, which has the same property as the one of approximate identity in many special function spaces. In this note, we use the idea of separation variables and atomic decomposition to extend single parameter to two-parameters and discuss the convergence of Calderón reproducing formulae of two-parameters in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Keywords: atomic decomposition, Calderón reproducing formula, Littlewood-Paley, Plancherel-Pôlya inequality

1. Introduction

The main purpose of this article is to construct some Calderón reproducing formulae for two parameters. For this, we recall some history about Calderón reproducing formulae and some works in this field. There are some type of Calderón reproducing formulae with applications to characterization of function spaces in harmonic analysis. The famous one is the reproducing formula generated by Poisson kernel

$$P_t(x) := C_n t(|x|^2 + t^2)^{-(n+1)/2}, \text{ for } t > 0 \text{ and } x \in \mathbb{R}^n.$$

See (Frazier, M., Jawerth, B. & Weiss, G., 1991) for details and we describe shortly here. To do this we choose a radial function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with the support of φ contained in the unit ball at the origin in $\mathbb{R}^n$ and $\int \varphi(x)dx = 0$. Using normalization, we always assume that $\int_0^\infty \widehat{\varphi}(se_1)e^{-s}ds = -1$, where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$, where $\widehat{\varphi}$ is the Fourier transform of φ. Then we have the Calderón reproducing formula in the following: for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$f(x) = \int_0^\infty \int_{\mathbb{R}^n} \Big\{t\frac{\partial}{\partial t}(P_t * f)(y)\Big\}\varphi_t(x-y)dy\frac{dt}{t},$$

where $\varphi_t = t^{-n}\varphi(t^{-1}x)$ as usual. Next let us make a discretization on the last equation. For this purpose we define a family of dyadic cubes in $\mathbb{R}^n$. We say that $Q \subseteq \mathbb{R}^n$ is a dyadic cube if there exist $\nu \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$ such that

$$Q = Q_{\nu\mathbf{k}} = \{x \in \mathbb{R}^n : 2^{-\nu}k_i \le x_i < 2^{-\nu}(k_i + 1),\ i = 1, 2, \cdots, n\}.$$

Set $\mathscr{Q} := \{Q_{\nu\mathbf{k}} : \nu \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ and $T(Q) = Q \times [\frac{\ell(Q)}{2}, \ell(Q)] \subseteq \mathbb{R}^{n+1}_+$, where $\ell(Q)$ is the side length of the given dyadic cube Q. Then $\mathbb{R}^{n+1}_+ = \cup_{Q\in\mathscr{Q}} T(Q)$ is a union of pairwise disjoint sets. If we set

$$s_Q := \Big(\iint_{T(Q)} \Big|t\frac{\partial}{\partial t}(P_t * f)(y)\Big|^2 dy\frac{dt}{t}\Big)^{1/2},$$

and if $s_Q \neq 0$, we also set

$$a_Q(x) = \frac{1}{s_Q}\iint_{T(Q)} t\frac{\partial}{\partial t}(P_t * f)(y)\varphi_t(x-y)dy\frac{dt}{t}.$$

Thus we obtain a decomposition $f(x) = \sum_{Q\in\mathscr{Q}} s_Q a_Q$, which is an atomic decomposition. Moreover, the function a_Q satisfies $a_Q \in C^\infty$, $\int a_Q(x)dx = 0$ and $\mathrm{supp}(a_Q) \subseteq 3Q$, where cQ denote the cube concentric with Q whose each edge is c times as long. We found that the coefficient s_Q and the Littlewood-Paley g_2 function

$$g_2(f)(x) := \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}(P_t * f)(y)\Big|^2 dy\frac{dt}{t}\Big)^{1/2}$$

are closely related, $\sum_{Q\in\mathcal{Q}} |s_Q|^2 = \|g_2(f)\|_{L^2}^2$. Therefore,

$$\|f\|_{L^2}^2 \approx \sum_{Q\in\mathcal{Q}} |s_Q|^2.$$

In general, if the Poisson kernel $P_t(x)$ is replaced by any compactly support kernel $\varphi \in L^1(\mathbb{R}^n)$ satisfying

(i) φ is real-valued and radial;

(ii) $\mathrm{supp}(\varphi) \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$;

(iii) $\varphi \in C^\infty(\mathbb{R}^n)$;

(iv) $\int_0^\infty |\widehat{\varphi}(t\xi)|^2 \frac{dt}{t} = 1$, for $\xi \in \mathbb{R}^n \setminus \{0\}$,

then we have a Calderón reproducing formula

$$f(x) = \int_0^\infty (\varphi_t * \varphi_t * f)(x)\frac{dt}{t}.$$

This is called a continuous version of Calderón reproducing formula. In (Frazier, M. & Jawerth, B., 1990), authors constructed a discrete version of Calderón reproducing formula which is called the φ-transform identity. Namely, choose $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\mathrm{supp}(\widehat{\varphi}) \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and $|\widehat{\varphi}| \geq c > 0$ when $3/5 \leq |\xi| \leq 5/3$. Then there is $\psi \in \mathscr{S}(\mathbb{R}^n)$ satisfying the same condition such that

$$\sum_{\nu\in\mathbb{Z}} \overline{\widehat{\varphi}(2^{-\nu}\xi)}\widehat{\psi}(2^{-\nu}\xi) = 1, \quad \xi \neq 0.$$

Equivalently,

$$f = \sum_{\nu\in\mathbb{Z}} \widetilde{\varphi}_{2^{-\nu}} * \psi_{2^{-\nu}} * f,$$

where $\widetilde{f}(x) = \overline{f(-x)}$. Using a sampling theorem, the last equality can be discretized as

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \psi_Q,$$

where, for a given function g and a dyadic cube $Q = Q_{\nu\mathbf{k}}$ with left corner $x_Q = 2^{-\nu}\mathbf{k}$, $g_Q(x) = |Q|^{-1/2} g(\frac{x-x_Q}{\ell(Q)})$.

In fact, the concept of Calderón reproducing formulae is to write a given function into a sum of convolutions $K_\nu * f$, where K is a kernel satisfying certain conditions. This was first appeared in a paper written by Calderón in (Calderón, A. P., 1964) dealing on complex interpolation in 1967. In 1975, Calderón introduced parabolic Hardy spaces in (Calderón, A. P., 1977) in sense of atomic decomposition which is more popular now. In 1980, Chang and R. Fefferman introduced atomic decomposition of H^1 on the bidisc (Chang, S.-Y. A. & Fefferman, R., 1980). In 1982, Uchiyama gave a concise form of Calderón reproducing formula in (Uchiyama, A., 1982). Frazier and Jawerth introduced the φ-transform identity and smooth atomic decomposition in (Frazier, M. & Jawerth, B., 1990). It is natural to ask what does the Calderón reproducing formula converges in some sense?

To describe the convergence of a Calderón reproducing formula of one parameter case, choose $H, h \in \mathscr{S}(\mathbb{R}^n)$ satisfying

(i) $\mathrm{supp}(\widehat{H}) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$;

(ii) $\mathrm{supp}(\widehat{h}) \subseteq \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$;

(iii) $\widehat{H}(\xi) + \sum_{\nu=1}^\infty \widehat{h}_\nu(\xi) = 1$, for all $\xi \in \mathbb{R}^n$;

(iii′) $\sum_{\nu=-\infty}^\infty \widehat{h}_\nu(\xi) = 1$, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

where $h_\nu(x) = 2^{\nu n} h(2^\nu x)$. For a given (generalized) function f, set $f_N := H * f + \sum_{\nu=1}^\infty h_\nu * f$ for $N \in \mathbb{N}$. Then we have the following well-known results which were given by Frazier, Jawerth and Weiss.

Proposition 1.1. *Let H and h satisfy conditions* (i), (ii) *and* (iii).

(a) *If* $f \in L^p(\mathbb{R}^n)$ *for* $1 \le p < \infty$, *then* f_N *converges to* f *a.e. and in* L^p *as* $N \to \infty$.

(b) *If* $f \in \mathscr{S}(\mathbb{R}^n)$, *then* f_N *converges to* f *in* $\mathscr{S}(\mathbb{R}^n)$ *as* $N \to \infty$.

(c) *If* $f \in \mathscr{S}'(\mathbb{R}^n)$, *then* f_N *converges to* f *in* $\mathscr{S}'(\mathbb{R}^n)$ *as* $N \to \infty$.

Proposition 1.2. *Let* h *satisfy conditions* (ii) *and* (iii′). *Also set* $f_N := \sum_{\nu=-N}^{\infty} h_\nu * f$ *for* $N \in \mathbb{N}$.

(a) *If* $f \in L^p(\mathbb{R}^n)$ *for* $1 \le p < \infty$, *then* f_N *converges to* f *a.e. as* $N \to \infty$.

(b) *If* $f \in L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$ *then* f_N *converges to* f *in* L^p *as* $N \to \infty$.

(c) *If* $f \in L^1(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} f(x)dx = 0$ *then* f_N *converges to* f *in* L^1 *as* $N \to \infty$.

(d) *If* $f \in \mathscr{S}_\infty(\mathbb{R}^n)$, *then* f_N *converges to* f *in* $\mathscr{S}(\mathbb{R}^n)$ *as* $N \to \infty$.

(e) *If* $f \in \mathscr{S}'(\mathbb{R}^n)$, *then* f_N *converges to* f *in* $\mathscr{S}'_k(\mathbb{R}^n)$ *as* $N \to \infty$, *where* $k = \deg \widehat{f}$.

There are some works concerning Calderón reproducing formulae on space of homogeneous type, see (Han, Y.-S., 1997; Deng, D.-G. & Han, Y.-S., 1995) for details. Also there are some concerning Calderón reproducing formulae associated to para-accretive functions, see (Han, Y.-S. & Yang, D.-C., 2005; Yang, D.-C., 2005) lfor details. We do not intend to complete the list concerning Calderón reproducing formulae here. In harmonic analysis, there are some study on singular integral operators on product spaces, see (Fefferman, R. & Stein, E. M., 1982; Journé, J.-L., 1985; Fefferman, R., 1981; Fefferman, R., 1987; Han, Y.-S., & et al. 2010) for more details. Here we will study the convergence of a Calderón reproducing formula of two parameters. In general, the case of m parameters is the same but more tedious.

The paper is organized as follows. In Section 2, we construct a Calderón reproducing formula for inhomogeneous case and then study the convergence of such a Calderón reproducing formula in $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, in $\mathscr{S}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and in $\mathscr{S}'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. In Section 3, we will do the same argument but for homogeneous case.

Throughout, we Use C to denote a universal constant that does not depend on the main variables but may differ from line to line. Also, Q and P always means the dyadic cubes in $\mathbb{R}^n$ or in $\mathbb{R}^{n_i}$, and for $r > 0$, we denote by rQ the cube concentric with Q whose side length is r times as long.

2. A Calderón Reproducing Formula: Inhomogeneous Cases

For convenience, we first give some notations. Let $\mathbb{Z}_+^n := \{\alpha = (\alpha_1, \cdots, \alpha_n) : \alpha_i \in \mathbb{Z}_+, i = 1, \ldots, n\}$. We say α is a multi-index means $\alpha \in \mathbb{Z}_+^n$ and its norm is the sum of its components, i.e., $|\alpha| = \sum_{i=1}^n \alpha_i$. Also D^α denote the partial differential operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

To emphasize, we use D_x^α instead of D^α for $x \in \mathbb{R}^n$. For $\alpha \in \mathbb{Z}_+^n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. For $\mathbf{k}^i \in \mathbb{Z}^{n_i}$, $i = 1, 2$, let us consider a rectangle $Q \times P \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $Q = Q_{j\mathbf{k}^1}$ is a dyadic cube in $\mathbb{R}^{n_1}$ and $P = P_{k\mathbf{k}^2}$ is a dyadic cube in $\mathbb{R}^{n_2}$. As usual, $x_Q = 2^{-j}\mathbf{k}^1$ and $y_P = 2^{-k}\mathbf{k}^2$ denote the left corner of Q and P, respectively. For any function f defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, let

$$f_{QP}(x, y) := |Q|^{-1/2}|P|^{-1/2} f(\frac{x - x_Q}{\ell(Q)}, \frac{y - y_P}{\ell(P)}) = 2^{jn_1/2} 2^{kn_2/2} f(2^j x - k^1, 2^k y - k^2),$$

$$f_{jk}(x, y) = 2^{jn_1} 2^{kn_2} f(2^j x, 2^k y),$$

$$\widetilde{f}(x, y) = \overline{f(-x, -y)}.$$

From these, it is easy to check $\widetilde{g}_{jk} * f(x_Q, y_P) = |Q|^{-1/2}|P|^{-1/2}\langle f, g_{QP}\rangle$ if it exists.

Choose $H^i, h^i \in \mathscr{S}(\mathbb{R}^{n_i})$, $i = 1, 2$, satisfy

(i) $\mathrm{supp}(\widehat{H^i}) \subseteq \{\xi_i \in \mathbb{R}^{n_i} : |\xi_i| \le 2\}$,

(ii) $\mathrm{supp}(\widehat{h^i}) \subseteq \{\xi_i \in \mathbb{R}^{n_i} : 1/2 \le |\xi_i| \le 2\}$,

(iii) $\widehat{H^i}(\xi_i) + \sum_{\nu=1}^{\infty} h_\nu^i(\xi_i) = 1$ for every $\xi_i \in \mathbb{R}^{n_i}$, where $h^i_\nu(x) = 2^{\nu n_i} h^i(2^\nu x)$ for $\nu \in \mathbb{Z}$. Then we can deduce an equation in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$:

$$\widehat{\Phi}(\xi_1, \xi_2) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \widehat{h}_{jk}(\xi_1, \xi_2) = 1 \quad \forall\, (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{2.1}$$

where

$$\begin{aligned}\widehat{\Phi}(\xi_1, \xi_2) &= \widehat{H}^1(\xi_1)\widehat{H}^2(\xi_2) + \widehat{H}^1(\xi_1) \sum_{k=1}^{\infty} h^2_k(\xi_2) + \widehat{H}^2(\xi_2) \sum_{k=1}^{\infty} h^1_j(\xi_1) \\ &= \widehat{H}^1(\xi_1)\widehat{H}^2(\xi_2) + \widehat{H}^1(\xi_1)(1 - \widehat{H}^2(\xi_2)) + \widehat{H}^2(\xi_2)(1 - \widehat{H}^1(\xi_1)).\end{aligned} \tag{2.2}$$

For $N = (N_1, N_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and a (generalized) function f defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we define a truncated Calderón formula for two parameters in the following way:

$$f_N(x, y) := \Phi * f(x, y) + \Big(\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} h^1_j h^2_k \Big) * f(x, y). \tag{2.3}$$

Let $\Phi_N(x, y) := 2^{N_1 n_1 + N_2 n_2} \Phi(2^{N_1} x, 2^{N_2} y)$ for $N = (N_1, N_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. From equation (2.1) we have

$$\widehat{\Phi}(\xi_1, \xi_2) = 1 - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \widehat{h}^1_j(\xi_1)\widehat{h}^2_k(\xi_2),$$

and hence

$$\begin{aligned}\widehat{\Phi}_N(\xi_1, \xi_2) &= 1 - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \widehat{h}^1(2^{-j-N_1}\xi_1)\widehat{h}^2(2^{-k-N_2}\xi_2) \\ &= 1 - \sum_{j=N_1+1}^{\infty} \sum_{k=N_2+1}^{\infty} \widehat{h}^1_j(\xi_1)\widehat{h}^2_k(\xi_2)\end{aligned} \tag{2.4}$$

The Fourier transform of f_N given in equation (2.3) is

$$\begin{aligned}\widehat{f_N} &= \widehat{\Phi}\widehat{f} + \Big(\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \widehat{h}^1_j \widehat{h}^2_k \Big) \widehat{f} \\ &= \widehat{f}\Big(\widehat{\Phi} + \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} h^1_j h^2_k \Big) \\ &= \widehat{f}\Big(\Big(1 - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \widehat{h}^1_j \widehat{h}^2_k\Big) + \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \widehat{h}^1_j \widehat{h}^2_k \Big) \\ &= \widehat{f}\Big(\Big[1 - \Big(\sum_{j=1}^{N_1} \widehat{h}^1_j + \sum_{j=N_1+1}^{\infty} \widehat{h}^1_j \Big)\Big(\sum_{k=1}^{N_2} \widehat{h}^2_k + \sum_{k=N_2+1}^{\infty} \widehat{h}^2_k \Big)\Big] + \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} h^1_j h^2_k \Big) \\ &= \widehat{f}\Big(\Big[1 - \Big(\sum_{j=N_1+1}^{\infty} \sum_{k=N_2+1}^{\infty} \widehat{h}^1_j \widehat{h}^2_k \Big)\Big] - \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} \widehat{h}^1_j \widehat{h}^2_k \Big) - \Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} \widehat{h}^1_j \widehat{h}^2_k \Big)\Big) \\ &= \Big(1 - \sum_{j=N_1+1}^{\infty} \sum_{k=N_2+1}^{\infty} \widehat{h}^1_j \widehat{h}^2_k \Big)\widehat{f} - \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} \widehat{h}^1_j \widehat{h}^2_k \Big)\widehat{f} - \Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} \widehat{h}^1_j \widehat{h}^2_k \Big)\widehat{f}.\end{aligned}$$

Hence we have

$$f_N = \Phi_N * f - \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h^1_j h^2_k \Big) * f - \Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} h^1_j h^2_k \Big) * f. \tag{2.5}$$

We say that $f_N \to f$ as $N \to \infty$ in certain sense means that $f_N \to f$ in certain sense as $N_1 \to \infty$ and $N_2 \to \infty$. To prove one of main theorems, we need the following two lemmas.

Lemma 2.1. *With the notations above, we have the following results.*

(a) *If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f$ converges to 0 pointwise and in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

(b) *If $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

(c) *If $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

Proof. For part (a), since $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) \to 0$ as $N \to \infty$ and using the continuity of convolution on L^p,

$$\Big| \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \Big| \to 0 \quad \text{as} \quad N \to \infty,$$

and hence, by Lebesgue dominated convergence theorem, $\Big\| \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \Big\|_{L^p} \to 0$ *as* $N \to \infty$.

To show part (b), we use Leibniz' formula for each multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$,

$$\begin{aligned} D^\alpha \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 * f \Big) &= D^\alpha \Big[\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) \Big] * f \\ &= \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * D^\alpha f, \end{aligned}$$

because $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) \to 0$ as $N \to \infty$ and the continuity of convolution on $\mathscr{S}$. Therefore, $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

The proof of part (c) follows from part (b) by a duality argument. Let $g \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we have

$$\Big(\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f, g \Big) = \Big(f, \Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} \widetilde{h_j^1} \widetilde{h_k^2} \Big) * g \Big).$$

Applying part(b), one has

$$\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} \widetilde{h_j^1} \widetilde{h_k^2} \Big) * g \to 0.$$

Hence

$$\Big(\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f, g \Big) \to 0 \text{ as } N \to \infty$$

for every $g \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and so $\Big(\sum_{j=1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. □

The proof of the convergence of $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} h_j^1 h_k^2 \Big)$ is similar with Lemma 2.1 in each case, hence we have the following results.

Lemma 2.2. *With the notations above, we have*

(a) *if $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} h_j^1 h_k^2 \Big) * f$ converges to 0 pointwise and in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$;*

(b) *if $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} h_j^1 h_k^2 \Big) * f \to 0$ as $N \to \infty$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$;*

(c) *if $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=1}^{N_2} h_j^1 h_k^2 \Big) * f \to 0$ as $N \to \infty$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

After we obtain Lemmas 2.1 and 2.2, we can show that the convergence of the Calderón reproducing formula on L^p, on $\mathscr{S}$ and $\mathscr{S}'$. We first consider that f_N converges on $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Theorem 2.3. *Let $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $1 \le p < \infty$. If f_N is defined in* (2.3) *then f_N converges to f a.e. and in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$.*

Proof. By (2.5), Lemma 2.1 part (a) and Lemma 2.2 part (a), it suffices to show that $\|\Phi_N * f\|_{L^p} \to f$ as $N \to \infty$. Note that Φ is an approximate identity since Φ satisfies $\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} \Phi(x,y)dxdy = 1$. Let $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathbf{s} = (t,s) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. By the continuity of norm, for any $\varepsilon > 0$, there exists $r > 0$ such that $\|f_{-\mathbf{s}} - f\|_{L^p} < \varepsilon/2$ when $|t| + |s| < r$. Since Φ is an approximate identity, there exists $M > 0$ such that $\iint_{|x|+|y|\ge r} |\Phi_N(x,y)|dxdy < \frac{\varepsilon}{4\|f\|_{L^p}}$ for $\min\{N_1, N_2\} > M$. For $1{=}\iint \Phi_N(t,s)dtds$,

$$f * \Phi_N(x,y) - f(x,y) = \iint (f(x-t, y-s) - f(x,y))\Phi_N(t,s)dtds.$$

Hence, by Minkowski integral inequality,

$$\begin{aligned}
\|f * \Phi_N - f\|_{L^p} &\le \Big(\iint\Big(\iint |f(x-t,y-s) - f(x,y)||\Phi_N(t,s)|dtds\Big)^p dxdy\Big)^{\frac{1}{p}} \\
&\le \iint \|f_{-\mathbf{s}} - f\|_p |\Phi_N(t,s)|dtds \\
&\le \iint_{|t|+|s|<r} \|f_{-\mathbf{s}} - f\|_{L^p}|\Phi_N(t,s)|dtds \\
&\quad + \iint_{|t|+|s|\ge r} \|f_{-\mathbf{s}} - f\|_{L^p}|\Phi_N(t,s)|dtds \\
&< \frac{\varepsilon}{2} + 2\|f\|_p \frac{\varepsilon}{4\|f\|_{L^p}} = \varepsilon,
\end{aligned}$$

when $\min\{N_1, N_2\} > M$. Consequently, $\Phi_N * f \to f$ as $N \to \infty$ in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. □

Next let us consider the convergence of f_N in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. To obtain the convergence of f_N, it suffice to show the convergence of $\widehat{f_N}$ by the continuity of inverse Fourier transform. More precisely, $f_N \to f$ is equivalent to $\widehat{f_N} \to \widehat{f}$ as $N \to \infty$.

Theorem 2.4. *Let f_N be given in* (2.3)*. Then*

(a) *if $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ then $f_N \to f$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$;*

(b) *if $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ then $f_N \to f$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$.*

Proof. For part (a), by (2.5), Lemma 2.1(b) and continuity of inverse Fourier transform, it is enough to show that $\widehat{\Phi}_N\widehat{f} \to \widehat{f}$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, this is to show that

$$\sup_{(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1 + |\xi_1| + |\xi_2|)^M |D^\gamma(\widehat{f} - \widehat{\Phi}_N\widehat{f})| \to 0 \text{ as } N \to \infty, \text{ for } M > 0, \gamma \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}.$$

Since $\widehat{f} - \widehat{\Phi}_N\widehat{f} = \widehat{f}\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h_j^1}\widehat{h_k^2}\Big)$, we need to estimate $D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h_j^1}\widehat{h_k^2}\Big)$, and $D^\rho\widehat{f}(\xi_1,\xi_2)$, where $\sigma, \rho \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$.

First let us observe $I_N := \sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)$

$$I_N = \begin{cases} 0, & \text{if } |\xi_1| < 2^{N_1} \text{ or } |\xi_2| < 2^{N_2} \\ \widehat{h^1_{N_1+1}}(\xi_1), & \text{if } 2^{N_1} \le |\xi_1| < 2^{N_1+1} \text{ and } |\xi_2| > 2^{N_2+1} \\ \widehat{h^2_{N_2+1}}(\xi_2), & \text{if } 2^{N_2} \le |\xi_2| < 2^{N_2+1} \text{ and } |\xi_1| > 2^{N_1+1} \\ \widehat{h^1_{N_1+1}}(\xi_1)\widehat{h^2_{N_2+1}}(\xi_2), & \text{if } 2^{N_1} \le |\xi_1| < 2^{N_1+1} \\ & \quad \text{and } 2^{N_2} \le |\xi_2| < 2^{N_2+1} \\ 1, & \text{if } |\xi_1| > 2^{N_1+1} \text{ and } |\xi_2| > 2^{N_2+1} \end{cases}.$$

Let $\sigma = (\alpha,\beta) \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$, and then $D^\sigma = D^\alpha_{\xi_1} D^\beta_{\xi_2}$. Note that when $|\xi_1| < 2^{N_1}$, $|\xi_2| < 2^{N_2}$, or (both $|\xi_1| > 2^{N_1+1}$ and $|\xi_2| > 2^{N_2+1}$),

$$D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big) = 0.$$

Also, when $2^{N_1} \le |\xi_1| < 2^{N_1+1}$ and $2^{N_2} \le |\xi_2| < 2^{N_2+1}$,

$$D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big) = 2^{-(N_1+1)|\alpha|}(D^\alpha\widehat{h^1})(2^{-N_1-1}\xi_1)2^{-(N_2+1)|\beta|}(D^\beta\widehat{h^2})(2^{-N_2-1}\xi_2).$$

For $2^{N_1} \le |\xi_1| < 2^{N_1+1}$ and $|\xi_2| > 2^{N_2+1}$,

$$D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big) = \begin{cases} 0, & \text{if } \beta \neq 0 \\ 2^{-(N_1+1)|\alpha|}(D^\alpha\widehat{h^1})(2^{-N_1-1}\xi_1), & \text{if } \beta = 0 \end{cases}.$$

Similarly, for $2^{N_2} \le |\xi_2| < 2^{N_2+1}$ and $|\xi_1| > 2^{N_1+1}$,

$$D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big) = \begin{cases} 0, & \text{if } \alpha \neq 0 \\ 2^{-(N_2+1)|\beta|}(D^\beta\widehat{h^2})(2^{-N_2-1}\xi_2), & \text{if } \alpha = 0 \end{cases}.$$

Hence we get for $N \in \mathbb{Z}_+ \times \mathbb{Z}_+$, and $\sigma \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$,

$$D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big) \le C_\sigma \chi_{\{(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}: |\xi_1|>2^{N_1} \text{ and } |\xi_2|>2^{N_2}\}}.$$

Now if $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ then $\widehat{f} \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. For any $M > 0$ and multi-index ρ, we have $|D^\rho\widehat{f}(\xi_1,\xi_2)| \le C_{\rho,M}(1 + |\xi_1| + |\xi_2|)^{-M-1}$. Therefore, by the Leibniz's formula, for any $\gamma \in \mathbb{Z}_+^{n_1} \times \mathbb{Z}_+^{n_2}$ and $M > 0$,

$$\begin{aligned}
|D^\gamma(\widehat{f} - \widehat{\Phi_N f})| &= \Big|\sum_{\sigma+\rho=\gamma} C_{\sigma,\rho} D^\sigma\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h_j^1}(\xi_1)\widehat{h_k^2}(\xi_2)\Big)D^\rho\widehat{f}(\xi_1,\xi_2)\Big| \\
&\le C_{\gamma,M}(1 + |\xi_1| + |\xi_2|)^{-M-1}\chi_{\{(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}: |\xi_1|>2^{N_1} \text{ and } |\xi_2|>2^{N_2}\}} \\
&= C_{\gamma,M}\frac{(1 + |\xi_1| + |\xi_2|)^{-M}}{(1 + |\xi_1| + |\xi_2|)}\chi_{\{(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}: |\xi_1|>2^{N_1} \text{ and } |\xi_2|>2^{N_2}\}} \\
&\le C_{\gamma,M}(1 + |\xi_1| + |\xi_2|)^{-M}2^{-N'},
\end{aligned}$$

where $N' = \min\{N_1, N_2\}$. Hence $\sup_{(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}(1+|\xi_1|+|\xi_2|)^M|D^\gamma(\widehat{f}-\widehat{\Phi_N f})| \to 0$ as $N \to \infty$ for any $M > 0$ and multi-index ρ, i.e., $\Phi_N \to f$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$.

For part (b), let $g \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and then, by a duality argument, we have

$$\begin{aligned}
\langle f_N, g\rangle &= \langle \Phi_N * f - \Big(\sum_{j=1}^{N_1}\sum_{k=N_2+1}^{\infty} h_j^1 h_k^2\Big) * f - \Big(\sum_{j=N_1+1}^{\infty}\sum_{k=1}^{N_2} h_j^1 h_k^2\Big) * f, g\rangle \\
&= \langle \Phi_N * f, g\rangle - \Big\langle\Big(\sum_{j=1}^{N_1}\sum_{k=N_2+1}^{\infty} h_j^1 h_k^2\Big) * f, g\Big\rangle - \Big\langle\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=1}^{N_2} h_j^1 h_k^2\Big) * f, g\Big\rangle \\
&= \langle f, \tilde{\Phi}_N * g\rangle - \Big\langle f, \Big(\sum_{j=1}^{N_1}\sum_{k=N_2+1}^{\infty} \tilde{h}_j^1 \tilde{h}_k^2\Big) * g\Big\rangle - \Big\langle f, \Big(\sum_{j=N_1+1}^{\infty}\sum_{k=1}^{N_2} \tilde{h}_j^1 \tilde{h}_k^2\Big) * g\Big\rangle.
\end{aligned}$$

By part (a), Lemma 2.1 (c), Lemma 2.2 (c) and continuity of tempered distributions, we have $\langle f_N, g\rangle \to \langle f, g\rangle$ as $N \to \infty$, for all g in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, i.e., $f_N \to f$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$. □

At the end of this section, we give a concrete example to show the existence of a Calderón reproducing formula for inhomogeneous case. Choose $\varphi(x,y) = \varphi^1(x)\varphi^2(y) \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, where $\varphi^i \in \mathscr{S}(\mathbb{R}^{n_i})$ satisfying $\mathrm{supp}(\widehat{\varphi^i}) \subseteq \{\xi^i \in \mathbb{R}^{n_i} : 1/2 \le |\xi^i| \le 2\}$ and $\widehat{\varphi^i}(\xi^i) \ge c > 0$ when $3/5 \le |\xi^i| \le 5/3$, for $i = 1,2$. Let

$$K(\xi^1,\xi^2) := \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \widehat{\varphi^1_j}(\xi^1)\widehat{\varphi^2_k}(\xi^2).$$

Then $K(\xi^1,\xi^2) \ge c > 0$. Define a function ψ by $\widehat{\psi}(\xi^1,\xi^2) := \frac{\widehat{\varphi}(\xi^1,\xi^2)}{K(\xi^1,\xi^2)}$. Then ψ satisfies the same conditions as φ and hence

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \widehat{\varphi}_{jk}\widehat{\psi}_{jk} = 1.$$

Define Φ by

$$\widehat{\Phi} := 1 - \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \widehat{\varphi}_{jk}\widehat{\psi}_{jk}.$$

Since $\widehat{\varphi}, \widehat{\psi} \ge 0$, $\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \widehat{\varphi}_{jk}\widehat{\psi}_{jk} \le 1$. Thus $\widehat{\Phi} \ge 0$, $\Phi \in \mathscr{S}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $\mathrm{supp}(\widehat{\Phi}) \subseteq \{(\xi^1,\xi^2) : |\xi^i| \le 2, i = 1,2\}$. Set $\widehat{\Psi} = \sqrt{\widehat{\Phi}}$. Then $\Psi \in \mathscr{S}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $\mathrm{supp}(\widehat{\Psi}) \subseteq \{(\xi^1,\xi^2) : |\xi^i| \le 2, i = 1,2\}$ which is a compact set in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Therefore,

$$\widehat{\Psi}\cdot\widehat{\Psi} + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \widehat{\varphi}_{jk}\widehat{\psi}_{jk} = 1.$$

By the theorems above, we can represent f as $\Psi * \Psi * f + \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \varphi_{jk} * \psi_{jk} * f$, and then, using the sampling theorem, we get

$$f(x,y) = \sum_{\substack{Q_i\\ \ell(Q_j)=1}}\sum_{\substack{P_k\\ \ell(P_k)=1}} \langle f, \Psi_{QP}\rangle \Psi_{QP} + \sum_{\substack{Q_i\\ \ell(Q_j)<1}}\sum_{\substack{P_k\\ \ell(P_k)<1}} \langle f, \varphi_{QP}\rangle \psi_{QP}. \tag{2.6}$$

3. The Calderón Reproducing Formula: Homogeneous Cases

In this section, we consider the convergence of a Calderón reproducing formula in homogeneous case. First choose $h^i \in \mathscr{S}(\mathbb{R}^{n_i})$, $i = 1,2$, satisfy

(i) $\mathrm{supp}(\widehat{h^i}) \subseteq \{\xi_i \in \mathbb{R}^{n_i} : 1/2 \le |\xi_i| \le 2\}$,

(ii) $\sum_{\nu=-\infty}^{\infty} h^i_\nu(\xi_i) = 1$ for every $\xi_i \in \mathbb{R}^{n_i} \setminus \{0\}$,

where $h^i_\nu(x) = 2^{\nu n_i} h^i(2^\nu x)$ for $\nu \in \mathbb{Z}$. By (2.1), we have

$$\widehat{\Phi}_N(\xi_1,\xi_2) = 1 - \sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h^1_j}(\xi_1)\widehat{h^2_k}(\xi_2) \qquad \text{for } N = (N_1,N_2) \in \mathbb{Z}_+\times\mathbb{Z}_+.$$

Consider

$$\begin{aligned}
\widehat{\Phi}_N - \widehat{\Phi}_{-N-1} &= \Big(1 - \sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h^1_j}\widehat{h^2_k}\Big) - \Big(1 - \sum_{j=-N_1}^{\infty}\sum_{k=-N_2}^{\infty} \widehat{h^1_j}\widehat{h^2_k}\Big)\\
&= -\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty} \widehat{h^1_j}\widehat{h^2_k}\Big) + \Big(\sum_{j=-N_1}^{N_1}\widehat{h^1_j} + \sum_{j=N_1+1}^{\infty}\widehat{h^1_j}\Big)\Big(\sum_{k=-N_2}^{N_2}\widehat{h^2_k} + \sum_{k=N_2+1}^{\infty}\widehat{h^2_k}\Big)\\
&= \Big(\sum_{j=-N_1}^{N_1}\widehat{h^1_j}\sum_{k=-N_2}^{N_2}\widehat{h^2_k}\Big) + \Big(\sum_{j=-N_1}^{N_1}\sum_{k=N_2+1}^{\infty}\widehat{h^1_j}\widehat{h^2_k}\Big) + \Big(\sum_{j=N_1+1}^{\infty}\sum_{k=-N_2}^{N_2}\widehat{h^1_j}\widehat{h^2_k}\Big)
\end{aligned}$$

Let f be a (generalized) function and we define a truncated function f_N by

$$f_N := \Big(\sum_{j=-N_1}^{N_1}\sum_{k=-N_2}^{N_2} h^1_j h^2_k\Big) * f.$$

Thus we get

$$f_N = (\Phi_N - \Phi_{-N-1}) * f - \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f - \Big(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2 \Big) * f. \tag{3.7}$$

From (3.7), to see the convergence of f_N, it suffices to show the convergence of $\Phi_N * f$, $\Phi_{-N-1} * f$, $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f$ and $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2 \Big) * f$. From Theorems 2.3 and 2.4 in last section, $\Phi_N * f$ converges to f as $N \to \infty$. It remains to consider the others.

Lemma 3.1. *With the notations above, we have the following results.*

(a) *If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ pointwise and in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ where $1 \le p < \infty$.*

(b) *If $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

(c) *If $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

Proof. For part (a), by Lebesgue dominated convergence theorem the convolution

$$\Big| \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \Big| \to 0 \text{ as } N \to \infty$$

pointwise, since $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big)$ converge to 0 pointwise as $N \to \infty$. By the pointwise convergence of $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f$ and Lebesgue dominated convergence theorem again, $\Big\| \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \Big\|_{L^p} \to 0$ as $N \to \infty$.

For part (b), we see, for all multi-index α,

$$\begin{aligned} D^\alpha \Big(\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \Big) &= D^\alpha \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \\ &= \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * D^\alpha f. \end{aligned}$$

Because $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) \to 0$ as $N \to \infty$, and continuity of convolution, $\Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

The proof of part (c) follows from part(b) by duality. Precisely, for g in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we have

$$\Big\langle \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2 \Big) * f, g \Big\rangle = \Big\langle f, \Big(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} \tilde{h}_j^1 \tilde{h}_k^2 \Big) * g \Big\rangle.$$

Then, by part (b) $(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} \widetilde{h_j^1} \widetilde{h_k^2}) * g \to 0$. Hence $\langle (\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2) * f, g \rangle \to 0$ as $N \to \infty$, for all g in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, and so $(\sum_{j=-N_1}^{N_1} \sum_{k=N_2+1}^{\infty} h_j^1 h_k^2) * f \to 0$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. □

The proof of the convergence of $(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2)$ is similar with Lemma 3.1 in each case, hence we have the following results.

Lemma 3.2. *With the notations above, we have the following results.*

(a) *If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2 \Big) * f \to 0$ pointwise and in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ where $1 \le p < \infty$.*

(b) *If $f \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2 \Big) * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

(c) *If* $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$*, then* $\Big(\sum_{j=N_1+1}^{\infty} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2\Big) * f \to 0$ *in* $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$*.*

As in Section 2, we first deal with the convergence of f_N in $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and then in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by Lemmas 3.1 and 3.2.

Theorem 3.3. *For* $N = (N_1, N_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ *and* $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$*, let* $f_N := \big(\sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2\big) * f$*. Then*

(a) *for* $1 \le p < \infty$*, then* $f_N \to f$ *a.e. as* $N \to \infty$*;*

(b) *for* $1 < p < \infty$*, then* $f_N \to f$ *in* $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *as* $N \to \infty$*;*

(c) *for* $p = 1$*, furthermore, assume* $\iint f(x,y)dxdy = 0$ *then* $f_N \to f$ *in* $L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *as* $N \to \infty$*.*

Proof. To prove part (a), by (3.7), Lemma 3.1 (a), Lemma 3.2 (a) and Theorem 2.3, it is suffices to prove $\Phi_{-N} * f \to 0$ a.e.as $N \to \infty$. Let q be the conjugate index of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality,

$$\begin{aligned}
|\Phi_{-N} * f(x,y)| &= |2^{-N_1 n_1 - N_2 n_2} \iint \Phi(2^{-N_1}x - 2^{-N_1}t, 2^{-N_2}y - 2^{-N_2}s) f(t,s)dtds| \\
&\le 2^{-N_1 n_1 - N_2 n_2} \Big(\iint |\Phi(2^{-N_1}x - 2^{-N_1}t, 2^{-N_2}y - 2^{-N_2}s)|^q dtds \Big)^{1/q} \|f\|_{L^p} \\
&= 2^{-N_1 n_1 - N_2 n_2} \Big(\iint |\Phi(2^{-N_1}x - t, 2^{-N_2}y - s)|^q 2^{N_1 n_1 + N_2 n_2} dtds \Big)^{1/q} \|f\|_{L^p} \\
&= 2^{-N_1 n_1 - N_2 n_2} \cdot 2^{\frac{N_1 n_1 + N_2 n_2}{q}} \|\Phi\|_{L^q} \|f\|_{L^p} \\
&= 2^{\frac{-(N_1 n_1 + N_2 n_2)}{p}} \|\Phi\|_{L^q} \|f\|_{L^p}.
\end{aligned}$$

Hence $\Phi_{-N} * f \to 0$ a.e. as $N \to \infty$.

Next let us prove part (b). Since Φ is an approximate identity and $\Phi \in \mathscr{S}$ then $|\Phi| = \sum_{j=1}^{\infty} a_j \chi_{Q_j}$, where a_j are positive and $Q_j = C_j \times D_j$ are rectangles of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. A calculation shows

$$\begin{aligned}
|f * a_j \chi_{Q_j}(x,y)| &\le a_j \int_{C_j} \int_{D_j} |f(x-t, y-s)| dsdt \\
&= a_j \frac{1}{|Q_j|} |Q_j| \int_{C_j} \int_{D_j} |f(x-t, y-s)| dsdt \\
&\le a_j |Q_j| (Mf)(x,y),
\end{aligned}$$

which implies

$$|f * \sum_{j=1}^{m} a_j \chi_{Q_j}(x,y)| \le (Mf)(x,y) \sum_{j=1}^{m} a_j |Q_j| \le (Mf)(x,y)$$

since $\sum_{j=1}^{\infty} a_j |Q| = 1$. Hence $|f * \Phi(x,y)| \le (Mf)(x,y)$ and so $Mf \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. By part (a) and Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{N\to\infty} \|\Phi_{-N} * f\|_{L^p}^p &= \lim_{N\to\infty} \iint |\Phi_{-N} * f|^p dxdy \\
&= \iint \lim_{N\to\infty} |\Phi_{-N} * f|^p dxdy = 0.
\end{aligned}$$

Finally, let us consider part (c), let $f \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with $\iint f(x,y)dxdy = 0$. Then

$$\begin{aligned}
\Phi_{-N} * f(x,y) = 2^{-N_1 n_1} 2^{-N_2 n_2} \iint \Big(&\Phi(2^{-N_1}x - 2^{-N_1}t_1, 2^{-N_2}y - 2^{N_2}t_2) \\
&- \Phi(2^{-N_1}x, 2^{-N_2}y) \Big) f(t_1, t_2) dt_1 dt_2.
\end{aligned}$$

Taking L^1-norm on both sides and then applying Fubiini's theorem, we have

$$\begin{aligned}\|\Phi_{-N} * f\|_{L^1} &= \iint |2^{-N_1n_1-N_2n_2} \iint \Big(\Phi(2^{-N_1}x - 2^{-N_1}t_1, 2^{-N_2}y - 2^{-N_2}t_2) \\ &\qquad - \Phi(2^{-N_1}x, 2^{-N_2}y)\Big) f(t_1,t_2)dt_1dt_2|dxdy \\ &\le \iint |f(t_1,t_2)|2^{-N_1n_1-N_2n_2} \iint |\Phi(2^{-N_1}x - 2^{-N_1}t_1, 2^{-N_2}y - 2^{-N_2}t_2) \\ &\qquad - \Phi(2^{-N_1}x, 2^{-N_2}y)|dt_1dt_2dxdy \\ &= \iint |f(t_1,t_2)| \iint |\Phi(x - 2^{-N_1}t_1, y - 2^{-N_2}t_2) - \Phi(x,y)|dt_1dt_2dxdy.\end{aligned}$$

For each $(t_1, t_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\lim_{N\to\infty} \iint |\Phi(x - 2^{-N_1}t_1, y - 2^{-N_2}t_2) - \Phi(x,y)|dxdy = 0,$$

by continuity of L^1-norm. Note that the integrand is dominated by $|f(t_1,t_2)|2\|\Phi\|_{L^1}$ in $L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. So by Lebesgue dominated convergence theorem, $\|\Phi_{-N} * f\|_{L^1} \to 0$ as $N \to \infty$. □

Let us recall a definition about degree of a tempered distribution.

Definition 3.1. Let $L \in \mathscr{S}'(\mathbb{R}^n)$. The least positive integer k is called the degree of L, denoted by $\deg L$, if for $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$|L(\varphi)| \le C \sum_{\substack{|\alpha|\le\ell \\ |\beta|\le k}} |||\varphi|||_{\alpha,\beta} =: C\|\varphi\|_{\ell,k}, \qquad \forall \ell \in \mathbb{Z}_+,$$

where $|||\varphi|||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)|$.

For $k \in \mathbb{Z}_+$, let

$$\mathscr{S}_k(\mathbb{R}^n) := \{g \in \mathscr{S}(\mathbb{R}^n) : \int x^\gamma g(x)dx = 0,\ \forall |\gamma| \le k\}$$

and $\mathscr{S}_\infty(\mathbb{R}^n) := \cap_{k\in\mathbb{Z}_+} \mathscr{S}_k(\mathbb{R}^n)$.

Now we give another main theorem in this section.

Theorem 3.4. *For $N = (N_1, N_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and a given function f, let*

$$f_N = \Big(\sum_{j=-N_1}^{N_1} \sum_{k=-N_2}^{N_2} h_j^1 h_k^2\Big) * f.$$

(a) *If $f \in \mathscr{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $f_N \to f$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$.*

(b) *If $f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $f_N \to f$ in $\mathscr{S}_k'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$ where $k = deg\widehat{f}$.*

Proof. By (3.7), Lemma 3.1 (b), Lemma 3.2 (b), Theorem 2.4, and the continuity of inverse Fourier transform, it is suffices to prove $\widehat{\Phi}_{-N}\widehat{f} \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

First, we put $f(x,y) = f(z)$, where $z \in \mathbb{R}^{n_1+n_2}$. Since $f \in \mathscr{S}_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then for $0 \le |\rho| \le k$, for any $k \in \mathbb{Z}_+$, and $\omega = (t_1, t_2)$,

$$\begin{aligned}D^\rho \widehat{f}(\omega) &= \int f(z)(-iz)^{|\rho|} e^{-iz\cdot\omega} dz \\ &= \int f(z)(-iz)^{|\rho|} \Big(e^{-iz\cdot\omega} - \sum_{j=0}^{k-|\rho|} \frac{(-iz\cdot\omega)^j}{j!}\Big) dz.\end{aligned}$$

By Taylor's theorem,

$$|e^{i\theta} - \sum_{j=0}^{m} \frac{(-i\theta)^j}{j!}| \le C \Big\| \frac{d^{m+1}}{d\theta^{m+1}} e^{i\theta} \Big\|_\infty |\theta|^{m+1}.$$

Thus

$$\begin{aligned}|D^{\rho}\widehat{f}(\omega)| &\le \int |f(z)||z|^{|\rho|}|z\cdot\omega|^{k+1-|\rho|}dz\\ &\le C|\omega|^{k+1-|\rho|}\int |f(z)||z|^{k+1}dz\\ &= C_f|\omega|^{k+1-|\rho|}\\ &\le C_f(|t_1|+|t_2|)^{k+1-|\rho|},\end{aligned}$$

by equivalence of norms in a finite-dimensional space.

Observe that,

$$\Phi_{-N}(t_1,t_2)=\begin{cases}1, & \text{if } |t_1|\le 2^{-N_1} \text{ or } |t_2|\le 2^{-N_2}\\ 1-\widehat{h}^1_{-N_1+1}(t_1), & \text{if } 2^{-N_1}\le |t_1|<2^{-N_1+1} \text{ and } |t_2|>2^{-N_2+1}\\ 1-\widehat{h}^2_{-N_2+1}(t_2), & \text{if } 2^{-N_2}\le |t_2|<2^{-N_2+1} \text{ and } |t_1|>2^{-N_1+1}\\ 1-\widehat{h}^1_{-N_1+1}(t_1)\widehat{h}^2_{-N_2+1}(t_2), & \text{if } 2^{-N_1}\le |t_1|<2^{-N_1+1}\\ & \quad\text{and } 2^{-N_2}\le |t_2|<2^{-N_2+1}\\ 0, & \text{if } |t_1|>2^{-N_1+1} \text{ and } |t_2|>2^{-N_2+1}\end{cases}.$$

Let $\sigma=(\alpha,\beta)\in\mathbb{Z}_+^{n_1}\times\mathbb{Z}_+^{n_1}$. Then $D^{\sigma}=D^{\alpha}_{t_1}D^{\beta}_{t_2}$. When $|t_1|\le 2^{-N_1}$, $|t_2|\le 2^{-N_2}$ or (both $|t_1|>2^{-N_1+1}$ and $|t_2|>2^{-N_2+1}$),

$$D^{\sigma}\Big(\sum_{j=-N_1+1}^{\infty}\sum_{k=-N_2+1}^{\infty}\widehat{h}^1_j(t_1)\widehat{h}^2_k(t_2)\Big)=0.$$

When $2^{-N_1}\le |t_1|<2^{-N_1+1}$ and $2^{-N_2}\le |t_2|<2^{-N_2+1}$,

$$D^{\sigma}\Big(\sum_{j=N_1+1}^{\infty}\sum_{k=N_2+1}^{\infty}\widehat{h}^1_j(t_1)\widehat{h}^2_k(t_2)\Big)=2^{(N_1-1)|\alpha|}(D^{\alpha}\widehat{h}^1)(2^{N_1-1}t_1)2^{(N_2-1)|\beta|}(D^{\beta}\widehat{h}^2)(2^{N_2-1}t_2).$$

For $2^{-N_1}\le |t_1|<2^{-N_1+1}$ and $|t_2|>2^{-N_2+1}$,

$$D^{\sigma}\Big(\sum_{j=-N_1+1}^{\infty}\sum_{k=-N_2+1}^{\infty}\widehat{h}^1_j(t_1)\widehat{h}^2_k(t_2)\Big)=\begin{cases}0, & \text{if } \beta\ne 0\\ 2^{(N_1-1)|\alpha|}(D^{\alpha}\widehat{h}^1)(2^{N_1-1}t_1), & \text{if } \beta=0\end{cases}.$$

For $2^{N_2}\le |t_2|<2^{-N_2+1}$ and $|t_1|>2^{-N_1+1}$,

$$D^{\sigma}\Big(\sum_{j=-N_1+1}^{\infty}\sum_{k=-N_2+1}^{\infty}\widehat{h}^1_j(t_1)\widehat{h}^2_k(t_2)\Big)=\begin{cases}0, & \text{if } \alpha\ne 0\\ 2^{(N_2-1)|\beta|}(D^{\beta}\widehat{h}^2)(2^{N_2-1}t_2), & \text{if } \alpha=0\end{cases}.$$

Thus we get for $N\in\mathbb{Z}_+\times\mathbb{Z}_+$, and $N'=\max\{N_1,N_2\}$,

$$D^{\sigma}\Big(1-\sum_{j=-N_1+1}^{\infty}\sum_{k=-N_2+1}^{\infty}\widehat{h}^1_j(t_1)\widehat{h}^2_k(t_2)\Big)\le C_{\sigma}2^{N'|\sigma|}\chi_{\{(t_1,t_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:|t_1|\le 2^{-N_1+1},\ |t_2|\le 2^{-N_2+1}\}}.$$

Hence by the Leibniz's lemma, if $|\gamma|=k$

$$\begin{aligned}&(1+|t_1|+|t_2|)^M|D^{\gamma}(\widehat{\Phi}_{-N}\widehat{f})(t_1,t_2)|\\ &\le (1+|t_1|+|t_2|)^M\sum_{\sigma+\rho=\gamma}C_{\sigma,\rho}|D^{\sigma}\widehat{\Phi}_{-N}(t_1,t_2)||D^{\rho}\widehat{f}(t_1,t_2)|\\ &\le (1+|t_1|+|t_2|)^M\sum_{\sigma+\rho=\gamma}C_{\sigma,\rho}2^{N'|\sigma|}(|t_1|+|t_2|)^{k+1-|\rho|}\\ &\qquad\times\chi_{\{(t_1,t_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}:|t_1|\le 2^{-N_1+1} \text{ and } |t_2|\le 2^{-N_2+1}\}}\\ &\le (1+|t_1|+|t_2|)^M\sum_{\sigma+\rho=\gamma}C_{\sigma,\rho}2^{N'|\sigma|}(2\cdot 2^{N'})^{k+1-|\sigma|}\\ &\le C\sum_{\sigma+\rho=\gamma}2^{N'(|\sigma|+|\rho|)}\cdot 2^{-N'k-N'}\cdot 2^{k+1-|\rho|}\\ &\le C\cdot 2^{-N'},\end{aligned}$$

Hence $\Phi_{-N} * f \to 0$ in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$.

For part (b), by (3.7), Lemma 3.1 (c), Lemma 3.2 (c), Theorem 2.4, and the continuity of inverse Fourier transform, it suffice to show $\widehat{\Phi_{-N} f} \to 0$ in $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. The proof follows from part (a) by a duality. Let $g \in \mathscr{S}_k(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Since $k = deg\widehat{f}$, there exists $M > 0$ such that

$$\begin{aligned}
|(\Phi_{-N} * f, g)| &= |(\widehat{\Phi}_{-N}\widehat{f}, \widehat{g})| \\
&= |(\widehat{f}, \widehat{\widehat{\Phi}}_{-N}\widehat{g})| \\
&\le C \sum_{|\gamma|\le k} \sup_{(t_1,t_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1 + |t_1| + |t_2|)^M |D^\gamma(\widehat{\widehat{\Phi}}_{-N}\widehat{f})(t_1, t_2)|,
\end{aligned}$$

By part (a), we have

$$\sup_{(t_1,t_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} (1 + |t_1| + |t_2|)^M |D^\gamma(\widehat{\widehat{\Phi}}_{-N}\widehat{f})(t_1, t_2)| \to 0,$$

for any g in $\mathscr{S}_k(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to 0$, and hence we have $\Phi_{-N} * f \to 0 \in \mathscr{S}'_k(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $N \to \infty$, by continuity of distributions. □

If we choose φ and ψ at the end of the last section, then we have

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \widehat{\varphi}_{jk}\widehat{\psi}_{jk} = 1 \quad \text{a.e.}$$

and

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \varphi_{jk} * \psi_{jk} * f.$$

Using the sampling theorem, we obtain

$$f = \sum_{Q\in\mathscr{Q}_1} \sum_{P\in\mathscr{Q}_2} \langle f, \varphi_{QP}\rangle \psi_{QP},$$

where $\mathscr{Q}_i$ is the collection of all dyadic cubes in $\mathbb{R}^{n_i}$, $i = 1, 2$.

References

Calderón, A. P. (1964). Intermediate spaces and interpolation, the complex method, *Studia Math.*, *24*, 113-190.

Calderón, A. P. (1977). An atomic decomposition of distributions in parabolic H^p spaces, *Adv. in Math.*, *25*, 216-225. https://doi.org/10.1016/0001-8708(77)90074-3

Chang, S.-Y. A., & Fefferman, R. (1980). A continuous version of duality of H^1 with BMO on the bi-disc, *Ann. of Math.*, *112*, 179-201. https://doi.org/10.2307/1971324

Coifman, R. R., & Weiss, G. (1977). Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, *83*, 569-645. https://doi.org/10.1090/S0002-9904-1977-14325-5

Deng, D.-G., & Han, Y.-S. (1995). Calder n type reproducing formula on spaces of homogeneous type. *Sci. China* Ser. A, *38*(1), 17-28.

Fefferman, R. (1981). Singular integrals on product domains, *Bull. Amer. Math. Soc.*, *4*, 195-201. https://doi.org/10.1090/S0273-0979-1981-14883-7

Fefferman, R. (1987). Harmonic analysis on product spaces, *Ann. of Math.*, *126*, 109-130. https://doi.org/10.2307/1971346

Fefferman, R., & Stein, E. M. (1982). Singular integrals on product spaces, *Adv. in Math.*, *45*, 117-143. https://doi.org/10.1016/S0001-8708(82)80001-7

Frazier, M., & Jawerth, B. (1990). A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.*, *34*, 34-170. https://doi.org/10.1016/0022-1236(90)90137-A

Frazier, M., Jawerth, B, & Weiss, G. (1991). *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series, *79*, A.M.S., Providence, RI.

Han, Y.-S. (1997). Inhomogeneous Caldeón reproducing formula on spaces of homogeneous type. *J. Geom. Anal.*, *7*(2), 259-284. https://doi.org/10.1007/BF02921723

Han, Y.-S., Lee, M.-Y., Lin, C.-C., & Lin, Y.-C. (2010). Calderón-Zygmund operators on product Hardy spaces, *J. Funct. Anal.*, *258*, 2834-2861. https://doi.org/10.1007/BF02921723

Han, Y.-S., & Yang, D.-C. (2005). Inhomogeneous discrete Caldern reproducing formulae associated to para-accretive functions on metric measure spaces. *Funct. Approx. Comment. Math.*, *33*, 15-55.

Jessen, B., Marcinkiewicz, J., & Zygmund, A. (1935). Note on the differentiability of multiple integrals, *Fund. Math.*, *25*, 55-91.

Journé, J.-L. (1985). Calderón-Zygmund operators on product spaces, *Rev. Mat. Iberoamericana*, *1*, 55-91. https://doi.org/10.4171/RMI/15

Uchiyama, A. (1982). A constructive proof of the Fefferman-Stein decomposition of $BMO(\mathbb{R}^n)$, *Acta Math.*, *148*, 215-241.

Yang, D.-C. (2005). Inhomogeneous Caldern reproducing formulae associated to para-accretive functions on metric measure spaces. *Taiwanese J. Math.*, *9*(4), 683-720.