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Some Remarks on the Stability Condition of Numerical Scheme of the KdV-type Equation

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Abstract

This paper has employed a comparative study between the numerical scheme and stability condition. Numerical calculations are carried out based on three different numerical schemes, namely the central finite difference, fourier leap-frog, and fourier spectral RK4 schemes. Stability criteria for different numerical schemes are developed for the KdV equation, and numerical examples are put to test to illustrate the accuracy and stability between the solution profile and numerical scheme.

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1. Introduction

The Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

is a nonlinear, dispersive partial differential equation for a function where $u(x,t)$ of two real variables, space x and time t. It is a mathematical model of waves on shallow water surfaces and particularly notable as the prototypical example of an exactly solvable model, i.e., a non-linear partial differential equation whose solutions can be exactly and precisely specified. The solitary solution of the equation was first observed in J. Russell (1837), and the equation itself was later derived by Korteweg and de Vries in D. Korteweg & G. de Vries (1895). Since then, it has been applied in many fields to describe a wide range of physical phenomena such as interaction of nonlinear waves M. J. Ablowitz & D.E. Baldwin (2012), collision-free hydro-magnetic waves in a cold plasma, ion-acoustic waves, interfacial electrohydrodynamics (M. Q. Tran, 1979).

The KdV equation was not studied much until Zabusky and Kruskal (1965) proposed an explicit numerical scheme and discovered numerically that its solutions seemed to decompose at large times into a collection of "solitons", a series of well separated solitary waves. The scheme is described as follows

$$\begin{aligned} u_i^{j+1} \quad &= \quad u_i^{j-1} - \frac{\Delta t}{\Delta x}\frac{\left(u_{i+1}^j + u_i^j + u_{i-1}^j\right)}{3}\left(u_{i+1}^j - u_{i-1}^j\right) \\ &\quad - \frac{\Delta t}{\Delta x^3}\left(u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j\right), \end{aligned}$$

with $j = 1, 2, \ldots$. Here central difference approximations were used for both the first space and first time derivatives to improve the accuracy for given step sizes Δx and Δt, respectively.

The study of non-linear waves would not have been so successful had it not done with stable numerical schemes, especially for time-dependent problems, stability guarantees that the numerical method produces a bounded solution whenever the solution of the exact differential equation is bounded. Stability, in general, can be difficult to investigate, especially when the equation under consideration is nonlinear. In this article, we analyze the stability of three numerical schemes on the KdV equation based on von Neumann method and conclude that Fourier RK4 scheme can meet the stability criterion with suitable spatial and time steps.

2. Finite Difference Scheme

In this section we present a finite difference scheme for the linearized KdV equation in order to proceed with the stability analysis. The equation is described as

$$u_t + u_{xxx} = 0. \tag{2}$$

We note that the KdV equation (1) is used to be considered as the weakly nonlinear, weakly dispersive behavior of the long wave case on the free surface. However, equation (2) not only can be solved explicitly using Fourier methods but also served as a tool for studying the nonlinear equation (1).

We consider a function $u(x_i, t_n)$ with the x-t plane subdivided into a rectangular grid or mesh with each rectangle having sides of length h and k, where $x_i = ih$, $t_n = nk$ with $i = 1, 2, \ldots, m$ and $n = 1, 2, \ldots, q$, for some integers m and q. The various mesh points may be labeled by a pair of integers. Let the point P_i^n be denoted by $(i,\ n)$. The value of u at P_i^n is approximated by u_i^n. Expressing the finite-difference approximation in terms of this notation, we have

$$\left(\frac{\partial u}{\partial x}\right)_{P_i^n} = \frac{u_{i+1}^n - u_i^n}{h}, \tag{3}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{P_i^n} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \tag{4}$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)_{P_i^n} = \frac{u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n}{2h^3}. \tag{5}$$

To analyse the numerical scheme for (2), we adopt the central finite difference (FD) scheme with leap-frog time-stepping formula as follows

$$u_i^{n+1} = u_i^{n-1} - \frac{\Delta t}{\Delta x^3}\left(u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n\right). \tag{6}$$

The characteristic equation for this recurrence relation is

$$g^2 - 2\frac{\Delta t}{\Delta x^3} i \left(\sin 2x - 2\sin x\right) g - 1 = 0, \tag{7}$$

which we obtain by inserting in (6) the ansatz $u_i^n = g^n$, and the condition for stability is that both complex roots must lie in the closed unit disk, with only simple roots permitted on the unit circle. As a result, we obtain

$$|g| = \left|-4i\frac{\Delta t}{\Delta x^3}\sin^2\frac{x}{2}\sin x\right| < 1, \tag{8}$$

or

$$\left|-4\frac{\Delta t}{\Delta x^3}\sin^2\frac{x}{2}\sin x\right| < 1. \tag{9}$$

When $x = 2\pi/3$, we have

$$\max\left|\sin^2\frac{x}{2}\sin x\right| = \frac{3\sqrt{3}}{8}. \tag{10}$$

Therefore we have obtained

$$\frac{\Delta t}{\Delta x^3} < \frac{2}{3\sqrt{3}} \approx 0.3849, \tag{11}$$

which is the stability criteion for linearized KdV equation (2) and can be used to calculate the numerical solution for (1).

3. Fourier Leap-frog Scheme

In this section, we first model the KdV equation (1) by a Fourier spectral method on $[-L,\ L]$. In practice, it can be transformed to

$$u_t + \frac{2\pi}{L}uu_x + \left(\frac{2\pi}{L}\right)^3 u_{xxx} = 0, x \in [-L,\ L], \tag{12}$$

where L is a given number representing the boundary point of the spatial domain. In practice, we need to transform u, u_x into Fourier space and discretize the equation. For any integer $N > 0$, we consider the collation points $x_j = j\Delta x = 2\pi j/N$, $j = 0, 1, \ldots, N-1$, and note that if $u(x,t)$ is the solution of the KdV equation, then we transform it into the discrete Fourier space as

$$\hat{u}(k,t) = F(u) = \frac{1}{N}\sum_{j=0}^{N-1} u\left(x_j, t\right) e^{-ikx_j},\ -\frac{N}{2} \le k \le \frac{N}{2} - 1. \tag{13}$$

From this, using the inversion formula, we get

$$u\left(x_j, t\right) = F^{-1}\left(\hat{u}\right) = \sum_{k=-N/2}^{N/2-1} \hat{u}\left(k, t\right) e^{-ikx_j}, 0 \le j \le N-1, \tag{14}$$

where we denote the discrete Fourier transform and the inverse Fourier transform by F and F^{-1} respectively. Therefore, we have

$$\frac{\partial^n u}{\partial x^n} = F^{-1}\left\{(ik)^n F\left(u\right)\right\}, \qquad n = 1, 2, \ldots, \tag{15}$$

in particular, we have

$$u_x = F^{-1}\left\{ikF\left(u\right)\right\}, \tag{16}$$

$$u_{xxx} = F^{-1}\left\{-ik^3 F\left(u\right)\right\}. \tag{17}$$

Then (12) can be transformed into a semi-discrete form as follows:

$$u_t = -\frac{2\pi}{L} u F^{-1}\left\{ikF\left(u\right)\right\} - \left(\frac{2\pi}{L}\right)^3 F^{-1}\left\{-ik^3 F\left(u\right)\right\}. \tag{18}$$

Taking into the consideration of the collation points, (12) can be further discretized into

$$\frac{du\left(x_j, t\right)}{dt} = -\frac{2\pi}{L} u\left(x_j, t\right) F^{-1}\left\{ikF\left(u\right)\right\} - \left(\frac{2\pi}{L}\right)^3 F^{-1}\left\{-ik^3 F\left(u\right)\right\}, \tag{19}$$

for $0 \le j \le N-1$. Now we denote $\mathbf{U} = \left[u\left(x_0, t\right), u\left(x_1, t\right), \ldots, u\left(x_{N-1}, t\right)\right]^T$, then (19) can be written in the vector form as

$$\mathbf{U}_t = \mathbf{F}\left(\mathbf{U}\right), \tag{20}$$

where $\mathbf{F}$ defines the right-hand side of (19).

For the stability analysis, it requires further information on the condition imposed on the time step Δt. Therefore, similar to (12), we could start with the following linearized equation

$$u_t + \frac{2\pi}{L} u_x + \left(\frac{2\pi}{L}\right)^3 u_{xxx} = 0, x \in \left[-L,\ L\right], \tag{21}$$

for some suitable boundaries L, and approximate the solution by using Fourier Leap-Frog (FLF) transforms

$$u\left(x, t+\Delta t\right) - u\left(x, t-\Delta t\right) = 2\Delta t \mathcal{F}\left(-\frac{2\pi}{L} u_x - \left(\frac{2\pi}{L}\right)^3 u_{xxx}\right), \tag{22}$$

where $\mathcal{F}$ is called the fourier transform operator. By using the discrete version of (13) and (14), the scheme is shown as follows

$$\left[\begin{array}{l} u\left(x, t+\Delta t\right) - u\left(x, t-\Delta t\right) + 2\left(\frac{2\pi}{L}\right)\Delta t F^{-1}\left\{ikF\left(u\right)\right\} \\ +2\left(\frac{2\pi}{L}\right)^3 \Delta t F^{-1}\left\{-ik^3 F\left(u\right)\right\} \end{array}\right] = 0. \tag{23}$$

For simplicity, let $v = k\left(2\pi/L\right)$, then (23) can be written as

$$u\left(x, t+\Delta t\right) - u\left(x, t-\Delta t\right) + 2v\Delta t F^{-1}\left\{iF\left(u\right)\right\} + 2v^3 \Delta t F^{-1}\left\{-iF\left(u\right)\right\} = 0. \tag{24}$$

To proceed, we look for a solution to (24) of the form

$$u\left(x, t\right) = \kappa^{t/\Delta t} e^{ikx}, \tag{25}$$

and subsitute it into (23) to get

$$\kappa^{(t+\Delta t)/\Delta t} e^{ikx} - \kappa^{(t-\Delta t)/\Delta t} e^{ikx} + 2iv\Delta t \kappa^{t/\Delta t} e^{ikx} - 2iv^3 \Delta t \kappa^{t/\Delta t} e^{ikx} = 0, \tag{26}$$

i.e.,

$$\kappa^2 - \kappa^{-1} + 2iv\Delta t - 2iv^3 \Delta t = 0, \tag{27}$$

or

$$\kappa^2 - \kappa^{-1} - 2if(v, \Delta t)\kappa - 1 = 0, \tag{28}$$

where

$$f(v, \Delta t) = v^3 \Delta t - v\Delta t. \tag{29}$$

The scheme is conditionally stable if and only if $f(v, \Delta t)$ is real and less than one in magnitude. Since the wave number v takes the values

$$v = 0, \pm 1, \pm 2, \ldots, \pm N/2, \tag{30}$$

and the interval is discretized with N equidistant mesh points, that is

$$\Delta x = 2\pi/N, \tag{31}$$

we want to find the largest value of Δt such that

$$|f(v, \Delta t)| < 1, \tag{32}$$

is true for all v. Here the most severe restriction on Δt is imposed for the v, which are largest in magnitude, i.e., for $v = \pm v_{\max}$ with $v_{\max} = N/2 = \pi/\Delta x$. Thus, we obtain

$$f(v_{\max}, \Delta t) = \Delta t\left(\frac{\pi}{\Delta x}\right)^3 - \Delta t\left(\frac{\pi}{\Delta x}\right). \tag{33}$$

therefore $|f(v, \alpha, \Delta t)| < 1$ implies to

$$\Delta t\left(\frac{\pi}{\Delta x}\right)^3 < 1, \tag{34}$$

and the stability condition becomes

$$\frac{\Delta t}{\Delta x^3} < \frac{1}{\pi^3} \approx 0.0322515. \tag{35}$$

4. Fourier RK4 Scheme

We have implemented the Fourier spectral method with fourth-order Runge–Kutta (RK4) time differencing to solve the fifth-order KdV equation. The RK4 method is known to have a truncation error of $O\left(\Delta t^4\right)$ and one of the most widely used methods for solving differential equations. Its algorithm is described below:

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \frac{\Delta t}{6}\left(\mathbf{f}_{k1} + 2\mathbf{f}_{k2} + 2\mathbf{f}_{k3} + \mathbf{f}_{k4}\right), \tag{36}$$

where

$$\mathbf{f}_{k1} = \mathbf{f}(t_k,\ \mathbf{y}_k), \tag{37}$$
$$\mathbf{f}_{k2} = \mathbf{f}(t_k + \Delta t/2,\ \mathbf{y}_k + \Delta t\mathbf{f}_{k1}/2), \tag{38}$$
$$\mathbf{f}_{k3} = \mathbf{f}(t_k + \Delta t/2,\ \mathbf{y}_k + \Delta t\mathbf{f}_{k2}/2), \tag{39}$$
$$\mathbf{f}_{k4} = \mathbf{f}(t_k + \Delta t,\ \mathbf{y}_k + \Delta t\mathbf{f}_{k3}). \tag{40}$$

For the study of the stability, we use the standard Fourier analysis to find the condition imposed on the time step Δt. For simplicity, we consider the linearized KdV equation as follows:

$$u_t + \alpha u_x + u_{xxx} = 0. \tag{41}$$

We could approximate this equation by using RK4 scheme in (36) with the ansatz

$$u(x, t) = \kappa^{t/\Delta t} e^{ivx},$$

which will not only extend the domain of the equation to the whole real line but also enable us to examine $\max|\kappa|$ to decide on stability of the numerical scheme. As a restult, we have

$$\mathbf{f}_{k1} = -iv\kappa^{t/\Delta t} e^{ivx}\left(-v^2 + \alpha\right), \tag{42}$$

$$\mathbf{f}_{k2} = -\frac{1}{2} v\kappa^{t/\Delta t} e^{ivx}\left(v^5\Delta t - 2\alpha v^3\Delta t + \alpha^2 v\Delta t - 2iv^2 + 2i\alpha\right) \tag{43}$$

$$\mathbf{f}_{k3} = \frac{1}{4}\nu\kappa^{t/\Delta t}e^{i\nu x}\begin{pmatrix} -i\nu^8\Delta t^2 + 3i\alpha\nu^6\Delta t^2 - 3i\alpha^2\nu^4\Delta t^2 \\ +i\alpha^3\nu^2\Delta t^2 - 2\nu^5\Delta t + 4\alpha\nu^3\Delta t \\ -2\alpha^2\nu\Delta t + 4i\nu^2 - 4i\alpha \end{pmatrix} \tag{44}$$

$$\mathbf{f}_{k4} = \frac{1}{4}\nu\kappa^{t/\Delta t}e^{i\nu x}\begin{pmatrix} \nu^{11}\Delta t^3 - 4\alpha\nu^9\Delta t^3 + 6\alpha^2\nu^7\Delta t^3 - 2i\nu^8\Delta t^2 \\ -4\alpha^3\nu^5\Delta t^3 + 6\alpha i\nu^6\Delta t^2 + \alpha^4\nu^3\Delta t^3 \\ -6i\alpha^2\nu^4\Delta t^2 + 2i\alpha^3\nu^2\Delta t^2 - 4\nu^5\Delta t \\ +8\alpha\nu^3\Delta t - 4\alpha^2\nu\Delta t + 4i\nu^2 - 4i\alpha \end{pmatrix} \tag{45}$$

Therefore (36) gives

$$-\frac{1}{24}\begin{pmatrix} \nu^{12}\Delta t^4 - 4\alpha\nu^{10}\Delta t^4 + 6\alpha^2\nu^8\Delta t^4 \\ +12i\alpha\nu^7\Delta t^3 - 4\alpha^3\nu^6\Delta t^4 - 4i\nu^9\Delta t^3 \\ +\alpha^4\nu^4\Delta t^4 - 24i\alpha\nu\Delta t \\ -12i\alpha^2\nu^5\Delta t^3 - 12\nu^6\Delta t^2 + 24\alpha\nu^4\Delta t^2 \\ -12\alpha^2\nu^2\Delta t^2 + 4i\alpha^3\nu^3\Delta t^3 + 24i\nu^3\Delta t \\ +24 - \kappa \end{pmatrix}\kappa^{t/\Delta t}e^{i\nu x} = 0. \tag{46}$$

We want to find the restriction on Δt such that

$$\max|\kappa| < 1, \tag{47}$$

is true with $\nu_{\max} = N/2 = \pi/\Delta x$. Detailed calculation for (47) gives the condition

$$\left|\begin{array}{l} 1 - \left\{\begin{array}{l} (1/6)\,\alpha\pi^{10}/\Delta x^{10} - (1/24)\,\pi^{12}/\Delta x^{12} \\ -(1/4)\,\alpha^2\pi^8/\Delta x^8 + (1/6)\,\alpha^3\pi^6/\Delta x^6 \\ -(1/24)\,\alpha^4\pi^4/\Delta x^4 \end{array}\right\}\Delta t^4 \\ +\left\{\begin{array}{l} (i/6)\pi^9/\Delta x^9 - (i/2)\,\alpha\pi^7/\Delta x^7 \\ -(i/6)\,\alpha^3\pi^3/\Delta x^3 + (i/2)\,\alpha^2\pi^5/\Delta x^5 \end{array}\right\}\Delta t^3 \\ +\left\{(1/2)\pi^6/\Delta x^6 - \alpha\pi^4/\Delta x^4 + (1/2)\,\alpha^2\pi^2/\Delta x^2\right\}\Delta t^2 \\ +\left\{i\alpha\pi/\Delta x - i\pi^3/\Delta x^3\right\}\Delta t \end{array}\right| < 1. \tag{48}$$

The analysis of the present article should be more broadly in line with the KdV equation (1). Briefly, different numbers of α in (48) will result in different stability regions, hence we assume that $\alpha = 1$ in (48) to get

$$CRI = \left(\frac{1}{576\Delta x^{24}}\left[\begin{array}{l} \pi^{24}\Delta t^8 - 8\pi^{22}\Delta x^2\Delta t^8 \\ +28\pi^{20}\Delta x^4\Delta t^8 \\ -56\pi^{18}\Delta x^6\Delta t^8 + 70\pi^{16}\Delta x^8\Delta t^8 \\ -56\pi^{14}\Delta x^{10}\Delta t^8 + 28\pi^{12}\Delta x^{12}\Delta t^8 \\ -8\pi^{10}\Delta x^{14}\Delta t^8 + \pi^8\Delta x^{16}\Delta t^8 \\ -8\pi^{18}\Delta x^6\Delta t^6 + 48\pi^{16}\Delta x^8\Delta t^6 \\ -120\pi^{14}\Delta x^{10}\Delta t^6 + 160\pi^{12}\Delta x^{12}\Delta t^6 \\ -120\pi^{10}\Delta x^{14}\Delta t^6 + 48\pi^8\Delta x^{16}\Delta t^6 \\ -8\pi^6\Delta x^{18}\Delta t^6 + 576\Delta x^{24} \end{array}\right]\right)^{1/2} < 1, \tag{49}$$

which is the stability criterion for Δx and Δt.

5. Numerical Test

One of the most interesting features of the KdV equation is the existence of infinitely many conservation laws. A conservation law in differential equation form can be written as $T_t + X_x = 0$, in which the "density" T and the "flux" X are polynomials in the solution u and its x-derivatives (P. G. Drazin & R. S. Johnson, 1989). If both T and X_x are integrable over the domain $(-\infty, +\infty)$, then the assumption that $X \longrightarrow 0$ as $x \longrightarrow \infty$ implies that the conservation law can be integrated over all x to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{+\infty} T\mathrm{d}x\right) = 0, \tag{50}$$

or

$$\int_{-\infty}^{+\infty} T\mathrm{d}x = C, \tag{51}$$

where C is a constant. The integral of T over the entire spatial domain is therefore invariant with time and usually called an invariant of motion or a constant of motion (Verheest and Hereman 1994; Goktas and Hereman 1999). The KdV equation (1) itself is already in conservation form, i.e.,

$$\partial_t u + \partial_x\left(\frac{1}{2}u^2 + u_{xx}\right) = 0, \tag{52}$$

so that we immediately see that the quantity

$$\int_{-\infty}^{+\infty} u\mathrm{d}x \tag{53}$$

is conserved. This corresponds to conservation of mass. It can be seen that

$$u\left(u_t + uu_x + u_{xxx}\right) = \left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + uu_{xx} - \frac{u_x^2}{2}\right)_x = 0. \tag{54}$$

This shows that the conserved density given by the funtional $T = u^2/2$ is again a polynomial conserved density and the quantity

$$\int_{-\infty}^{+\infty} \frac{u^2}{2}\mathrm{d}x$$

corresponds to the conservation of momentum. Similarly, if one multiplies the KdV equation by u^2 and u_x respectively to get

$$u^2\left(u_t + uu_x + u_{xxx}\right) + u_x\left(u_t + uu_x + u_{xxx}\right) = 0, \tag{55}$$

one obtains

$$\left(\frac{u^3}{3} + u_x^2\right)_t + \left[-\frac{u^4}{4} - u^2u_{xx} + 2uu_x^2 - 2u_xu_{xxx} + u_{xx}^2\right]_x = 0. \tag{56}$$

Therefore the quantity

$$\int_{-\infty}^{+\infty} \left(\frac{u^3}{3} + u_x^2\right)\mathrm{d}x, \tag{57}$$

must be conserved. This corresponds to conservation of energy. It turns out that there is an infinite number of conserved quantities and amongst the first three conserved quantities represent the most important physical quantities. In this section, we resort to the numerical conservation law to check the accuracy and stability for our numerical schemes. Based on the phenoemenon that rounding errors on the computer may prevent any further improvement in the calculations, we choose the first conservation law in (1) for our numerical scheme test as it costs smallest floating-point operations.

Consider the initial condition

$$u(x, 0) = 3a^2 \sec\mathrm{h}^2\left(\left(a\left(x - x_0\right) - a^3t\right)/2\right), \tag{58}$$

for any real a and x_0, which is an exac solution of the KdV equation (1). Our first case is focused on the study of performance of FD scheme on the KdV equation. We set $a = 25$, $x_0 = -2$ and compute the solutions with the central finite difference scheme in (6) with 256 mesh points in the domain $[-\pi, \pi]$, Δt =1e-7 and plot the solutions from time $t = 0$ to t =6e-3. Solution profile is plotted in time period $t \in$[0, 6e-3] and illustrated in Fig. 2. The solution profile clearly shows that, despite the fact that ripples emanating from the wave, the solution waves still manage to propagate at some constant speeds. From Fig. 3, it illustrates the comparison of exact solitary wave with numerical solution profile at the final time of calculation, and one can see that the waves were struggling to maintain their shapes as time progressed and finally paid the price with vast expenses of accuracy with regard to its height and speed after surviving for 60,000 integrations of numerical calculations. This clearly indicates the FD scheme can only remain relatively stable with a certain degrees of accuracy. In addition, the error of the conservation law in (53) measured by the difference of the values away from its initial value is illustrated in Fig. 4, supporting the fact that the FD scheme remains relatively stable with variations around the center datum line to a certain degree in which the accuracy will be compromised. Nevertheless, it is fair to point out that the stability requirement in (11) refers to be as $\Delta t/\Delta x^3 \approx 0.00676$ with given values of $\Delta x = 2\pi/256$ and Δt=1e-7 and allows the numerical calculations.

On the second case where the KdV equation is numericall solved with FLF numerical scheme, we carried out a similar study of adopting the same numerical settings as in (58) by setting $a = 25$, $x_0 = -2$, use 256 grid points in the domain $[-\pi, \pi]$, Δt =1e-7 and plot the solution from time $t = 0$ to t =6e-3. As a result, solution profile from t=0 to t=6e-3 is plotted in Fig. 5. The results clearly shows that no visible ripples are emanating from the wave, and the wave propagates at an approximately constant speed to the right. Indeed, Fig. 6 shows the comparison of exact solitary wave with numerical solution profile at the final time of calculation, together with the difference between these two waves. It can be seen that the formation of numerical solutions has been substantially maintained and survived after 60,000 integrations with little expense of accuracy with regard to its height and speed. It is observed that only fewer than 1 in 2,000 of the magnitude of wave amplitude has been lost which shows the FLF scheme is reliable and accuracy has been progressed compared to the FD scheme. Furthermore, the accuracy check of the scheme measured by the conservation law (53) is illustrated in Fig. 7 and shows that the error is accurate to $O\left(10^{-12}\right)$ in a remarkable performance. This not only demonstrates the numerical results here should be regarded as genuine, not numerical artifacts, but also that the numerical wave solutions are solitons traveling to the right. In addition, the stability requirement in (35) gives the value $\Delta t/\Delta x^3 \approx 0.00676 < 0.0322$ to further support the results and performance of numerical calculations.

On the third case, we discuss the problem when the KdV equation is solved numerically by spectral methods, the pattern is usually the same: spectral differentiation in space, RK4 differences in time. Here we consider the same initial condition as described in (58) and compute the solutions with the numerical scheme in (23) with 256 mesh points in domain $[-\pi, \pi]$, time step Δt =1e-7 and plot the solutions from time $t = 0$ to t =6e-3. Solution profile is plotted at time $t \in$[0, 6e-3] in Fig. 8. The results illustrate the phenomenon that the wave propagates at a constant speed coherently. Moreover, neither ripples nor dispersive wave components are emanating from the calculations which brings satisfactory results. On the other hand, Fig. 9 shows the results of the exact and numerical solitary wave solutions and differences in between these two waves. It is seen that the numerical error is maintained to be accurate to 10^{-3} in the magnitude of 2000 of wave amplitude. Hence under the FRK4 scheme, the numerical solutions have been undergoing for 60,000 integrations and obtained without any significant expense of accuracy with regard to its height and speed. Furthermore, the velocities of the solutions and their shapes are almost unchanged compared to the exact solution. Moreover, accuracy test for the numerical scheme in Fig. 10 shows that the errors have been remarkably kept as small as $O\left(10^{-12}\right)$. This not only shows that the FRK4 scheme is significantly stable but also the accuracy of the scheme is guranteed to produce numerical calculations. Consequently, the criterion equation in (49) gives $CRI = 0.99 < 1$ and supports the fact that the computation through FRK4 becomes the choice of numerical methods.

6. Concluding Remarks

Fig. 1 illustrates the stability regions from (11), (35) and (49) for different schemes on the KdV equation with Δt against Δx. Three numerical schemes including the finite difference, fourier leap-frog and fourier RK4 procedures are presented for the equation. By using the analysis on the linearized equation, we deduce that the marginal curves of stability region are all nonlinear curves. It is known that the fourier spectral method consisting of space space differentiation and integrating procedure in time can be swiftly convergent and spectrally accurate in spatial domain. In particular, the region of stability of the fourth order Runge-Kutta method is so complicated that it can not be expressed in terms of a closed form of algebraic equation but can be characterized through the symbolic computation. The result also shows that the stability region for FLF scheme is stricter that that of the FRK4 scheme.

Stability criteria of numerical schemes on the KdV equation are developed, namely the central finite difference, fourier leap-frog, and fourier spectral RK4 schemes. Each of the schemes gives us an estimated region for allowed values of Δx and Δt. The errors measured by the spread of the values of conservation law from its initial value are recorded, which gives an accuracy test for the numerical schemes. This paper also carried out a comparative study of solitary wave solution for different schemes and numerical settings to test the accuracy and stability condition. It is seen that fourier spectral RK4 scheme strikes the right balance between the finite difference and fourier leap-frog schemes for numerical methods.

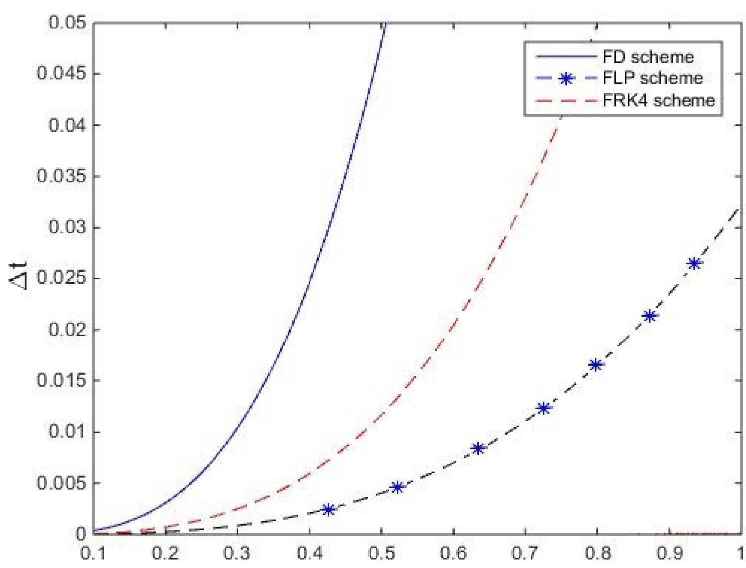

Figure 1. Stability regions for the KdV equation with different schemes

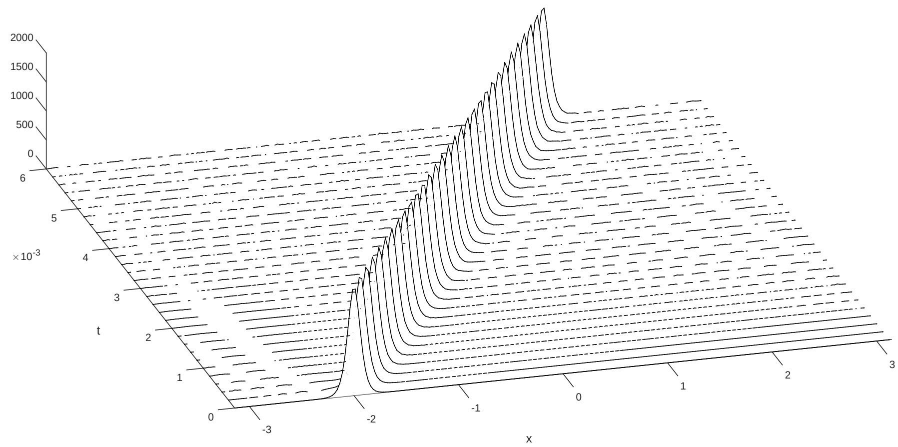

Figure 2. Solution profiles of the KdV equation using finite difference scheme

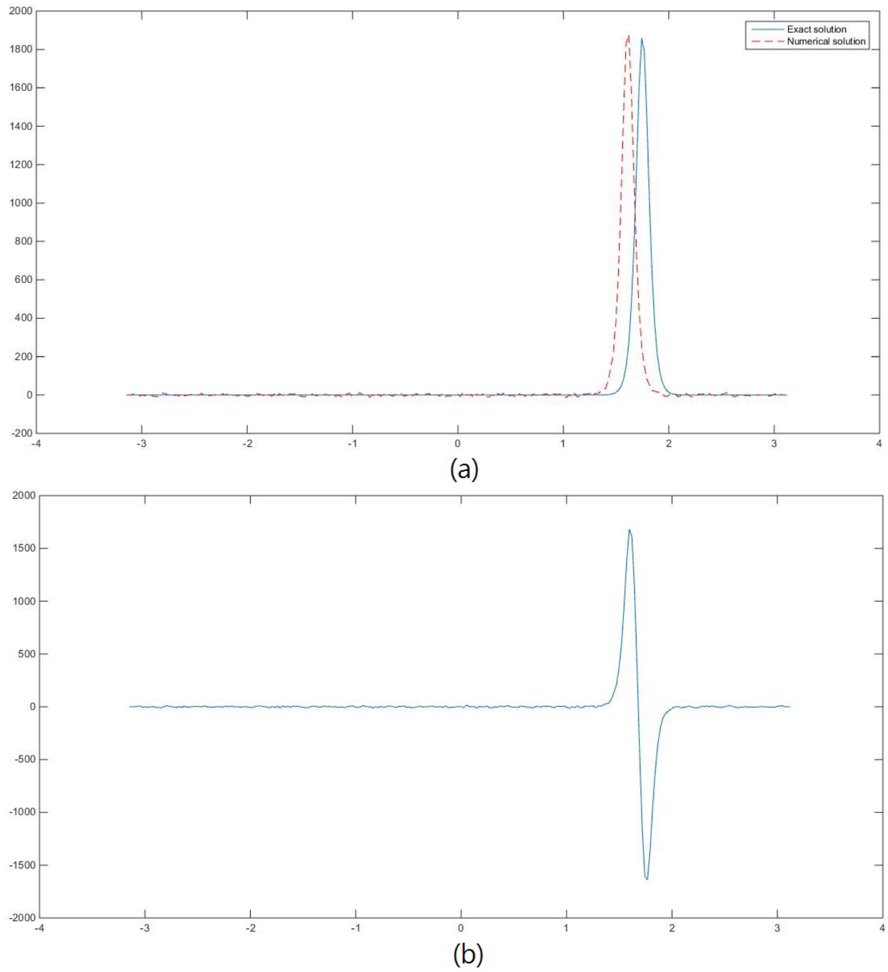

Figure 3. Solution profile for FD scheme depicted at time t = 0.006, showing differences between exact and numerical solutions

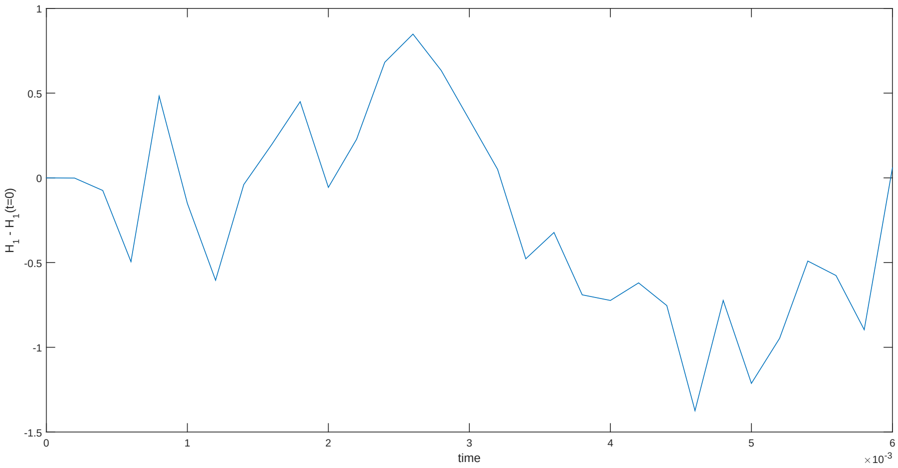

Figure 4. Numerical evaluation of conservation law (53) for KdV by using FD scheme with initial condition as single solitary wave solution

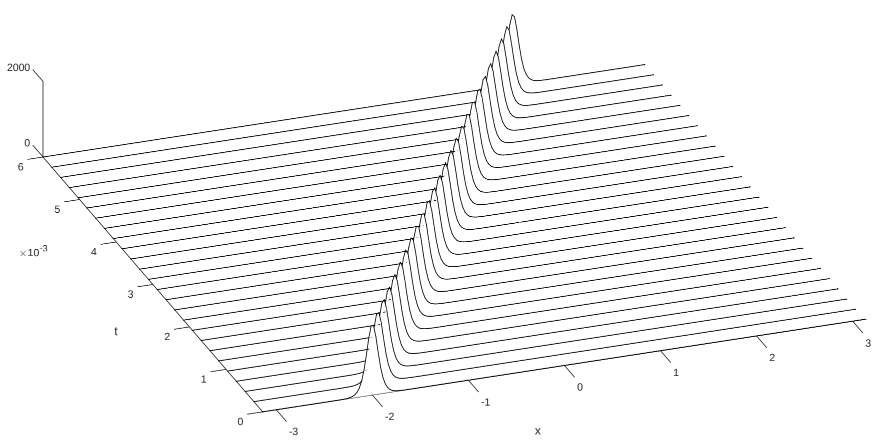

Figure 5. Solution profiles of the KdV equation using Fourier Leap Frog scheme

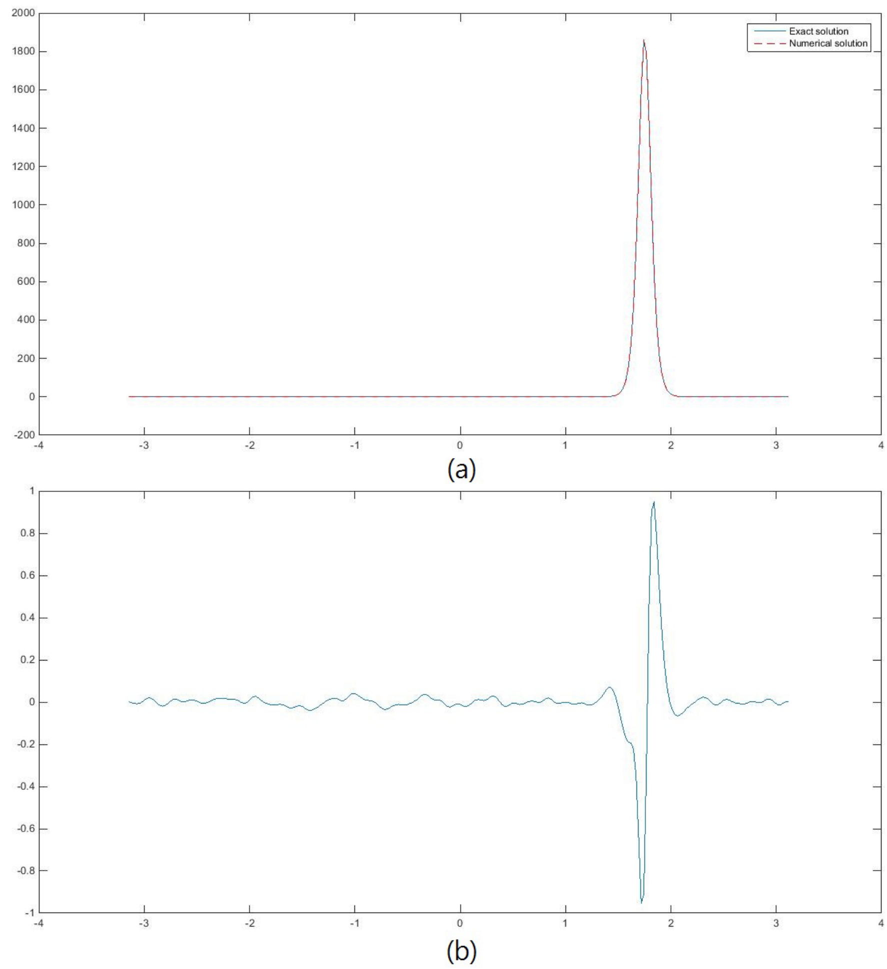

Figure 6. Solution profilefor FLF scheme depicted at time t = 0.006, showing differences between exact and numerical solutions

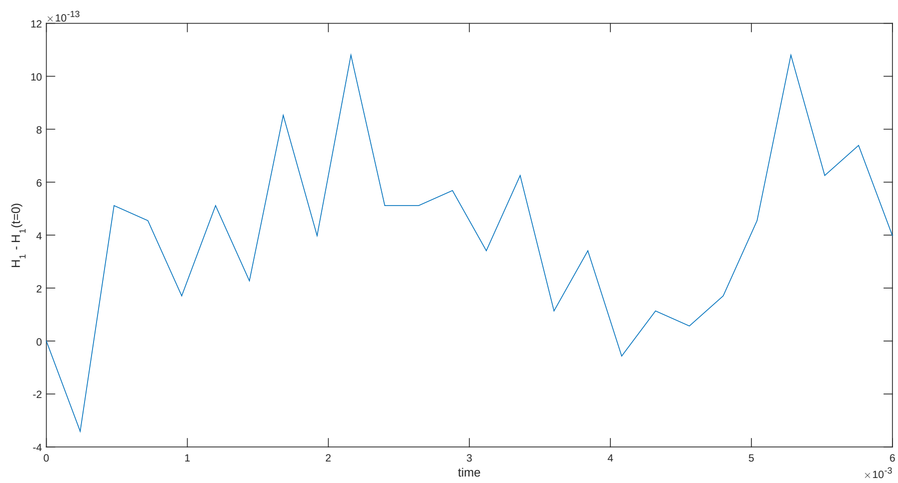

Figure 7. Numerical evaluation of conservation law (53) for KdV by using FLF scheme with initial condition as single solitary wave solution

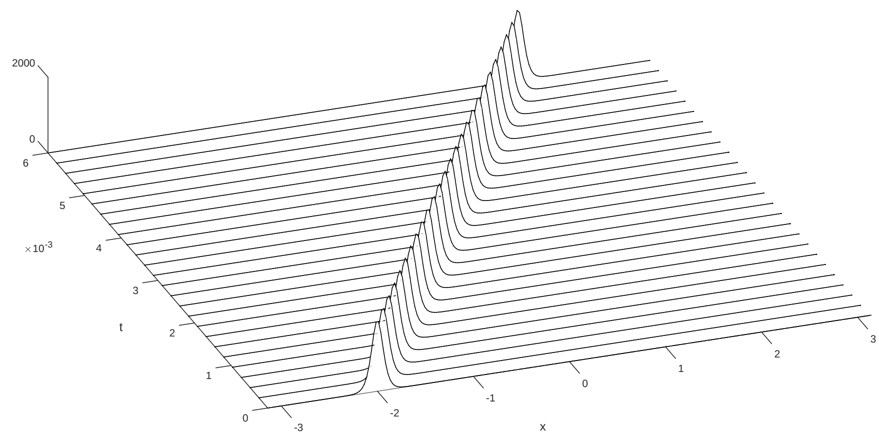

Figure 8. Solution profiles of the KdV equation using Fourier RK4 scheme

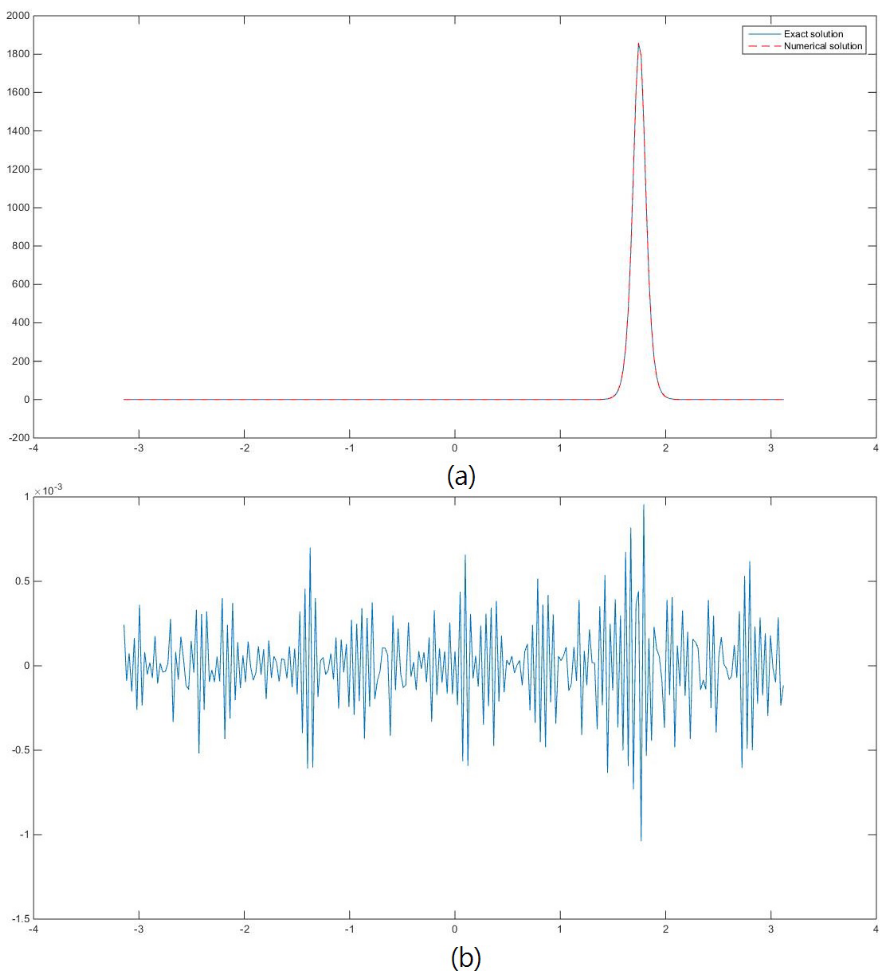

Figure 9. Solution profile for FRK4 scheme depicted at time t = 0.006 for FRK4 scheme, showing differences between exact and numerical solutions

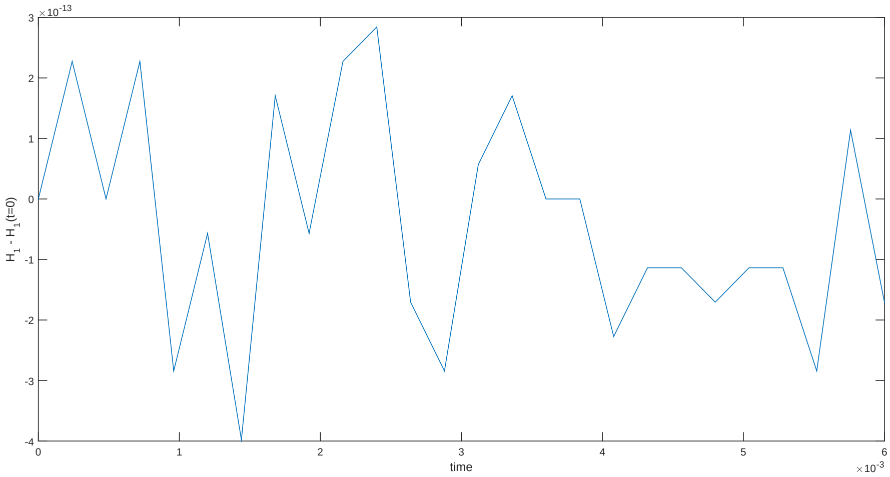

Figure 10. Numerical evaluation of conservation law (53) for KdV by using FRK4 scheme with initial condition as single solitary wave solution.

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