Lévy Process, Proportional Transaction Costs and Foreign Exchange

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Abstract

We analyse optimal portfolio selection problem of maximizing the utility of an agent who invests in a stock and money market account in the presence of proportional transaction cost $\lambda > 0$ and foreign exchange rate. The stock price follows a (generalized) Geometric Itô-Lévy process. The utility function is $U(c) = c^p/p$ for all $c \geq 0$, $p < 1$, $p \neq 0$.

Keywords: Itô-Lévy diffusions, Portfolio optimization, foreign exchange, stochastic control, utility functions

1. Introduction

The study of dynamic portfolio choice has a long history, tracking back to Merton (1971). The introduction of the jumps to the Merton model where the dynamics of stock price are modelled by a Lévy process has attracted a lot of researchers; Øksendal and Sulem (2004, 2005, 2014), Tankov (2005) and many more. But so far when the jumps are included, the determination of the optimal portfolio has not been amenable to a closed-form solution and this is a long-standing open problem in continuous time finance. Lévy processes were introduced by Lévy (1937) who pioneered the theory of infinite divisibility. Modern accounts of the probability theory of Lévy processes are given in Sato (1999) and Applebaum (2004).

Our work is motivated by Øksendal and Sulem (2014) on their quick introduction to some important tools in the modern research within mathematical finance, with emphasis on applications to portfolio optimization and risk minimization, Liu (2004) by successfully analyzing and deriving the optimal transaction policy in an explicit form when $n \geq 1$ risky assets are correlated and subject to fixed transaction costs in an infinite horizon, Janeček and Shreve (2004) and Bichuch (2011) by considering the cases where an agent invests in a stock and a money market account with the hope of maximizing his/her wealth in an infinite time horizon and at the final time T, respectively, in the presence of the proportional transaction costs. Their main idea were to provide a heuristic and a rigorous derivation of the value function in powers of $\lambda^{\frac{1}{3}}$. In this paper, we intend to make contribution in modeling choices and bring powerful yet simple mathematical tools through discussing the impact of foreign exchange when the risky asset is a generalised Itô-Lévy process in the face of transaction costs. We discuss several propositions based on different kinds of jumps in our optimal portfolio problem.

The rest of the paper is organised as follows. In section 2, we present in detail out model. In section 3, we discuss the optimal portfolio in the absence of the transaction costs. In section 4, we provide an optimal investment in the presence of transaction costs. Conclusions are in section 5 followed by references.

2. The Model

We consider a market consisting of two investment opportunities, the money market account a bond and a stock. Assume that there are two riskless assets BW money market and US money market in BW pula and US dollars, respectively with interest rates $r_1 > 0$ and $r_2 > 0$. Due to uncertainty about the future exchange rates, the asset in a US money market is not riskless with respect to the BW pula investor, nor is the asset in BW pula market riskless with respect to the US dollar investor, thus choice of numeraire determines which asset is riskless. Let D_t be the rate of exchange at time $t \in [0, T]$. The dynamics of D_t at time $t \in [0, T]$ follow the diffusion process.

$$dD_t = D_t[\mu_1 dt + \sigma_1 dB_1(t)], \tag{1}$$

where $B_1(t)$ is the Brownian motion. Let bond share price R_t^b at time $t \in [0, T]$ reported in units of bond be given by

$$dR_t^b = r_1 R_t^b dt, \ R_0^b = 1, \tag{2}$$

Thus, the share price of the money market at time t in dollar is $R_t^b D_t = R_t$. From equation (1) and (2) we obtain the stochastic differential equation (SDE) given by

$$dR_t = R_t[(r_1 + \mu_1)dt + \sigma_1 dB_1(t)], \ R(0) = r_0 > 0.$$

In the point of view of the bond investor; in the bond markets, neither the bond nor the exchange rate is tradable. However, $\{R_t\}_{t\geq0}$ can be thought as a bond tradable. Suppose the share price of stock, S_t, which is in dollars at any time $t \geq 0$ is a geometric Itô-Lévy process given by

$$dS_t = S_{t-}\left[\mu_2 dt + \sigma_2 dB_2(t) + \int_{\mathbb{R}} g(t,\zeta)\tilde{N}(dt,d\zeta)\right]; \quad S(0) = S_0 > 0. \tag{3}$$

Taking $g > -1$ then $S(t)$ can never jump to 0 or a negative value (Øksendal and Sulem, 2014). Solution to (3) is given by

$$S_t = S_0 \exp\left\{\left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t) + \int_0^t \int_{\mathbb{R}}(ln(1+g) - g)\nu(d\zeta)ds + \int_0^t \int_{\mathbb{R}} ln(1+g)\tilde{N}(dt,d\zeta)\right\} \tag{4}$$

where $\mu_{1,2} > r_{1,2} > 0$ and $\sigma_{1,2} > 0$ are (constants) known as mean rates of return and volatilities of the exchange rate and stock, respectively. $\tilde{N}$ denote the compensated jump measure of S, defined as $\tilde{N}(dt,d\zeta) = N(dt,d\zeta) - \nu(d\zeta)dt$, $N(dt,d\zeta)$ is the differential notation of the random measure $N([0,t],A)$. ζ is regarded as a generic jump size. The jump measure $N([0,t],A)$ gives the number of jumps of S up to time t with jump size in the set $A \subset \mathbb{R} - \{0\}$. Moreover, if we assume that $\bar{A} \subset \mathbb{R} - \{0\}$, then it can be shown that A contains only finitely many jumps in any finite time interval (Øksendal and Sulem, 2014). The Lévy measure $\nu(\cdot)$ of S is defined by $\nu(A) = \mathbb{E}[N([0,1],A)]$ and satisfies $\int_{\mathbb{R}} \min\{1,|x|\}\nu(dx) < \infty$, so the jumps have finite variation. S_{t-} means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula. $\{B_i(t)\}_{t\geq0}$ for $i = 1,2$ is the standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})$ with $B_i(0) = 0$ for $i = 1,2$ almost surely. The correlation between $\{B_1(t)\}_{t\geq0}$ and $\{B_2(t)\}_{t\geq0}$ is given by

$$B_1(t) = \beta B_2(t) + \sqrt{1 - \beta^2}B_t, \quad -1 \leq \beta \leq 1,$$

where β is the correlation coefficient and B_t is a Brownian motion independent of $B_2(t)$.

Define the dynamics of the bond and stock holdings in monetary terms as

$$dX_t = (r_1 + \mu_1)X_t dt + \sigma_1 X_t dB_1(t) - (1+\lambda)dL_t + (1-\lambda)dM_t, \quad X_0 = x, \tag{5}$$

$$dY_t = Y_{t-}\left[\mu_2 dt + \sigma_2 dB_2(t) + \int_{\mathbb{R}} g(t,\zeta,\omega)\tilde{N}(dt,d\zeta)\right] + dL_t - dM_t, \quad Y_0 = y, \tag{6}$$

where $\lambda \in [0,1)$ accounts for proportional transaction costs paid from the money market, L_t represents the cumulative dollar value of stock purchased up to time t, while M_t is the cumulative dollar value of stock sold. The agent must choose a policy consisting of two adapted processes L_t and M_t that are nondecreasing and right-continuous with left limits and $L_{0-} = M_{0-} = 0$. Note that, purchase of dL_t units of stock requires a payment of $(1+\lambda)dL_t$ from the money market account while sale of dM_t units of stock realizes only $(1-\lambda)dM_t$ in cash. The investor's net wealth in monetary terms at time $t \in [0,T]$ is

$$W_t = X_t + Y_t - \lambda|Y_t|.$$

Define the solvency region

$$Q_\lambda := \{(x,y); \; x + (1+\lambda)y \geq 0, \; x + (1-\lambda)y \geq 0\}$$

as the set of positions from which the agent can move to a positive wealth in both assets. The policy $(L_s, M_s)_{s\in[t,T]}$ is admissible for the initial position (t,x,y) if (X_s, Y_s) starting from $(X_{t-}, Y_{t-}) = (x,y)$ and given (5) and (6) in $\overline{Q_\lambda} \; \forall s \in [t,T]$. Since the agent may choose to re-balance his position, then we set the initial time to be $t-$. We let $\mathcal{A}(t,x,y)$ be the set of all such admissible policies.

Define the agent's utility function by $U(c) = \frac{c^p}{p}$ for all $c \geq 0$, $p < 1$, $p \neq 0$. Define the value function as the supremum of the utility of the final cash position, after the agent liquidates her stock holdings

$$v(t,x,y) = \sup_{(L,M)\in\mathcal{A}(t,x,y)} \mathbb{E}^{x,y}[U(X_T + Y_T - \lambda|Y_T|)|\mathcal{F}_t], \tag{7}$$

for $(t,x,y) \in [0,T] \times \overline{Q_\lambda}$. Define the auxiliary value function

$$v_\rho(t,x,y) = \sup_{(L,M)\in\mathcal{A}(t,x,y)} \mathbb{E}^{x,y}[e^{-\rho(T-t)}U(X_T + Y_T - \lambda|Y_T|)|\mathcal{F}_t], \tag{8}$$

for $(t,x,y) \in [0,T] \times \overline{Q_\lambda}$ and $\rho \geq 0$. Here $\rho \geq 0$ is the discounting factor and $\mathbb{E}^{x,y}$ denotes the conditional expectation at time t given that initial endowment is $X_t = x$ and $Y_t = y$. Therefore

$$v(t,x,y) = e^{\rho(T-t)}v_\rho(t,x,y), \; \rho \geq 0, \; (t,x,y) \in [0,T] \times \overline{Q_\lambda}.$$

The paper concentrates on finding v_ρ. The problem with foreign exchange rate, jumps and no transaction costs ($\lambda = 0$) has solution

$$v_\rho(t, x, y) = \frac{1}{p} e^{pA(T-t)} (x+y)^p, \ \rho \geq 0, \ (t, x, y) \in [0, T] \times \overline{\mathcal{S}_\lambda}, \tag{9}$$

where

$$
\begin{aligned}
A \ &= \ -\frac{\rho}{p} + \mu + \frac{(p-1)}{2}(\sigma_1^2 - \pi^{\star 2}(\sigma_1^2 + \sigma_2^2)) \\
&+ \ \frac{1}{p} \int_\mathbb{R} \{(1 + \pi^\star g(t, \zeta))^p - 1 + p\pi^\star g(t, \zeta)((p-1)\pi^\star g(t, \zeta) - 1)\} \nu(d\zeta)
\end{aligned}
\tag{10}
$$

and the wealth proportion satisfies the equation

$$\pi^\star = \frac{\mu - \mu_2 + (p-1)\sigma_1^2}{(p-1)(\sigma_1^2 + \sigma_2^2)} + ((p-1)(\sigma_1^2 + \sigma_2^2))^{-1} \int_\mathbb{R} g(t, \zeta)[(1 + \pi^\star g(t, \zeta))^{p-1} - 1]\nu(d\zeta) \tag{11}$$

with $\mu = r_1 + \mu_1$.

3. No Transaction Costs Case

Let $\pi(t)$ denote a portfolio representing the portion of the total wealth invested in the risky asset at time $t \in [0, T]$. If we assume that $\pi(t)$ is self-financing, the corresponding wealth $W(t) = W_\pi(t)$ satisfies

$$dW_t = W_t \left[(\mu_2 \pi + (1 - \pi)\mu)dt + (1 - \pi)\sigma_1 dB_1(t) + \pi\sigma_2 dB_2(t) + \pi \int_\mathbb{R} g(t, \omega)\tilde{N}(dt, d\zeta) \right], \tag{12}$$

where $\mu = r_1 + \mu_1$. We seek to maximize $\mathbb{E}[e^{-\rho(T-t)}U(W_\pi(T))|\mathcal{F}_t]$ over all $\pi \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of all admissible portfolios and U is the power utility function.

From Theorem 3.2 in Øksendal and Sulem (2014) and taking $U(w) = w^p/p$, $p < 1$, $p \neq 0$. Then the problem is to maximize $\mathbb{E}[e^{-\rho(T-t)}(W_\pi(T))^p/p|\mathcal{F}_t]$. Thus,

$$
\begin{aligned}
(\mathcal{L}f)(t, w) \ &= \ \frac{\partial f}{\partial t}(t, w) \ + \ w((\mu_2\pi + (1-\pi)\mu))\frac{\partial f}{\partial w}(t, w) + \frac{1}{2}w^2[(1-\pi)^2\sigma_1^2 + \pi^2\sigma_2^2]\frac{\partial^2 f}{\partial w^2}(t, w) \\
&+ \ \int_\mathbb{R} \{f(t, w + w\pi g(t, \zeta)) - f(w) - \nabla f(w)w\pi g(t, \zeta)\}\nu(d\zeta) = 0.
\end{aligned}
\tag{13}
$$

We guess that $f(t, w) = e^{-\rho t}(h(t)w)^p/p$, where $h(t)$ is a deterministic function. We also assume that $g(t, z)$ is deterministic. Then the optimal portfolio $\pi^\star$ is the solution of the equation

$$\pi^\star = \frac{\mu - \mu_2 + (p-1)\sigma_1^2}{(p-1)(\sigma_1^2 + \sigma_2^2)} + ((p-1)(\sigma_1^2 + \sigma_2^2))^{-1} \int_\mathbb{R} g(t, \zeta)[(1 + \pi^\star g(t, \zeta))^{p-1} - 1]\nu(d\zeta). \tag{14}$$

Now, if we assume that the measure ν has light tails, then we can apply Taylor expansion

$$1 - (1 + \pi^\star g(t, \zeta))^{-(1-p)} = (1-p)\pi^\star g(t, \zeta) + o(g(t, \zeta)^2)$$

which leads to the optimal portfolio (13) evolving as

$$\pi^\star \approx \frac{1}{p-1} \cdot \frac{\mu - \mu_2 + (p-1)\sigma_1^2}{(\sigma_1^2 + \sigma_2^2) - \int_\mathbb{R} g^2(t, \zeta)\nu(d\zeta)}. \tag{15}$$

We note that for pure diffusive case, $\nu = 0$, we obtain the Merton solution

$$\pi^\star = \frac{\mu - \mu_2 + (p-1)\sigma_1^2}{(p-1)(\sigma_1^2 + \sigma_2^2)}.$$

Therefore, with jumps it is optimal to place a larger wealth fraction in the risky investment. The resulting value is smaller than in the no-jump case (see Øksendal and Sulem (2005)). Next, we find $h(t)$ (our assumed solution of (13)) by substituting corresponding derivatives into (13) and setting $h(T) = 1$ to get

$$h(t) = \exp\{A(T - t)\},$$

where

$$
\begin{aligned}
pA = -\rho \quad & + \quad p(\mu_2\pi^\star + (1-\pi^\star)\mu) + \frac{p(p-1)}{2}((1-\pi^\star)^2\sigma_1^2 + \pi^{\star 2}\sigma_2^2) \\
& + \quad \int_{\mathbb{R}}\{(1+\pi^\star g(t,\zeta))^p - 1 - p\pi^\star g(t,\zeta)\}\nu(d\zeta) \\
& = \quad -\rho + p\mu + \frac{p(p-1)}{2}(\sigma_1^2 - \pi^{\star 2}(\sigma_1^2 + \sigma_2^2)) \\
& + \quad \int_{\mathbb{R}}\{(1+\pi^\star g(t,\zeta))^p - 1 + p\pi^\star g(t,\zeta)((p-1)\pi^\star g(t,\zeta) - 1)\}\nu(d\zeta).
\end{aligned}
\tag{16}
$$

Finally, the value function is given by

$$
f(t,w) = v(t,w) = e^{-\rho t}e^{pA(T-t)}w^p/p, \quad p < 1, \quad p \neq 0.
$$

Substituting the constants $\pi^\star$ and A in (12), yields the optimal wealth process $W_t^\star$ of the Ito-Lévy process.

4. Optimal Investment in the Presence of Transaction Costs

Theorem 1 *Since X, Y are jointly Markov, the value function $v(t, x, y)$ defined by equation (8) is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation on $[0, T] \times Q_\lambda$*

$$
\min\{-v_t - \mathcal{K}(v), -(1-\lambda)v_x + v_y, (1+\lambda)v_x - v_y\} = 0
\tag{17}
$$

where $\mathcal{K}$ is the second-order differential operator given by

$$
\begin{aligned}
\mathcal{K}(v) \quad & = \quad -\rho v(t,x,y) + \frac{1}{2}[\sigma_2^2 y^2 v_{yy}(t,x,y) + \sigma_1^2 x^2 v_{xx}(t,x,y)] \\
& + \quad \mu x v_x(t,x,y) + \mu_2 y v_y(t,x,y) + \beta\sigma_1\sigma_2 xy v_{xy}(t,x,y) \\
& - \quad \int_{\mathbb{R}}\{v(t,x,y+g(y,\zeta)) - v(t,x,y) - v_y(t,x,y)g(y,\zeta)\}\nu(d\zeta)
\end{aligned}
\tag{18}
$$

with the terminal condition

$$
v(T,x,y) = U(x+y-\lambda|y|), \quad (x,y) \in \overline{Q_\lambda}.
\tag{19}
$$

Noting that the value function v has the homotheticity property for $\gamma > 0$,

$$
v(t,\gamma x,\gamma y) = \gamma^p v(t,x,y)
\tag{20}
$$

where the admissible policy $\mathcal{A}(t,\gamma x,\gamma y) = \{(\gamma L, \gamma M) : (L,M) \in \mathcal{A}(t,x,y)\}$. We can reduce the problem into that of two variables with the transformation; $\xi = \frac{y}{x+y}$, $1 - \xi = \frac{x}{x+y}$. Define

$$
\phi(t,\xi) = v(t, 1-\xi, \xi), \quad (t,\xi) \in [0,T] \times \overline{Q_\phi}
\tag{21}
$$

where $\overline{Q_\phi} = \left[\frac{-1}{\lambda}, \frac{1}{\lambda}\right]$, gives

$$
v(t,x,y) = (x+y)^p \phi(t,\xi), \quad (t,\xi) \in [0,T] \times Q_v.
\tag{22}
$$

Theorem 2 *ϕ is a viscosity solution of the HJB equation $\mathcal{L}(\phi) = 0$ on $[0,T] \times \overline{Q_\phi}$;*

$$
\mathcal{L}(\phi) = \min\{-\phi_t - \mathcal{P}(\phi), \lambda p\phi(t,\xi) + (1-\lambda\xi)\phi_\xi(t,\xi), \lambda p\phi(t,\xi) - (1+\lambda\xi)\phi_\xi(t,\xi)\},
\tag{23}
$$

such that

(24)

$$
\begin{aligned}
\mathcal{P}(\phi) \;=\; & \{-\rho + p(\mu(1-\xi) + \mu_2\xi) + \frac{p(p-1)}{2}(\sigma_2{}^2\xi^2 + \sigma_1{}^2(1-\xi)^2 + 2\beta\sigma_1\sigma_2\xi(1-\xi))\}\phi(t,\xi) \\
+ \;& \xi(1-\xi)[\mu_2 - \mu + (p-1)(\beta\sigma_1\sigma_2(1-2\xi) + \xi(\sigma_1{}^2 + \sigma_2{}^2))]\phi_\xi(t,\xi) \\
- \;& \frac{\xi^2(1-\xi)^2}{2}(\sigma_2{}^2 + \sigma_1{}^2 + 2\beta\sigma_1\sigma_2)\phi_{\xi\xi}(t,\xi) \\
- \;& \int_{\mathbb{R}}\{\phi(t,\xi + h(\xi,\zeta)) - \phi(t,\xi)(1 + ph(\xi,\zeta)) - \phi_\xi(t,\xi)h(\xi,\zeta)\}\nu(d\zeta), \\
=\;& \{pA - \Lambda_0 + \frac{p(p-1)}{2}((\sigma_1{}^2 + \sigma_2{}^2)(\xi - \pi)^2 + 2\beta\sigma_1\sigma_2\xi(1-\xi))\}\phi(t,\xi) \\
+ \;& (p-1)\xi(1-\xi)[(\xi-\pi)(\sigma_1{}^2 + \sigma_2{}^2) + \beta\sigma_1\sigma_2(1-2\xi) + \sigma_1{}^2 + \int_{\mathbb{R}}g^2(t,\zeta)\nu(d\zeta)]\phi_\xi(t,\xi) \\
- \;& \frac{\xi^2(1-\xi)^2}{2}(\sigma_2{}^2 + \sigma_1{}^2 + 2\beta\sigma_1\sigma_2)\phi_{\xi\xi}(t,\xi) \\
- \;& \int_{\mathbb{R}}\{\phi(t,\xi + h(\xi,\zeta)) - \phi(t,\xi)(1 + ph(\xi,\zeta)) - \phi_\xi(t,\xi)h(\xi,\zeta)\}\nu(d\zeta),
\end{aligned}
$$

where

$$
\begin{aligned}
h(\xi,\zeta) \;=\;& (x+y)^{-1}g(y,\zeta) \\
=\;& (x+y)^{-1}yk(t,\zeta) \\
=\;& \xi k(t,\zeta),
\end{aligned}
$$
(25)

and

$$
\Lambda_0 = \int_{\mathbb{R}}\{(1 + \pi^\star g(t,\zeta))^p - 1 + p\pi^\star g(t,\zeta)((p-1)\pi^\star g(t,\zeta) - 1)\}\nu(d\zeta),
$$

subject to the terminal condition

$$
\phi(T,\xi) = U(1 - \lambda|\xi|), \;\; \xi \in \overline{Q_\phi}.
$$

For convenience we shall from now on impose the following additional integrability condition on $\nu(\cdot)$:

$$
\int_{\mathbb{R}}g^2(t,\zeta)\nu(d\zeta) < \infty.
$$

Remark

1. This is equivalent to Bichuch (2011) when there are no jump term and no foreign exchange fluctuations (that is, $\sigma_1 = 0$ and $\mu_1 = 0$).

2. In the presence of the jumps, we then study the scenario when the impact of the foreign exchange is given by

$$
\begin{aligned}
\Lambda_0\phi(t,\xi) \;=\;& \Big((p-1)\xi(1-\xi)\int_{\mathbb{R}}g^2(t,\zeta)\nu(d\zeta)\Big)\phi_\xi(t,\xi) \\
- \;& \int_{\mathbb{R}}\{\phi(t,\xi + h(\xi,\zeta)) - \phi(t,\xi)(1 + ph(\xi,\zeta)) - \phi_\xi(t,\xi)h(\xi,\zeta)\}\nu(d\zeta).
\end{aligned}
$$
(26)

We will be concerned with the nature of the solution to equation (26) as the impact of foreign exchange. We assume $\phi(t,\xi) = f(\xi)l(t)$ and $g(t,\zeta) = \zeta$. Since $l(t) \neq 0$, then (26) simplifies to

$$
\begin{aligned}
\int_{\mathbb{R}}f(\xi(1+\zeta))\nu(d\zeta) \;=\;& \Big((p-1)\xi(1-\xi)\int_{\mathbb{R}}\zeta^2\nu(d\zeta) + \xi\int_{\mathbb{R}}\nu(d\zeta)\Big)f_\xi(\xi) \\
+ \;& \Big(\int_{\mathbb{R}}(1 - p\xi\zeta)\nu(d\zeta) - \Lambda_0\Big)f(\xi).
\end{aligned}
$$
(27)

For the ease of the computations, we set the following:

$$
\int_{\mathbb{R}}f(\xi(1+\zeta))\nu(d\zeta) = 0,
$$

so we have

$$
\begin{aligned}
f(\xi) &= A_0 \exp\left\{ \int_{\overline{Q_\phi}} \frac{\int_{\mathbb{R}}(1 - p\xi\zeta)\nu(d\zeta) - \Lambda_0}{\left(\int_{\mathbb{R}}((1-p)(1-\xi)\zeta^2 + 1)\nu(d\zeta)\right)\xi} d\xi \right\} \\
&= A_0 \exp\left\{ R\log(\xi) + P\log\left(\int_{\mathbb{R}}((1-p)(1-\xi)\zeta^2 + 1)\nu(d\zeta)\right) \right\}
\end{aligned}
\tag{28}
$$

where

$$
R = \frac{\int_{\mathbb{R}}\nu(d\zeta) - \Lambda_0}{\int_{\mathbb{R}}((1-p)\zeta^2 + 1)\nu(d\zeta)},
$$

$$
P = \frac{\Lambda_0 - \int_{\mathbb{R}}\nu(d\zeta)}{\int_{\mathbb{R}}((1-p)\zeta^2 + 1)\nu(d\zeta)} + \frac{p\int_{\mathbb{R}}\nu(d\zeta)}{(1-p)\int_{\mathbb{R}}\zeta^2\nu(d\zeta)}
$$

and A_0 is a constant. Satisfying the following condition

$$
\int_{\mathbb{R}}((1-p)(1-\xi)\zeta^2 + 1)\nu(d\zeta) > 0.
$$

We need to specify the Lévy measure $\nu(d\zeta)$ driving the common jumps; then we can successfully compute the integrals in (27). This is considered under different cases below:

4.1 Uniform Jumps

Proposition 1 *Suppose the value function is $\phi(t, \zeta) = f(\zeta)l(t)$, $l(t) \neq 0$ and $g(t, \zeta) = \zeta$, then under uniform jumps, (28) is valid if*

$$
0 < \xi < \frac{(1-p)\zeta^2 + 3}{(1-p)\zeta^2}\bigg|_{\mathbb{R}}.
\tag{29}
$$

Proof. Set $\nu(d\zeta) = \tau d\zeta$, where τ is arbitrary constant describing the intensity of the jump. Then we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} f(\xi(1+\zeta))d\zeta &= \left(\frac{p-1}{3}(1-\xi)\zeta^2 + 1\right)\xi\zeta\bigg|_{\mathbb{R}} f_\xi(\xi) \\
&+ \left((1 - p\xi\zeta)\zeta\bigg|_{\mathbb{R}} - \Lambda_0^\star\right)f(\xi),
\end{aligned}
\tag{30}
$$

where

$$
\Lambda_0^\star = \left(\frac{(1+\pi\zeta)^{1+p}}{(1+p)\pi} - \frac{p(1-p)}{3}\pi^2\zeta^3 + \frac{p}{2}\pi\zeta^2 + \zeta\right)\bigg|_{\mathbb{R}}.
$$

Assuming that the integral on the right of (30) varnishes, we then obtain the following solution

$$
f(\xi) = B_0 \exp\left\{ \int_{\overline{Q_\phi}} \frac{(1 - p\xi\zeta)\zeta\big|_{\mathbb{R}} - \Lambda_0^\star}{\left(\frac{1-p}{3}(1-\xi)\zeta^2 + 1\right)\xi\zeta\big|_{\mathbb{R}}} d\xi \right\},
\tag{31}
$$

where B_0 is a constant of integration and

$$
\int_{\overline{Q_\phi}} \frac{(1 - p\xi\zeta)\zeta\big|_{\mathbb{R}} - \Lambda_0^\star}{\left(\frac{1-p}{3}(1-\xi)\zeta^2 + 1\right)\xi\zeta\big|_{\mathbb{R}}} d\xi = P^\star(\zeta)\log\left(\frac{1-p}{3}(1-\xi)\zeta^2 + 1\right)\bigg|_{\mathbb{R}}
$$
$$
+ R^\star(\zeta)\log(\xi)\bigg|_{\mathbb{R}},
\tag{32}
$$

where

$$
R^\star(\zeta) = \frac{3(1 - \Lambda_0^\star)}{(1-p)\zeta^2 + 3}
\tag{33}
$$

and

$$P^\star(\zeta) = \frac{3(\Lambda_0^\star - \zeta)}{((1-p)\zeta^2 + 3)\zeta} + \frac{3p}{(1-p)\zeta}. \tag{34}$$

such that the inequality

$$0 < \xi < \frac{(1-p)\zeta^2 + 3}{(1-p)\zeta^2}\bigg|_{\mathbb{R}}$$

is met.

4.2 Power Law Jumps

We assume that the arrival of jumps of size ζ follows a power law, dampened by an exponential function:

$$\nu(d\zeta) = \begin{cases} \tau^+ e^{-\beta_+\zeta}\zeta^{-\alpha-1}d\zeta, & \text{if } \zeta \in (0, \infty) \\ -\tau^- e^{-\beta_-|\zeta|}|\zeta|^{-\alpha-1}d\zeta, & \text{if } \zeta \in (-\infty, 0), \end{cases} \tag{35}$$

with the parameters $\alpha \in [-1, 2)$ and $\beta_\pm, \tau^\pm \in \mathbb{R}^+$. Setting $\beta_\pm = 0$, that is, without exponential dampening, the Lévy density uniquely determines an α-stable Lévy motion that generates the α-stable distribution (see Mandelbrot (1963) and Fama (1965)). For strictly dampening $\beta_\pm > 0$, the exponential functions $e^{-\beta_+\zeta}$ and $e^{-\beta_-|\zeta|}$ dampen the Lévy density so that the arrival of jumps decays faster as the absolute jump size $|\zeta|$ increases (see Wu (2006)). We also note the following:

1. When $\alpha < 0$: $\int_{\mathbb{R}} \nu(d\zeta) < 0$. Then we have large and rare events.

2. Infinite activity when $\alpha \geq 0$: Both small and large jumps. Jumps frequency increases with declining jump size and approaches infinity as $\zeta \to 0$.

3. Infinite Variation when $\alpha \geq 1$: many small jumps.

We shall consider a special case where the exponential dampening parameter, $\beta = 1$, and propose as follows;

Proposition 2 *Suppose the value function is $\phi(t, \xi) = f(\xi)l(t)$, $l(t) \neq 0$ and $g(t, \zeta) = \zeta$, $t \in [0, T]$, then (28) is valid under dampen power law jumps with $\beta^\pm = 1$ if*

$$0 < \xi < \frac{(1-p)(1-\alpha)\alpha - 1}{(1-p)(1-\alpha)\alpha}. \tag{36}$$

Proof. Setting $\beta^\pm = 1$, $\tau^\pm = \tau > 0$ and $\alpha \in [-1, 2)$ in (35). Then for all $t \in [0, T]$, the value function is given by

$$\phi(t, \xi) = A_0 l(t) \exp\left\{R\log(\xi) + P\log\left(2\tau\Gamma(-\alpha)(1 - \alpha(1-\alpha)(1-p)(1-\xi))\right)\right\} \tag{37}$$

where $\Gamma(\cdot)$ is a gamma function,

$$R = \frac{1 - \pi((1-p)\pi\alpha - p)(1-\alpha)}{1 - (1-p)(1-\alpha)\alpha}, \tag{38}$$

$$P = \frac{p}{(p-1)(1-\alpha)\alpha} - R, \tag{39}$$

and A_0 is an arbitrary constant. The value function $\phi(t, \xi)$ is defined only when

$$0 < \xi < \frac{(1-p)(1-\alpha)\alpha - 1}{(1-p)(1-\alpha)\alpha}. \tag{40}$$

Which completes the short proof.

4.3 Fixed Jump Size

Proposition 3 *Suppose the value function is $\phi(t, \xi) = f(\xi)l(t)$, $l(t) \neq 0$ and $g(t, \zeta) = \zeta$, then (28) is valid under fixed jump size if*

$$0 < \xi < \frac{11 - 2p}{2(1-p)}. \tag{41}$$

Proof. Set $\nu(d\zeta) = \delta(\zeta = \bar{\zeta})d\zeta$, for some $\bar{\zeta} \in [-1, 1]$ as the Dirac measure. This measure is associated to a point $\zeta \in \mathbb{R}$ defined as follows:

$$\nu(d\zeta) = \begin{cases} d\zeta, & \zeta \in [-1, 1]; \\ 0, & \zeta \notin [-1, 1]. \end{cases} \tag{42}$$

It then follows that (28) becomes

$$f(\xi) = \begin{cases} A_0 \exp\left\{\frac{3(2-\Lambda_2)}{2(1-p)}\log(\xi) + \frac{(1-p)\Lambda_2 - 2(p^2-3p+1)}{(2-p)(1-p)}\log\left(\frac{2}{3}((1-p)(1-\xi)+3)\right)\right\}, & \zeta \in [-1,1]; \\ \\ 0, & \zeta \notin [-1,1]. \end{cases} \tag{43}$$

where

$$\Lambda_2 = \begin{cases} \frac{1}{(1+p)\pi}\left((1+\pi)^{1+p} - (1-\pi)^{1+p}\right) - \frac{2}{3}((1-p)\pi^2 + 3), & \zeta \in [-1,1]; \\ 0, & \zeta \notin [-1,1], \end{cases} \tag{44}$$

A_0 is a constant and

$$0 < \xi < \frac{11-2p}{2(1-p)}.$$

Since the jump sizes are fixed, it becomes too restrictive and less suitable to use in modelling stock price (for instance). But it can be used as a building block to construct richer models.

5. Results

Even though the stock price may experience some jumps, if they are small in absolute value then the total payoff increases exponentially. It is also observed that in a jump setting, if the jump measure is zero then we have Merton model setting and optimal portfolio matches. The effect of jumps, transaction costs and foreign exchange is significant as it makes the model realistic.

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