Journal of Mathematics Research; Vol. 9, No. 4; August 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On Theory of logarithmic Poisson Cohomology

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Received: May 30, 2017 Accepted: June 14, 2017 Online Published: July 25, 2017

doi:10.5539/jmr.v9n4p209 URL: https://doi.org/10.5539/jmr.v9n4p209

Abstract

We define the notion of logarithmic Poisson structure along a non zero ideal $\mathcal{I}$ of an associative, commutative algebra $\mathcal{A}$ and prove that each logarithmic Poisson structure induce a skew symmetric 2-form and a Lie-Rinehart structure on the $\mathcal{A}$-module $\Omega_K(\log \mathcal{I})$ of logarithmic Kähler differential. This Lie-Rinehart structure define a representation of the underline Lie algebra. Applying the machinery of Chevaley-Eilenberg and Palais, we define the notion of logarithmic Poisson cohomology which is a measure obstructions of Linear representation of the underline Lie algebra for which the grown ring act by multiplication.

Keywords: Poisson structure, Logarithmic Poisson structure, Logarithmic Poisson cohomology, Logarithmic form, Logarithmic derivation, prequantization

1. Introduction

The first Poisson bracket on the algebra of smooth functions on $\mathbb{R}^{2n}$;

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right) \tag{1}$$

was defined by S.D. Poisson in 1809. This bracket plays a fundamental role in the analytical mechanics. One century later that A. Lichnerowicz in (Lichnerowicz, A. (1977)) and A. Weinstein in (Weinstein, A. (1983)) extended it in a large theory now known as the Poisson Geometry. It has been remarked by A. Weinstein that the theory can be traced back to S. Lie in (Lie, S. (1888)). The Poisson bracket (1) is derived from a symplectic structure on $\mathbb{R}^{2n}$ and it appears as one of the main ingredients of symplectic geometry.

The basic properties of the bracket (1) are that it yields the structure of a Lie algebra on the space of functions and it has a natural compatibility with the usual associative product of functions.

These facts are of algebraic nature and it is natural to define an abstract notion of a *Poisson algebra*.

Following A. Vinogradov and I. Krasil'shchik in (Vinogradov, A. M., & Krasil'shchik, I. S. (1981)), J. Braconnier in (Braconnier, J. (1977)) has developed the algebraic version of Poisson geometry.

One of the most important notion related to the Poisson geometry is Poisson cohomology which was introduced by A. Lichnerowicz in (Lichnerowicz, A. (1977)) and in algebraic setting by I. Krasil'shchik in (Krasil'shchik, I. (1988)). Unlike the De Rham cohomology, Poisson cohomology spaces are almost irrelevant to the topology of the manifold and moreover they have bad functorial properties. They are very large and their actual computation is both more complicated and less significant than in the case of the De Rham cohomology. However they are very interesting because they allow us to describe various results concerning Poisson structures in particular one important result about the *geometric quantization* of the manifold. Algebraic aspects of this theory were developed by J. Huebschmann in (Huebschmann, J. (2013)) and in the geometrical setting by I. Vaisman in (Vaisman, I. (1991).).

This paper deals with Poisson algebras but Poisson algebras of another kind. More precisely we study the *logarithmic Poisson structures*. If the Poisson structures draw their origins from symplectic structures, logarithmic Poisson structure are inspired by log symplectic structures which are based on the theory of logarithmic differential forms. The logarithmic differential forms was introduced by P. Deligne in (Deligne, P. (2006)) who defined them in the case of a normal crossings divisor of a given complex manifold. But the theory of logarithmic differential forms along a divisor without necessarily normal crossings was introduced by K. Saito in (Saito, K. (1980)). Explicitly if $\mathcal{I}$ is an ideal in a commutative algebra $\mathcal{A}$ over a commutative ring R a derivation D of $\mathcal{A}$ is called logarithmic along $\mathcal{I}$ if $D(\mathcal{I}) \subset \mathcal{I}$. We denote by $Der_{\mathcal{A}}(\log \mathcal{I})$

[1]J.D Was partially supported by MINESUP

the $\mathcal{A}$-module of derivations of $\mathcal{A}$ logarithmic along $\mathcal{I}$. A Poisson structure $\{.,.\}$ on $\mathcal{A}$ is called logarithmic [2] along $\mathcal{I}$ if for all $a \in \mathcal{A}$ we have $\{a,.\} \in Der_{\mathcal{A}}(\log \mathcal{I})$. In addition suppose that $\mathcal{I}$ is generated by $\{u_1, ..., u_p\} \subset \mathcal{A}$ and let $\Omega_{\mathcal{A}}$ be the $\mathcal{A}$-module of Kähler differential. The $\mathcal{A}$-module $\Omega_{\mathcal{A}}(\log \mathcal{I})$ generated by $\{\frac{du_1}{u_1}, ..., \frac{du_p}{u_p}\} \cup \Omega_{\mathcal{A}}$ is called the module of Kähler differentials logarithmic along $\mathcal{I}$.

J. Huebschmann's program of algebraic construction of the Poisson cohomology can be summarized as follows:

Let $\mathcal{A}$ be a commutative algebra over a commutative ring R. A Lie-Rinehart algebra on $\mathcal{A}$ is an $\mathcal{A}$-module which is an R-Lie algebra acting on $\mathcal{A}$ with suitable compatibly conditions. J. Huebschmann observes that each Poisson structure $\{.,.\}$ gives rise to a structure of Lie-Rinehart algebra in the sense of G. Rinehart in (Rinehart, G. S. (1963)) on the $\mathcal{A}$-module $\Omega_{\mathcal{A}}$ in natural fashion. But it was proved in (Palais, R. (1961)) that any Lie-Rinehart algebra L on $\mathcal{A}$ gives rise to a complex $Alt_{\mathcal{A}}(L, \mathcal{A})$ of alternating forms which generalizes the usual De Rham complex of manifold and the usual complex computing Chevalley-Eilenberg in (Chevalley, C., & Eilenberg, S. (1948)) Lie algebra cohomology. Moreover extending earlier work of Hochshild Kostant and Rosenberg in (Hochschild, G., Kostant, B., & Rosenberg, A. (2009)). G. Rinehart has shown that when L is projective as an $\mathcal{A}$-module the homology of the complex $Alt_{\mathcal{A}}(L, \mathcal{A})$ may be identified with $Ext^*_{U(\mathcal{A},L)}(\mathcal{A}, \mathcal{A})$ over a suitably defined universal algebra $U(\mathcal{A}, L)$ of differential operators. But the latter defines a Lie algebra cohomology $H^*(L, \mathcal{A})$ of L. So, since $\Omega_{\mathcal{A}}$ is free $\mathcal{A}$-module, it is projective. Therefore the homology of the complex $Alt_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A})$ computing the cohomology of the underlying Lie algebra of the Poisson algebra $(\mathcal{A}, \{.,.\})$. Then the Poisson cohomology of $(\mathcal{A}, \{.,.\})$ is the homology of $Alt_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A})$.

It follows from the definition of logarithmic Poisson structure that the image of Hamiltonian map of logarithmic principal Poisson structure is sub-module of $Der_{\mathcal{A}}(\log \mathcal{I})$. Inspired by this fact we introduce the notion of logarithmic Lie-Rinehart structure. A Lie-Rinehart algebra L on $\mathcal{A}$ is said to be logarithmic along an ideal $\mathcal{I}$ of $\mathcal{A}$ if it acts by logarithmic derivations on $\mathcal{A}$.

If $\mathcal{I}$ is an ideal of an associative commutative algebra $\mathcal{A}$ over a field of characteristic zero, denoted by $\widehat{Der_K(\log \mathcal{I})}$ the submodule of $Der_K(\log \mathcal{I})$ constituted by $v \in Der_K(\log \mathcal{I})$ such that $v(u) \in u\mathcal{A}$, we prove the following:

- $\widehat{Der_K(\log \mathcal{I})} = \{\delta \in Der_K(\log \mathcal{I}) : \delta(u) \in u\mathcal{A}, \forall u \in \mathcal{I}\}$.
$$\begin{array}{rcl} \hat{\varphi} : \widehat{Der_K(\log \mathcal{I})} & \rightarrow & \Omega_K(\log \mathcal{I})^\vee \\ \delta & \mapsto & \hat{\tilde{\delta}} \end{array}$$
is an homomorphism of $\mathcal{A}$-modules. Where $\hat{\tilde{\delta}} : \frac{du}{u} \mapsto \frac{1}{u}\tilde{\delta}(d(u))$.
- For each logarithmic Poisson structure $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, there exist a unique $\mathcal{A}$-module homomorphism $H : \Omega_K(\mathcal{A}) \rightarrow Der_K(\log \mathcal{I})$ such that $H \circ d = ad$.
- Let $\{-,-\}$ be a logarithmic Poisson structure along $\mathcal{I}$. The map $[-,-] : (x, y) \mapsto [x, y] = -(\omega(x, y)) + \mathcal{L}_{\bar{H}(x)}(y) - \mathcal{L}_{\bar{H}(y)}(x)$ is a Lie algebra structure on $\Omega_K(\log \mathcal{I})$.

2. On the Lie-Rinehart Algebra of Logarithmic Principal Differential Form

2.1 Logarithmic Derivation and Logarithmic Formal Differential.

In this section, we recall the notion of logarithmic derivation along a non zero ideal of an associative, commutative and unitary algebra $\mathcal{A}$.

2.1.1 Module of Logarithmic Derivations

Let k be a field of characteristic zero and $\mathcal{A}$ a k-algebra. $Der_k(\mathcal{A})$ the $\mathcal{A}$-module of derivations on $\mathcal{A}$, $\mathcal{I}$ a non zero ideal of $\mathcal{A}$ and $d_{\mathcal{A}/K}$ the universal derivation associated to the $\mathcal{A}$-module of Kähler differentials $\Omega_k(\mathcal{A})$.

Definition 2.1. *A k-derivation logarithmic along I is an element d of $Der_k(\mathcal{A})$ such that $d(\mathcal{I}) \subset \mathcal{I}$.*

The set of k-derivations logarithmic along $\mathcal{I}$ is denoted by $Der_K(\log \mathcal{I})$.

It follows from the definition and the fact that $\mathcal{A}$ is commutative, that $Der_K(\log \mathcal{I})$ is sub $\mathcal{A}$-module of $Der_K(\mathcal{A})$

The map $\sigma_{\mathcal{A}} : \psi \mapsto \psi \circ d_{\mathcal{A}/k}$ is an $\mathcal{A}$-module isomorphism from $\Omega_K(\mathcal{A})^\vee$ to $Der_K(\mathcal{A})$, where $\Omega_K(\mathcal{A})^\vee$ denote the dual of $\Omega_K(\mathcal{A})$.
Indeed, for all $\psi \in \Omega_K(\mathcal{A})^\vee$ and $a, b \in \mathcal{A}$,

[2] The statement *the Poisson structure is logarithmic along $\mathcal{I}$* also expresses as *$\mathcal{I}$ is a Poisson ideal of $\mathcal{A}$*. For example any smooth Poisson manifolds is logarithmic along the ideal of the smooth functions which vanish on a given symplectic leaf

$$\sigma_{\mathcal{A}}(\psi)(ab) = \psi(ad(b) + bd(a)) = a\sigma_{\mathcal{A}}(b) + b\sigma_{\mathcal{A}}(a).$$

Then $\sigma_{\mathcal{A}}(\psi) \in Der_K(\mathcal{A})$. On the other hand, The map $\varphi : d \mapsto \tilde{d}$, where $\tilde{d}$ is the unique $\mathcal{A}$-module homomorphism from $\Omega_K(\mathcal{A})$ to $\mathcal{A}$ such that $\tilde{d} \circ d = d$. Where φ denote the inverse of $\sigma_{\mathcal{A}}$.

we have the following proposition.

Proposition 2.2. $\sigma_{\mathcal{A}} : Der_K(\mathcal{A}) \backsimeq \Omega_K(\mathcal{A})^{\vee}$

In what follow, we will designate φ the inverse of $\sigma_{\mathcal{A}}$.

2.1.2 Module of Formals Logarithmic Differentials

In this subsection we define the module of logarithmic Kähler differential.

Let $\Im := \mathcal{I}^* \cup \{1_{\mathcal{A}}\}$. We denote $\Im^{-1}\Omega_K(\mathcal{A})$ the localized of $\Omega_K(\mathcal{A})$. Since $\Omega_K(\mathcal{A})$ is generated by $d(a)$ and $a \in \mathcal{A}$, $\Im^{-1}\Omega_K(\mathcal{A})$ is generated by $\{\frac{d(a)}{u}\}_{a\in\mathcal{A},u\in\Im}$. We denote $\Omega_K(\log \mathcal{I})$ the $\mathcal{A}$-submodule of $\Im^{-1}\Omega_K(\mathcal{A})$ generated by $\Omega_K(\mathcal{A}) \cup \{\frac{d(u)}{u}\}_{u\in\Im}$.

Definition 2.3. $\Omega_K(\log \mathcal{I})$ *is called $\mathcal{A}$-module of logarithmic formals differentials or module of logarithmic Kähler differential along $\mathcal{I}$.*

As $\mathcal{A}$-module, $\Omega_K(\log \mathcal{I})$ is not free in general. Indeed, for all $u \in \Im$, $u\frac{d(u)}{u} - d(u) = 0$, then the subset $\Omega_K(\mathcal{A}) \cup \{\frac{d(u)}{u}\}_{u\in\Im}$ is not free.

Follow the K. Saito in (Saito, K. (1980)), when $\Omega_K(\log \mathcal{I})$ is free $\mathcal{A}$-module, $\mathcal{I}$ is called a "*free ideal*"

It is proved in (12) that for all $\mathcal{A}$-module M, each $\delta \in Der_K(\mathcal{A}, M)$ induce a homomorphism of $\mathcal{A}$-modules $\tilde{\delta} : \Omega_K(\mathcal{A}) \to M$; such that $\tilde{\delta} \circ d = \delta$. If $M = \mathcal{A}$ and $\delta(u) \in u\mathcal{A}$ for all $u \in \mathcal{I}$, then $\tilde{\delta}(d(u)) \in u\mathcal{A}$. In this case, we consider the homomorphism $\hat{\tilde{\delta}} : \frac{d(u)}{u} \mapsto \frac{1}{u}\tilde{\delta}(d(u))$. Since $\tilde{\delta}(d(u)) \in u\mathcal{A}$, $\frac{1}{u}\tilde{\delta}(d(u)) \in \mathcal{A}$. Therefore, $\hat{\tilde{\delta}} \in \Omega_K(\log \mathcal{I})^{\vee}$. Let us denote $\widehat{Der_K(\log \mathcal{I})}$ the submodule of $Der_K(\log \mathcal{I})$ constituted by $\nu \in Der_K(\log \mathcal{I})$ such that $\nu(u) \in u\mathcal{A}$. The above construction induce an homomorphism

$$\begin{array}{cccc} \hat{\varphi} : \widehat{Der_K(\log \mathcal{I})} & \to & \Omega_K(\log \mathcal{I})^{\vee} \\ \delta & \mapsto & \hat{\tilde{\delta}} \end{array}.$$

We have prove the following lemma.

Lemma 2.4. *Let* $\widehat{Der_K(\log \mathcal{I})} = \{\delta \in Der_K(\log \mathcal{I}) : \delta(u) \in u\mathcal{A}, \forall u \in \mathcal{I}\}$.

$$\begin{array}{cccc} \hat{\varphi} : \widehat{Der_K(\log \mathcal{I})} & \to & \Omega_K(\log \mathcal{I})^{\vee} \\ \delta & \mapsto & \hat{\tilde{\delta}} \end{array}$$

is an homomorphism of $\mathcal{A}$-modules.

3. Logarithmic Poisson Structures

In this section, we introduce the notion of logarithmic Poisson structure and give some of it properties.

3.1 Definition and First Properties

Firstly, we recall that a Poisson structure on an algebra $\mathcal{A}$ is a skew-symmetric K-bilinear map on $\mathcal{A}$ that satisfy the Leibnitz role and Jacobi identity.

Definition 3.1. *A logarithmic Poisson structure along a non zero ideal $\mathcal{I}$ of $\mathcal{A}$ is a skew-symmetric k-bilinear map $\{-, -\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ that is bi-derivative, satisfy the Jacobi identity and such that $\{a, u\} \in u\mathcal{A}$ for all $a \in \mathcal{A}$, $u \in \mathcal{I}$.*

It follows from this definition that the image of associated adjoint map

$ad : \mathcal{A} \to Der_K(\mathcal{A})$ is a submodule of $Der_K(\log \mathcal{I})$. Indeed, for all $a \in \mathcal{A}$, $ad(a) := \{a, -\}$ and for all $u\mathcal{I}$, $\{a, u\} \in u\mathcal{A} \subset \mathcal{I}$. Then $ad(a)(\mathcal{I}) \subset \mathcal{I}$ and $ad(\mathcal{A})$. This end the proof of the following proposition.

Proposition 3.2. *Let $\{-, -\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be a logarithmic Poisson structure along $\mathcal{I}$. Then For all $a \in \mathcal{A}$, $ad(a) \in Der_K(\log \mathcal{I})$.*

We deduce the following corollaries.

Corollary 3.3. *Let $\{-,-\} : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ be a logarithmic Poisson structure along $\mathcal{I}$. $ad : \mathcal{A} \to Der_K(\log\mathcal{I})$ is homomorphism of Lie algebras and a derivation with values in the $\mathcal{A}$-module $Der_K(\log\mathcal{I})$.*

From the universal property of $\Omega_K(\mathcal{A})$, we deduce

Corollary 3.4. *For each logarithmic Poisson structure $\{-,-\} : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, there exist a unique $\mathcal{A}$-module homomorphism $H : \Omega_K(\mathcal{A}) \to Der_K(\log\mathcal{I})$ such that $H \circ d = ad$.*

Proof. Since $ad \in Der_k(\mathcal{A}, Der_K(\log\mathcal{I}))$, from universal property of $(\Omega_K(\mathcal{A}), d)$, there exist $H \in Hom_{\mathcal{A}}(\Omega_K(\mathcal{A}), Der_K(\mathcal{A}))$ such that $H \circ d = ad$.
For all $x \in \Omega_K(\mathcal{A})$, $\quad x = \sum_{i=1}^n x_i d(a_i)$.
Then $H(x) = \sum_{i=1}^n x_i H(d(a_i)) = \sum_{i=1}^n x_i H\circ d(a_i) = \sum_{i=1}^n x_i ad(a_i) \in Der_K(\log\mathcal{I})$. Therefore, $H(\Omega_K(\mathcal{A})) \subset Der_K(\log\mathcal{I})$. □

3.2 Logarithmic Poisson 2-form.

Proposition 3.5. *Each logarithmic Poisson structure along $\mathcal{I}$ induce an $\mathcal{A}$-module homomorphism $\bar{H}$ from $\Omega_K(\log\mathcal{I})$ to $Der_K(\log\mathcal{I})$, defined by*

$$\bar{H}(\tfrac{d(u)}{u}) = \tfrac{1}{u}H(d(u))$$

Proof.

For all $u \in \mathcal{I}$, $\bar{H}(\frac{d(u)}{u}) = \frac{1}{u}H(d(u)) = \frac{1}{u}\{u,-\} \in Der_K(\log\mathcal{I})$, since for all $a \in \mathcal{A}$, there exist $b \in \mathcal{A}$ such that $\{u, a\} = ub$. We extended $\bar{H}$ on $\Omega_K(\log\mathcal{I})$ by linearity. □

It follows from this proposition and definition of $\{-,-\}$ that $\bar{H}(\frac{d(u)}{u}) \in \widehat{Der_K(\log\mathcal{I})}$. Then, from lemma 2.4 $\hat{\varphi}\circ\bar{H}$ is a homomorphism from $\Omega_K(\log\mathcal{I})$ to $\Omega_K(\log\mathcal{I})^\vee$. We have the following lemma.

Lemma 3.6. *Each logarithmic Poisson structure induce a homomorphism of $\mathcal{A}$-modules*

$$\begin{array}{rcl} \Phi : \Omega_K(\log\mathcal{I}) & \to & \Omega_K(\log\mathcal{I})^\vee \\ \omega & \mapsto & \hat{\varphi}\circ\bar{H}(\omega) \end{array}$$

We deduce that

Proposition 3.7. *Each logarithmic Poisson structure along $\mathcal{I}$ induce a 2-form ω_0 on $\Omega_K(\log\mathcal{I})$.*

Proof. For all $x, y \in \Omega_K(\log\mathcal{I})$, $\omega_0 := [\Phi(x)]y$ □

The following proposition will be very useful after, since it shows that $\omega_0 \in Alt(\Omega_K(\log\mathcal{I}), \mathcal{A})$.

Proposition 3.8. *ω_0 is skew-symmetric.*

Proof. Let $x \in \Omega_K(\log\mathcal{I})$, $x = \sum_1^p x_i \dfrac{d(a_i)}{a_i} + \sum_{p+1}^n x_i d(a_i)$ We have,

$$\begin{array}{rcl}
[\Phi(x)](x) & = & [\sum\limits_1^p \dfrac{x_i}{a_i}[\hat{\varphi}\circ\bar{H}\circ d](a_i) + \sum\limits_{p+1}^n x_i[\hat{\varphi}\circ\bar{H}\circ d](a_i)](x) \\
& = & \sum\limits_1^p \dfrac{x_i}{a_i}[\hat{\varphi}\circ\bar{H}\circ d](a_i)[\sum\limits_{j=1}^p x_j \dfrac{d_{A/K}(a_j)}{a_j} + \sum\limits_{j=p+1}^n x_j d] + \\
& + & \sum\limits_{p+1}^n x_i[\hat{\varphi}\circ\bar{H}\circ d](a_i)[\sum\limits_{j=1}^p x_j \dfrac{d_{A/K}(a_j)}{a_j} + \sum\limits_{i,j=p+1}^n x_i d(a_j)] \\
& = & \sum\limits_{i,j=1}^p \dfrac{x_i x_j}{a_i a_j}\hat{\varphi}[\bar{H}\circ d(a_i)]\circ d(a_j) + \\
& + & \sum\limits_{i,j=p+1}^n \dfrac{x_i x_j}{a_i}\hat{\varphi}[\bar{H}\circ d(a_i)]\circ d(a_j) + \\
& + & \sum\limits_{i,j=p+1}^n \dfrac{x_i x_j}{a_j}\hat{\varphi}[\bar{H}\circ d(a_i)]\circ d(a_j) + \\
& + & \sum\limits_{i,j=p+1}^n x_i x_j \hat{\varphi}[\bar{H}\circ d(a_i)]\circ d(a_j) \\
& = & \sum\limits_{i,j=1}^p \dfrac{x_i x_j}{a_i a_j}\{a_i; a_j\} + \sum\limits_{1\le i\le p, p+1\le j\le n}^n \dfrac{x_i x_j}{a_i}\{a_i; a_j\} \\
& + & \sum\limits_{1\le j\le p, p+1\le i\le n}^n \dfrac{x_i x_j}{a_j}\{a_i; a_j\} + \sum\limits_{i,j=p+1}^n x_i x_j\{a_i; a_j\} = 0
\end{array}$$

□

Definition 3.9. *ω_0 is called Poisson 2-form logarithmic along $\mathcal{I}$.*

Since $\Omega_K(\mathcal{A}) \subset \Omega_K(\log \mathcal{I})$, the restriction $\Phi_{1_{\Omega_K(\mathcal{A})}} : \Omega_K(\mathcal{A}) \to \Omega_K(\mathcal{A})^{\vee}$ induce a skew-symmetric 2-form $\tilde{\omega}_0$. $\tilde{\omega}_0[(d(u))](d(v)) = \{u, v\}$. Indeed,

$$\begin{aligned} \tilde{\omega}_0(d(u), d(v)) &= [\Phi(d(u))](d(v)) \\ &= \hat{\varphi}(\bar{H}(d(u)))(d(v)) \\ &= [\hat{\varphi}(\{u, -\})](d(v)) \\ &= \hat{\hat{ad}}(u)[d(v)] \\ &= \{u, v\} \end{aligned}$$

We remark that $\tilde{\omega}_0$ is equal to the 2-form $\pi_{\{,\}}$ defined in (Huebschmann, J. (2013)) for an arbitrary Poisson structure $\{-, -\}$ on $\mathcal{A}$. By a simple computation, we have

Theorem 3.10. *For all $u, v \in \mathcal{I}^*$ and $a, b \in \mathcal{A}$ we have the following:*

1. $\omega_0(a\frac{d(u)}{u}, b\frac{d(v)}{v}) = \frac{ab}{uv}\{u, v\}$
2. $\omega_0(ad(u), b\frac{d(v)}{v}) = \frac{ab}{v}\{u, v\}$
3. $\omega_0(ad(u), bd(v)) = ab\{u, v\}$

From this theorem, it follow that the restriction of ω_0 on $\Omega_K(\mathcal{A}) \times \Omega_K(\mathcal{A})$ is equal to $\tilde{\omega}_0$.

4. Complex of Logarithmic Differential Form

In this section, we recall the notion of Lie-Rinehart algebra thanks to him we can define the complex of logarithmic differentials forms.

4.1 Lie-Rinehart Algebra and Logarithmic Lie-Rinehart Algebra

We conserve the above notations that, a Lie algebra on k is a pair $(\mathfrak{g}; [-, -])$, where $\mathfrak{g}$ is a k-module and $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ k-bilinear skew symmetric map satisfy the Jacobi identity. A Lie-Rinehart algebra is a pair $(\mathfrak{g}, \rho)$ where $\mathfrak{g}$ is an $\mathcal{A}$-module and a k-Lie algebra, and $\rho : \mathfrak{g} :\to Der_K(\mathcal{A})$ is a morphism of $\mathcal{A}$-modules and k-Lie algebras, such that

$$[g, ag'] = [(\rho(g))(a)]g' + a[g, g'] \tag{2}$$

for all $g, g' \in \mathfrak{g}$ and all $a \in \mathcal{A}$, see Rinehart (1963). We can observe that the Lie-Rinehart algebra is the algebraic analogue of a Lie algebroide, it is also known as a Lie pseudo-algebra or a Lie-Cartant pair.

When $\mathfrak{g}$ is a subset of $Der_K(\mathcal{A})$ and $\rho : \mathfrak{g} :\to Der_K(\mathcal{A})$ is the inclusion map, the pair $(\mathfrak{g}, \rho)$ is a Lie-Rinehart algebra if and only if $\mathfrak{g}$ is closed under the $\mathcal{A}$-module and k-Lie algebra structures of $Der_K(\mathcal{A})$. We will be mainly interested in Lie-Rinehart algebras of this type. An example of this type of Lie-Rinehart algebra is $(Der_K(\log \mathcal{I}), i)$. Which will be referred to Lie-Rinehart structure the map $\rho : \mathfrak{g} \to Der_K(\mathcal{A})$ such that $(\mathfrak{g}, \rho)$ is Lie-Rinehart. Since $Der_K(\log \mathcal{I})$ is subset of $Der_K(\mathcal{A})$ closed under the $\mathcal{A}$-module and k-Lie algebra structures of $Der_K(\mathcal{A})$, it is possible for a giving Lie-Rinehart algebra $(\mathfrak{g}, \rho)$ to verify $\rho(\mathfrak{g}) \subset Der_K(\log \mathcal{I})$. in this case, we obtain a particular type of Li-Rinehart algebra. More generally, if $\mathcal{I}$ is an ideal of $\mathcal{A}$, we have the following definition.

Definition 4.1. *A logarithmic Lie-Rinehart algebra along $\mathcal{I}$ or shortly log-Lie-Rinehart algebra, is a pair $(\mathfrak{g}, \rho)$, where $\rho : \mathfrak{g} \to Der_K(\log \mathcal{I})$ is a morphism of $\mathcal{A}$-modules and k-Lie algebras, satisfying (2).*

It is clear that each log-Lie-Rinehart algebra is a Lie-Rinehart algebra.

In general we can replace in the definition of Lie-Rinehart structure $Der_K(\mathcal{A})$ by the $\mathcal{A}$-module of first order differentials operators on some $\mathcal{A}$-module M; see (Dongho, J.(2012)).

If $\mathfrak{g}$ is a Lie algebra with zero torsion, then each morphism of $\mathcal{A}$-modules $\rho : \mathfrak{g} \to Der_K(\log \mathcal{I})$ satisfy (2) is a Lie algebra homomorphism. Indeed, $(\rho[x, y] - [\rho(x), \rho(y)])(a).z = \rho[x, y](a).z - [\rho(x), \rho(y)](a).z =$
$= \rho[x, y](a).z - \rho(x)[\rho(y)(a)].z + \rho(y)[\rho(x)(a)].z$
$= [[x, y], az] - a[[x, y], z] - [x, \rho(y)(a).z] + \rho(y)(a)[x, z] + [y, \rho(x)(a)z] - \rho(x)(a)[y, z]$
$= [[x, y], az] - a[[x, y], z] - [x, [y, az]] + [x, a[y, z]] + [y, a[x, z]] - a[y, [x, z]] + [y, [x, az]] - [y, a[x, z]] - [x, a[y, z]] + a[x, [y, z]]$
$= -([az, [x, y]] + [x, [y, az]] + [y, [x, az]]) - a([[x, y], z] + [[y, z], x] + [[z, x], y]) = 0$ for all $z \in \mathfrak{g}$; then $(\rho[x, y] - [\rho(x), \rho(y)])(a)$ $= 0$ for all $a \in \mathcal{A}$. Therefore, $\rho[x, y] = [\rho(x), \rho(y)]$. This end the proof of the following proposition

Proposition 4.2. *Let $\mathfrak{g}$ be a Lie algebra such that $Ann(\mathfrak{g}) = 0$, the module of annulation of $\mathfrak{g}$. Then each morphism of $\mathcal{A}$-modules, $\rho : \mathfrak{g} \to Der_K(\log \mathcal{I})$ satisfying (2) is a logarithmic Lie-Rinehart structure.*

Let $(\mathfrak{g}, \rho)$ be a Lie-Rinehart algebra with associated Lie-bracket $[-,-]$. $\rho : \mathfrak{g} \to Der_K(\mathcal{A})$ defines a representation of $\mathfrak{g}$ by derivation on $\mathcal{A}$, in the case of log-Lie-Rinehart algebra, we have the representation by logarithmic derivation. Using the machinery of Chevalley-Eilenberg and Palais, we define a differential complex $(Alt^*(\mathfrak{g}, \mathcal{A}), d_\rho)$, where

$$\begin{aligned}(d_\rho f)(l_1, ..., l_{p+1}) = & \sum_{i=1}^{p+1} (-1)^{i+1} \rho(l_i)(f(l_1, ..., \hat{l}_i, ..., l_{p+1})) + \\ & \sum_{1 \le i < j \le j}^{p+1} (-1)^{i+j} f([l_i, x_j], l_1, ..., \hat{l}_i, ..., \hat{l}_j, ..., l_{p+1})\end{aligned} \tag{3}$$

The associated cohomology is denoted by $H^*(Alt(\mathfrak{g}, \mathcal{A}), \mathcal{A})$. This generalized the De Rham cohomology when $\mathcal{A}$ is the algebra of smooth functions on a smooth manifold. Indeed, the De Rham cohomology correspond to the case $\mathfrak{g} = Der_K(\mathcal{A})$. Similarly, the case where $\mathfrak{g} = Der_X(\log D)$, the O_X-module of vector field logarithmic along a reduced divisor D of a complex manifolds X, give the logarithmic of De Rham cohomology.

We can also think about the notion of logarithmic-Lie-Rinehart-Closed, which is homologue of notion of Lie-Rinehart-Poisson defined in (Huebschmann, J. (2013)). A logarithmic-Lie-Rinehart-closed structure on a logarithmic Lie-Rinehart algebra $(\mathfrak{g}, \rho, \mathcal{I})$ is a skew symmetric 2-form $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathcal{A}$ such that $d_\rho(u) = 0$. We remark that a logarithmic-Lie-Rinehart-closed structure is just a 2-cocycle of $(Alt^*(\mathfrak{g}, \mathcal{A}), d_\rho)$. We referring to Log-Lie-Rinehart-Closed (Log LRC) algebra a quadruplet $(\mathfrak{g}, \rho, \mathcal{I}, \mu)$ where μ is a logarithmic-Lie-Rinehart-Closed structure on $(\mathfrak{g}, \rho, \mathcal{I})$. A Log-LRC $(\mathfrak{g}, \rho, \mathcal{I}, \mu)$ is said to be symplectic if the map $\mathfrak{g} \to \mathfrak{g}^*; x \mapsto i_x \mu$ is an isomorphism.

One of the much important Lie-Rinehart-Closed-Symplectic algebra is the algebra O_X of holomorphic map on a logarithmic complex manifold of complex dimension $2n$ (X, D) with Lie-Rinehart-Closed structure a closed 2-form ω logarithmic along a reduced divisor D of X such that $\omega^n \neq 0 \in H^{2n}(X, \Omega_X^{2n}[D])$. In the field of symplectic geometry, such structure are called log symplectic structure and the underline complex manifold is called log symplectic manifold.

5. Logarithmic Poisson Cohomology

5.1 Logarithmic Lie Derivative

It is well known that, see (Braconnier, J. (1977)), the map $d : \mathcal{A} \to \Omega_K(\mathcal{A})$ is k-derivation with values in the $\mathcal{A}$-module $\Omega_K(\mathcal{A})$. Since $\Omega_K(\mathcal{A})$ is an $\mathcal{A}$-submodule of $\Omega_K(\log \mathcal{I})$, $d : \mathcal{A} \to \Omega_K(\log \mathcal{I})$ is element of $Der_k(\mathcal{A}, \Omega_K(\log \mathcal{I}))$.

We denote $\bigwedge_{\mathcal{A}}[\Omega_K(\log \mathcal{I})] = \bigoplus_{n \in \mathbb{N}} \bigwedge_{\mathcal{A}}^n [\Omega_K(\log \mathcal{I})]$ the exterior $\mathcal{A}$-algebra of the $\mathcal{A}$-module $\Omega_K(\log \mathcal{I})$. d prolonged to a derivation of degree $+1$, we also defined d;

$$d : \bigwedge_{\mathcal{A}} [\Omega_K(\log \mathcal{I})] \to \bigwedge_{\mathcal{A}} [\Omega_K(\log \mathcal{I})] \tag{4}$$

such that $(\bigwedge_{\mathcal{A}}[\Omega_K(\log \mathcal{I})], d)$ become a differential complex.
Let $\delta \in Der_K(\log \mathcal{I})$, for all $p \geq 1$, the map

$$\begin{aligned}\sigma_\delta : [\Omega_K(\log \mathcal{I})]^p &\to \bigwedge\nolimits_{\mathcal{A}}^p [\Omega_K(\log \mathcal{I})] \\ (x_1, ..., x_p) &\mapsto \sum_{i=1}^{p} (-1)^{i-1} [\hat{\varphi}(\delta)](x_i) x_1 \wedge x_2 ... \wedge \hat{x}_i \wedge ... \wedge x_p\end{aligned}$$

is $\mathcal{A}$-multilineal alternate. We denote

$$i_\delta : \bigwedge_{\mathcal{A}} [\Omega_K(\log \mathcal{I})] \to \bigwedge_{\mathcal{A}} [\Omega_K(\log \mathcal{I})] \tag{5}$$

the unique morphism such that
$i_\delta(x_1 \wedge x_2 \wedge ... \wedge x_p) = \sum_{i=1}^{p} (-1)^{i-1} [\hat{\varphi}(\delta)](x_i) x_1 \wedge x_2 ... \wedge \hat{x}_i \wedge ... \wedge x_p$
for all $p \geq 1$.
The map i_δ is a derivation of degree -1.
Therefore, $i_\delta \circ d + d \circ i_\delta : \bigwedge_{\mathcal{A}}[\Omega_K(\log \mathcal{I})] \to \bigwedge_{\mathcal{A}}[\Omega_K(\log \mathcal{I})]$ is a derivation of degree zero. We denote $\mathcal{L}_\delta = i_\delta \circ d + d \circ i_\delta$; to be the logarithmic Lie derivation with respect to δ.
$\mathcal{L}_\delta$ have the following properties.

Proposition 5.1. *For all $x \in \Omega_K(\log \mathcal{I})$, $a \in \mathcal{A}$*

a) $\mathcal{L}_\delta(ax) = \delta(a)x + a\mathcal{L}_\delta(x)$

b) $\mathcal{L}_{a\delta}(x) = a\mathcal{L}_\delta(x) + \hat{\hat{\delta}}(x)d(a)$

c) $\mathcal{L}_\delta(d)(a)) = d(\delta(a))$

Proof. This is straightforward and left to the reader. □

With those properties, we can describe $\mathcal{L}_\delta$ on the generators of $\Omega_K(\log \mathcal{I})$.

Firstly, we shall mark that for all $\frac{x}{a} \in \Im^{-1}\Omega_K(\mathcal{A})$, $\mathcal{L}_\delta(\frac{x}{a}) = \frac{1}{a}\mathcal{L}_\delta(x) - \frac{\delta(a)}{a}\frac{x}{a}$. Indeed, for all $x \in \Omega_K(\mathcal{A}), a \in \mathcal{I}^*, \mathcal{L}_\delta(x) = \mathcal{L}_\delta(a\frac{x}{a}) = a\mathcal{L}_\delta(\frac{x}{a}) + \delta(a)\frac{x}{a}$. Then $\mathcal{L}_\delta(\frac{x}{a}) = \frac{1}{a}\mathcal{L}_\delta(x) - \frac{\delta(a)}{a}\frac{x}{a}$. This end the proof of the following lemma

Lemma 5.2. *For all* $\frac{x}{a} \in \Im^{-1}\Omega_K(A), \delta \in Der_K(\log \mathcal{I}), \mathcal{L}_\delta(\frac{x}{a}) = \frac{1}{a}\mathcal{L}_\delta(x) - \frac{\delta(a)}{a}\frac{x}{a}$

Immediately, we deduce the following corollary

Corollary 5.3. *for all* $a \in \mathcal{I}^*$, $\delta \in Der_K(\log \mathcal{I})$ $\mathcal{L}_\delta(\frac{d(a)}{a}) = d(\frac{\delta(a)}{a})$

Proof. .

$$\begin{aligned}\mathcal{L}_\delta(\frac{d(a)}{a}) &= \frac{1}{a}\mathcal{L}_\delta(d(a)) - \frac{\delta(a)}{a}\frac{d(a)}{a}\\ &= \frac{1}{a}d(\delta(a)) - \frac{\delta(a)}{a}\frac{d(a)}{a}\end{aligned}$$

Since $\delta \in \widehat{Der_K(\log \mathcal{I})}$, there exist $c \in \mathcal{A}^*$ such that $\delta(a) = ac$.
Therefore,

$$\begin{aligned}\frac{d\delta(a)}{a} &= d(c) + \frac{d(a)}{a}\\ &= d(\frac{\delta(a)}{a}) + \frac{\delta(a)}{a}\frac{d(a)}{a}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_\delta(\frac{d(a)}{a}) &= d(\frac{\delta(a)}{a}) + \frac{\delta(a)}{a}\frac{d(a)}{a} - \frac{\delta(a)}{a}\frac{d(a)}{a}\\ &= d(\frac{\delta(a)}{a})\end{aligned}$$

□

We have shown in subsection 3.2 that each logarithmic Poisson structure induce a map $\overline{H} : \Omega_K(\log \mathcal{I}) \to \widehat{Der_K(\log \mathcal{I})}$. So, we can compute $\mathcal{L}_{\overline{H}(\frac{d(a)}{a})}(\frac{d(b)}{b})$. The following corollary give compute it.

Corollary 5.4. *For all logarithmic Poisson structure along* $\mathcal{I}$,
$\mathcal{L}_{\bar{H}(\frac{d(a)}{a})}(\frac{d(b)}{b}) = d(\frac{1}{ab}\{a, b\})$ *for all* $a, b \in \mathcal{I}^*$

Proof. $\bar{H}(\frac{d(a)}{a}) = \frac{1}{a}H \circ d(a) = \frac{1}{a}\{a, -\} =: \varphi$
then

$$\begin{aligned}\mathcal{L}_{\bar{H}(\frac{d(a)}{a})}(\frac{d(b)}{b}) &= \mathcal{L}_\varphi\frac{d(b)}{b}\\ &= \mathcal{L}_\varphi(\frac{\varphi(b)}{b})\\ &= d(\frac{1}{ab}\{a, b\})\end{aligned}$$

□

The following proposition explicite $\mathcal{L}$ on generators of $\Omega_K(\log \mathcal{I})$.

Proposition 5.5. *For all $a \in \mathcal{A}, u, v \in \mathcal{I}$*

1. $\mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(\frac{d(v)}{v}) = ad(\frac{1}{uv}\{u,v\}) + \frac{1}{uv}\{u,v\}d(a)$

2. $\mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(b\frac{d_{\mathcal{A}/K}(v)}{v}) = \frac{1}{u}\{u,b\}\frac{d(v)}{v} + \frac{b}{uv}\{u,v\}d(a) + bad(\frac{1}{uv}\{u,v\})$

3. $\mathcal{L}_{\bar{H}(b\frac{d(v)}{v})}(a\frac{d_{\mathcal{A}/K}(u)}{u}) = \frac{b}{v}\{v,a\}\frac{d(u)}{u} + \frac{a}{uv}\{v,u\}d(b) + abd(\frac{1}{uv}\{v,u\})$

4. $d(\omega(a\frac{d(u)}{u}, b\frac{d_{\mathcal{A}/K}(v)}{v})) = abd(\frac{1}{uv}\{u,v\}) + \frac{b}{uv}\{u,v\}d(a) + \frac{a}{uv}\{u,v\}d(b)$

Proof. Let $a, b \in A; u, v \in \mathcal{I}^*$

1. We have:

$$\begin{aligned}
&\mathcal{L}_{\bar{H}[a\frac{d(u)}{u}]}(\frac{d(v)}{v}) \\
&= a\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(\frac{d(v)}{v}) + [\bar{H}(\widetilde{\frac{\hat{d}(u)}{u}})]\left(\frac{d(v)}{v}\right)d(a) \\
&= a\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(\frac{d(v)}{v}) + \frac{1}{u}\widetilde{H \circ \hat{d}(u)}(\frac{d(v)}{v})d(a) \\
&= a\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(\frac{d(v)}{v}) + \frac{1}{u}\widetilde{\{u,\hat{-}\}}(\frac{d(v)}{v})d(a) \\
&= ad(\tfrac{1}{uv}\{u,v\}) + \frac{1}{uv}\{u,v\}d(a)
\end{aligned}$$

2. From proposition 5.1 we deduce that;

$$\begin{aligned}
&\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(b\frac{d(v)}{v}) = [\bar{H}(\frac{d(u)}{u})](b)\frac{d(u)}{u} + \mathcal{L}_{\bar{H}[\frac{d(u)}{v}]}(\frac{d(v)}{v}) \\
&= \frac{1}{u}\{u,b\}\frac{d(v)}{v} + bd(\frac{1}{uv}\{u,v\})
\end{aligned}$$

therefore,

$$\begin{aligned}
&\mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(b\frac{d(v)}{v}) = a\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(b\frac{d(v)}{v}) + \bar{H}(\widetilde{\frac{\hat{d}(u)}{u}})\left(b\frac{d(v)}{v}\right)d(a) \\
&= \frac{a}{u}\{u,b\}\frac{d(v)}{v} + \frac{b}{uv}\{u,v\}d(a)abd(\frac{1}{uv}\{u,v\})
\end{aligned}$$

3. Changing the role of u and v, we obtain:

$$\mathcal{L}_{\bar{H}(b\frac{d(v)}{v})}(a\frac{d_{\mathcal{A}/K}(u)}{u}) = \frac{b}{v}\{v,a\}\frac{d(u)}{u} + \frac{a}{uv}\{v,u\}d(b) + abd(\frac{1}{uv}\{v,u\})$$

4. Since $\omega_0(x,y) := [\Phi(x)]y$ for all $x, y \in \Omega_K(\log \mathcal{I})$
$\omega(a\dfrac{d(u)}{u}, b\dfrac{d(v)}{v}) = \dfrac{ab}{uv}\{u,v\}$ and then
$$\begin{aligned} & d\omega(a\frac{d(u)}{u}, b\frac{d(v)}{v}) \\ &= d[\frac{ab}{uv}\{u,v\}] \\ &= abd[\frac{1}{uv}\{u,v\}] + d(ab).(\frac{1}{uv}\{u,v\}) \\ &= abd[\frac{1}{uv}\{u,v\}] + bd(a).(\frac{1}{uv}\{u,v\}) + ad(b).(\frac{1}{uv}\{u,v\}) \end{aligned}$$

□

From this proposition, we deduce the following corollary

Corollary 5.6. *Let* $a \in \mathcal{A}, u, v \in \mathcal{I}^*$
$$-d\omega_0(a\frac{d(u)}{u}, b\frac{d(v)}{v}) + \mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(b\frac{d(v)}{v}) - \mathcal{L}_{\bar{H}[b\frac{d(u)}{u}]}(a\frac{d(v)}{v}) = \frac{a}{u}\{u,b\}\frac{d(v)}{v} + \frac{b}{v}\{a,v\}\frac{d(u)}{u} + abd(\frac{1}{uv}\{u,v\}$$

Proof. This is obvious from proposition 5.1 □

Corollary 5.7. $-d\omega_0(ad(u), bd(v)) + \mathcal{L}_{\bar{H}(ad(u))}(bd(v)) - \mathcal{L}_{\bar{H}[bd(u)]}(ad(v)) = a\{u,b\}d(v) + b\{a,v\}d(u) + abd(\{u,v\})$

Proof. This coming from those equalities:

E.1 $d[\omega_0(ad(u), bd(v))] = a\{u,v\}d(b) + b\{u,v\}d(a) + abd[\{u,v\}]$.

E.2 $\mathcal{L}_{\bar{H}(ad(u))}(bd(v)) = abd[\{u,v\}] + a\{u,b\}d(v) + b\{u,v\}d(a)$

E.3 $\mathcal{L}_{\bar{H}(bd(v))}(ad(u)) = abd[\{v,u\}] + b\{v,a\}d(u) + a\{v,u\}d(b)$

□

We also have:

Corollary 5.8. $-d\omega(a\dfrac{d(u)}{u}, bd(v)) + \mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(bd(v)) - \mathcal{L}_{\bar{H}[bd(u)]}(a\dfrac{d(v)}{v}) = \dfrac{a}{u}\{u,b\}d(v) + b\{a,v\}\dfrac{d(u)}{u} + abd(\dfrac{1}{u}\{u,v\}$

Proof. For all $a, b \in \mathcal{A}, u, v \in \mathcal{I}^*$ we have:
$$d[\omega_0(a\frac{d(u)}{u}, bd(v))] = d[\frac{ab}{u}\{u,v\}] = [\frac{1}{u}\{u,v\}]d(ab) + abd[\frac{1}{u}\{u,v\}] = \frac{a}{u}\{u,v\}d(b) + \frac{b}{u}\{u,v\}d(a) + abd[\frac{1}{u}\{u,v\}]$$
$$\begin{aligned} \mathcal{L}_{\bar{H}[a\frac{d(u)}{u}]}(bd(v)) &= a\mathcal{L}_{\bar{H}[\frac{d(u)}{u}]}(bd(v)) + \bar{H}[\widetilde{\frac{d(u)}{u}}](bd(v))d(a) \\ &= a[b\mathcal{L}_{\bar{H}(\frac{d(u)}{u})}(d(v)) + \bar{H}(\frac{d(u)}{u})(b)d(v)] + \frac{b}{u}\{u,v\}d(a) \\ &= abd(\frac{1}{u}\{u,v\}) + \frac{a}{u}\{u,b\}d(v) + \frac{b}{u}\{u,v\}d(a) \\ \mathcal{L}_{\bar{H}[(bd(v))]}(a\frac{d(u)}{u}) &= b\mathcal{L}_{\bar{H}(d(v))}(a\frac{d(u)}{u}) + \widetilde{\bar{H}(d)(v)}(a\frac{d(u)}{u})d(b) \\ &= b[a\mathcal{L}_{\bar{H}(d(v))}(\frac{d(u)}{u}) + \bar{H}(d(v))(a)\frac{d(u)}{u}] + \frac{a}{u}\{v,u\}d(b) \\ &= bad(\frac{1}{u}\{v,u\}) + b\{v,a\}\frac{d(u)}{u} + \frac{a}{u}\{v,u\}d(b) \end{aligned}$$
It follow that; $-d\omega_0(a\dfrac{d(u)}{u}, bd(v)) + \mathcal{L}_{\bar{H}(a\frac{d(u)}{u})}(bd(v)) - \mathcal{L}_{\bar{H}[bd(u)]}(a\dfrac{d(v)}{v}) = \dfrac{a}{u}\{u,b\}d(v) + b\{a,v\}\dfrac{d(u)}{u} + abd\left(\dfrac{1}{u}\{u,v\}\right)$ □

Theorem 5.9. *Let $\{-,-\}$ be a logarithmic Poisson structure along $\mathcal{I}$. The map $[-,-] : (x,y) \mapsto [x,y] = -d(\omega(x,y)) + \mathcal{L}_{\bar{H}(x)}(y) - \mathcal{L}_{\bar{H}(y)}(x)$ is a Lie structure on $\Omega_K(\log \mathcal{I})$.*

We need the following lemmas to prove the above theorem.

Lemma 5.10. *Let $\{-,-\}$ be a Poisson structure on $\mathcal{I}$. There exist an unique $\{-,-\}$ on the quotient field $F_{\mathcal{A}}$ of $\mathcal{A}$; also denoted $\{-,-\}$ and satisfying $\{\frac{a}{b}, c\} = \frac{1}{b}\{a,c\} - \frac{a}{b^2}\{b,c\}$*

Lemma 5.11. *$[-,-]$ is k-bilinear and skew symmetric.*

Proof. It is straightforward computation. □

It is remaining the Jacobian identity to end the proof of the theorem. In this goal, we state the above lemma.

Lemma 5.12. *Let $a, b \in \mathcal{A}, u, v \in \mathcal{I}^*$*

1. $\left[a\frac{du}{u}, b\frac{dv}{v}\right] = \frac{a}{u}\{u,b\}\frac{dv}{v} + \frac{b}{v}\{a,v\}\frac{du}{u} + abd(\frac{1}{uv}\{u,v\})$
2. $\left[a\frac{du}{u}, bdv\right] = \frac{a}{u}\{u,b\}dv + b\{a,v\}\frac{du}{u} + abd(\frac{1}{u}\{u,v\})$
3. $[adu, bdv] = a\{u,b\}dv + b\{a,v\}du + abd(\{u,v\})$
4. $\left[adu, b\frac{dv}{v}\right] = a\{u,b\}\frac{du}{u} + \frac{b}{v}\{a,v\}du + abd(\frac{1}{v}\{u,v\})$

Since $\Omega_K(\mathcal{A})$ is $\mathcal{A}$-submodule of $\Omega_K(\log \mathcal{I})$ we can consider the restriction of $[-,-]$ on $\Omega_K(\mathcal{A})$. On the other hand, as a Poisson structure in the general meaning, $\{-,-\}$ induce on $\Omega_K(\mathcal{A})$ a Lie structure. In Theorem 3.8 of (Huebschmann, J. (2013)) it is prove that this bracket is defined by $[adu, bdv] = a\{u,b\}dv + b\{a,v\}du + abd(\{u,v\})$ which is equal to relation 3) of 5.12. We can then conclud that the two brackets coincide on $\Omega_K(\mathcal{A})$.

In the particular case where $a = b = 1$, we obtain the following.

Corollary 5.13. *For all $u, v \in \mathcal{I}^*$*

1. $\left[\frac{du}{u}, \frac{dv}{v}\right] = d(\frac{1}{uv}\{u,v\})$
2. $\left[du, \frac{dv}{v}\right] = d(\frac{1}{v}\{u,v\})$
3. $\left[\frac{du}{u}, dv\right] = d(\frac{1}{u}\{u,v\})$
4. $[du, dv] = d(\{u,v\})$

The following lemma prove the Theorem on the subset of $\Omega_K(\log \mathcal{I})$ generated by element of the form $\frac{du}{u}, u\mathcal{I}$.

Lemma 5.14. *For all $u, v, w \in \mathcal{I}^*$, on a:*
$$\left[\left[\frac{du}{u}, \frac{dv}{v}\right], \frac{dw}{w}\right] + \left[\left[\frac{dv}{v}, \frac{dw}{w}\right], \frac{du}{u}\right] + \left[\left[\frac{dw}{w}, \frac{du}{u}\right], \frac{dv}{v}\right] = 0$$

Proof. Let u, v, w as in the lemma.
we have show in 5.13 that $\left[\frac{du}{u}, \frac{dv}{v}\right] = d(\frac{1}{uv}\{u,v\})$; and $\left[du, \frac{du}{u}\right] = d(\frac{1}{v}\{u,v\})$.
Then $\left[\left[\frac{du}{u}, \frac{dv}{v}\right], \frac{dw}{w}\right] = \left[d(\frac{1}{uv}\{u,v\}), \frac{dw}{w}\right] = d(\frac{1}{w}\{\frac{1}{uv}\{u,v\}, w\})$
Applying lemma [5.10], we obtain:

$$\begin{aligned}\frac{1}{w}\{\frac{1}{uv}\{u,v\},w\} &= \frac{1}{\omega}(\frac{1}{uv}\{\{u,v\},w\}-\frac{1}{u^2v^2}\{u,v\}\{uv,w\})\\ &= \frac{1}{uvw}\{\{u,v\},w\}-\frac{1}{wu^2v}\{u,v\}\{u,w\}-\frac{1}{wuv^2}\{u,v\}\{v,w\}\end{aligned}$$

In the same manner, we have

$$\begin{aligned}\frac{1}{u}\{\frac{1}{wv}\{v,w\},u\} &= \frac{1}{uvw}\{\{v,w\},u\}-\frac{1}{wuv^2}\{v,w\}\{v,u\}-\frac{1}{w^2uv}\{v,w\}\{w,u\}\\ \frac{1}{v}\{\frac{1}{uw}\{w,u\},v\} &= \frac{1}{uvw}\{\{w,u\},v\}-\frac{1}{w^2uv}\{w,u\}\{w,v\}-\frac{1}{wu^2v}\{w,u\}\{u,v\}\end{aligned}$$

Therefore, the Jacobian identity of $\{-,-\}$ imply

$$\left[\left[\frac{du}{u},\frac{dv}{v}\right],\frac{dw}{w}\right]+\left[\left[\frac{dv}{v},\frac{dw}{w}\right],\frac{du}{u}\right]+\left[\left[\frac{dw}{w},\frac{du}{u}\right],\frac{dv}{v}\right]=\frac{1}{uvw}\{\{u,v\},w\}-\frac{1}{wu^2v}\{u,v\}\{u,w\}-\frac{1}{wuv^2}\{u,v\}\{v,w\}+\frac{1}{uvw}\{\{v,w\},u\}-\frac{1}{wuv^2}\{v,w\}\{v,u\}-\frac{1}{w^2uv}\{v,w\}\{w,u\}+\frac{1}{uvw}\{\{w,u\},v\}-\frac{1}{w^2uv}\{w,u\}\{w,v\}-\frac{1}{wu^2v}\{w,u\}\{u,v\}=0$$ □

we have also the following.

Lemma 5.15. *For all $u,v,w\in\mathcal{I}^*$*

$$\left[\left[\frac{du}{u},\frac{dv}{v}\right],dw\right]+\left[\left[\frac{dv}{v},dw\right],\frac{du}{u}\right]+\left[\left[dw,\frac{du}{u}\right],\frac{dv}{v}\right]=0$$

Proof. Let $u,v,w\in\mathcal{I}^*$.

$$\left[\left[\frac{du}{u},\frac{dv}{v}\right],dw\right]=\left[d(\frac{1}{uv}\{u,v\}),dw\right]=d\left(\{\frac{1}{uv}\{u,v\},w\}\right)$$

yet $\left\{\frac{1}{uv}\{u,v\},w\right\}=\frac{1}{uv}\{\{u,v\},w\}-\frac{1}{uv^2}\{u,v\}\{v,w\}-\frac{1}{vu^2}\{u,v\}\{u,w\}$

then $\left[\left[\frac{du}{u},\frac{dv}{v}\right],dw\right]=d\left(\frac{1}{uv}\{\{u,v\},w\}-\frac{1}{uv^2}\{u,v\}\{v,w\}-\frac{1}{vu^2}\{u,v\}\{u,w\}\right)$

similarly,

$$\left[\left[\frac{dv}{v},dw\right],\frac{du}{u}\right]=\left[d\left(\frac{1}{v}\{v,w\}\right),\frac{du}{u}\right]=d\left(\frac{1}{u}\left\{\frac{1}{v}\{v,w\},u\right\}\right)$$

yet $\frac{1}{u}\left\{\frac{1}{v}\{v,w\},u\right\}=\frac{1}{u}\left(\frac{1}{v}\{\{v,w\},u\}-\frac{1}{v^2}\{v,w\}\{v,u\}\right)=\frac{1}{uv}\{\{v,w\},u\}-\frac{1}{uv^2}\{v,w\}\{v,u\}$

then $\left[\left[\frac{dv}{v},dw\right],\frac{du}{u}\right]=d\left(\frac{1}{uv}\{\{v,w\},u\}-\frac{1}{uv^2}\{v,w\}\{v,u\}\right)$

and then,

$$\left[\left[dw,\frac{du}{u}\right],\frac{dv}{v}\right]=\left[d\left(\frac{1}{u}\{w,u\}\right),\frac{dv}{v}\right]=d\left(\frac{1}{v}\left\{\frac{1}{u}\{w,u\},v\right\}\right).$$

yet $\frac{1}{v}\left\{\frac{1}{u}\{w,u\},v\right\}=\frac{1}{vu}\{\{w,u\},v\}-\frac{1}{vu^2}\{w,u\}\{u,v\}$

then

$$\left[\left[dw,\frac{du}{u}\right],\frac{dv}{v}\right]=d\left(\frac{1}{vu}\{\{w,u\},v\}-\frac{1}{vu^2}\{w,u\}\{u,v\}\right).$$

consequently, Jacobian identity implies,

$$\left[\left[\frac{du}{u},\frac{dv}{v}\right],dw\right]+\left[\left[\frac{dv}{v},dw\right],\frac{du}{u}\right]+\left[\left[dw,\frac{du}{u}\right],\frac{dv}{v}\right]=0$$ □

Similarly, we prove that

E4 $\left[\left[\frac{du}{u},dv\right],dw\right]+\left[[dv,dw],\frac{du}{u}\right]+\left[\left[dw,\frac{du}{u}\right],dv\right]=0$

Using those lemmas, we can prove the following proposition that is a part of Theorem 5.9.

Proposition 5.16. *Let $\omega_1=a_1\frac{du_1}{u_1}+b_1dv_1$, $\omega_2=a_2\frac{du_2}{u_2}+b_2dv_2$ and $\omega_3=a_3\frac{du_3}{u_3}+b_3dv_3$ be elements of $\Omega_K(\log\mathcal{I})$ then,* $[[\omega_1,\omega_2],\omega_3]+[[\omega_2,\omega_3],\omega_1]+[[\omega_3,\omega_1],\omega_2]=0$

Proof. From the above lemmas, we have

$\left[a_1\frac{du_1}{u_1}, a_2\frac{du_2}{u_2}\right] = \frac{a_1}{u_1}\{u_1, a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{a_1, u_2\}\frac{du_1}{u_1} + a_1a_2d\left(\frac{1}{u_1, u_2}\{u_1, u_2\}\right)$ and then

$$\left[\left[a_1\frac{du_1}{u_1}, a_2\frac{du_2}{u_2}\right], a_3\frac{du_3}{u_3}\right] = \left[\frac{a_1}{u_1}\{u_1, a_2\}\frac{du_2}{u_2}, a_3\frac{du_3}{u_3}\right] + \left[\frac{a_2}{u_2}\{a_1, u_2\}\frac{du_1}{u_1}, a_3\frac{du_3}{u_3}\right] + \left[a_1a_2d\left(\frac{1}{u_1, u_2}\{u_1, u_2\}\right), a_3\frac{du_3}{u_3}\right]$$

$$\left[\frac{a_1}{u_1}\{u_1, a_2\}\frac{du_2}{u_2}, a_3\frac{du_3}{u_3}\right] = \frac{a_1}{u_1u_2}\{u_1, a_2\}\{u_2, a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{\frac{a_1}{u_1}\{u_1, a_2\}, u_3\}\frac{du_2}{u_2} + \frac{a_1a_3}{u_1}\{u_1, a_2\}d\left(\frac{1}{u_2u_3}\{u_2, u_3\}\right)$$

$$\left[\frac{a_2}{u_2}\{a_1, u_2\}\frac{du_1}{u_1}, a_3\frac{du_3}{u_3}\right] = \frac{a_2}{u_1u_2}\{a_1, u_2\}\{u_1, a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{\frac{a_2}{u_2}\{a_1, u_2\}, u_3\}\frac{du_1}{u_1} + \frac{a_2a_3}{u_2}\{a_1, u_2\}d\left(\frac{1}{u_3u_1}\{u_1, u_3\}\right)$$

$$\left[a_1a_2d(\frac{1}{u_1, u_2}\{u_1, u_2\}), a_3\frac{du_3}{u_3}\right] = \frac{a_3}{u_3}\{a_1a_2, u_3\}d(\frac{1}{u_1u_2}\{u_1, u_2\}) + a_1a_2\{\frac{1}{u_1u_2}\{u_1, u_2\}, a_3\}\frac{du_3}{u_3} + a_1a_2a_3d\left(\frac{1}{u_3}\{\frac{1}{u_1u_2}\{u_1, u_2\}, u_3\}\right)$$

therefore,

$$E_1 \quad : \left[\left[a_1\frac{du_1}{u_1}, a_2\frac{du_2}{u_2}\right], a_3\frac{du_3}{u_3}\right] + \circlearrowleft =$$
$$\frac{a_1}{u_1u_2}\{u_1, a_2\}\{u_2, a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{\frac{a_1}{u_1}\{u_1, a_2\}, u_3\}\frac{du_2}{u_2} + \frac{a_1a_3}{u_1}\{u_1, a_2\}d\left(\frac{1}{u_2u_3}\{u_2, u_3\}\right) + \frac{a_2}{u_1u_2}\{a_1, u_2\}\{u_1, a_3\}\frac{du_3}{u_3} +$$
$$\frac{a_3}{u_3}\{\frac{a_2}{u_2}\{a_1, u_2\}, u_3\}\frac{du_1}{u_1} + \frac{a_2a_3}{u_2}\{a_1, u_2\}d\left(\frac{1}{u_3u_1}\{u_1, u_3\}\right) + \frac{a_3}{u_3}\{a_1a_2, u_3\}d(\frac{1}{u_1u_2}\{u_1, u_2\}) + a_1a_2\{\frac{1}{u_1u_2}\{u_1, u_2\}, a_3\}\frac{du_3}{u_3} +$$
$$a_1a_2a_3d\left(\frac{1}{u_3}\{\frac{1}{u_1u_2}\{u_1, u_2\}, u_3\}\right) + \frac{a_2}{u_2u_3}\{u_2, a_3\}\{u_3, a_1\}\frac{du_1}{u_1} + \frac{a_1}{u_1}\{\frac{a_2}{u_2}\{u_2, a_3\}, u_1\}\frac{du_3}{u_3} + \frac{a_2a_1}{u_2}\{u_2, a_3\}d\left(\frac{1}{u_3u_1}\{u_3, u_1\}\right) +$$
$$\frac{a_3}{u_2u_3}\{a_2, u_3\}\{u_2, a_1\}\frac{du_1}{u_1} + \frac{a_1}{u_1}\{\frac{a_3}{u_3}\{a_2, u_3\}, u_1\}\frac{du_2}{u_2} + \frac{a_3a_1}{u_3}\{a_2, u_3\}d\left(\frac{1}{u_1u_2}\{u_2, u_1\}\right) + \frac{a_1}{u_1}\{a_2a_3, u_1\}d(\frac{1}{u_2u_3}\{u_2, u_3\}) +$$
$$a_2a_3\{\frac{1}{u_2u_3}\{u_2, u_3\}, a_1\}\frac{du_1}{u_1} + a_2a_3a_1d\left(\frac{1}{u_1}\{\frac{1}{u_2u_3}\{u_2, u_3\}, u_1\}\right) + \frac{a_3}{u_3u_1}\{u_3, a_1\}\{u_1, a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{\frac{a_3}{u_3}\{u_3, a_1\}, u_2\}\frac{du_1}{u_1} +$$
$$\frac{a_3a_2}{u_3}\{u_3, a_1\}d\left(\frac{1}{u_1u_2}\{u_1, u_2\}\right) + \frac{a_1}{u_3u_1}\{a_3, u_1\}\{u_3, a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{\frac{a_1}{u_1}\{a_3, u_1\}, u_2\}\frac{du_3}{u_3} + \frac{a_1a_2}{u_1}\{a_3, u_1\}d\left(\frac{1}{u_2u_3}\{u_3, u_2\}\right) +$$
$$\frac{a_2}{u_2}\{a_3a_1, u_2\}d(\frac{1}{u_3u_1}\{u_3, u_1\}) + a_3a_1\{\frac{1}{u_3u_1}\{u_3, u_1\}, a_2\}\frac{du_2}{u_2} + a_3a_1a_2d\left(\frac{1}{u_2}\{\frac{1}{u_3u_1}\{u_3, u_1\}, u_2\}\right)$$

To simplify this expression, we need the following properties of Lie's brackets

Lemma 5.17. *With the same hypothesis we have.:*

1. $d(\frac{1}{u_3}\{\frac{1}{u_1u_2}\{u_1, u_2\}, u_3\} + \frac{1}{u_1}\{\frac{1}{u_2u_3}\{u_2, u_3\}, u_1\} + \frac{1}{u_2}\{\frac{1}{u_3u_1}\{u_3, u_1\}, u_2\}) = 0$

2. $\frac{a_1}{u_1}\{\frac{a_2}{u_2}\{u_2, a_3\}, u_1\}\frac{du_3}{u_3} = \frac{a_1a_2}{u_1u_2}\{\{u_2, a_3\}, u_1\}\frac{du_3}{u_3} + \frac{a_1}{u_1u_2}\{a_2, u_1\}\{u_2, a_3\}\frac{du_3}{u_3} - \frac{a_1a_2}{u_1u_2^2}\{u_2, a_3\}\{u_2, u_1\}\frac{du_3}{u_3}$

3. $\frac{a_3}{u_3}\{\frac{a_1}{u_1}\{u_1, a_2\}, u_1\}\frac{du_2}{u_2} = \frac{a_3a_1}{u_1u_3}\{\{u_1, a_2\}, u_3\}\frac{du_2}{u_2} + \frac{a_3}{u_3u_1}\{a_1, u_3\}\{u_1, a_2\}\frac{du_2}{u_2} - \frac{a_3a_1}{u_3u_1^2}\{u_1, a_2\}\{u_1, u_3\}\frac{du_2}{u_2}$

4. $\frac{a_3}{u_3}\{\frac{a_2}{u_2}\{a_1,u_2\},u_3\}\frac{du_1}{u_1} = \frac{a_3a_2}{u_3u_2}\{\{a_1,u_2\},u_3\}\frac{du_1}{u_1} +$
$\frac{a_3}{u_3u_2}\{a_1,u_2\}\{a_2,u_3\}\frac{du_1}{u_1} - \frac{a_2a_3}{u_3u_2^2}\{a_1,u_2\}\{u_2,u_3\}\frac{du_1}{u_1}$

5. $\frac{a_1}{u_1}\{\frac{a_3}{u_3}\{a_2,u_3\},u_1\}\frac{du_2}{u_2} = \frac{a_1a_3}{u_1u_3}\{\{a_2,u_3\},u_1\}\frac{du_2}{u_2} +$
$\frac{a_1}{u_1u_3}\{a_2,u_3\}\{a_3,u_1\}\frac{du_2}{u_2} - \frac{a_1a_3}{u_1u_3^2}\{a_2,u_3\}\{u_3,u_1\}\frac{du_2}{u_2}$

6. $\frac{a_2}{u_2}\{\frac{a_3}{u_3}\{u_3,a_1\},u_2\}\frac{du_1}{u_1} = \frac{a_2a_3}{u_2u_3}\{\{u_3,a_1\},u_2\}\frac{du_1}{u_1} +$
$\frac{a_2}{u_2u_3}\{a_3,u_2\}\{u_3,a_1\}\frac{du_1}{u_1} - \frac{a_2a_3}{u_2u_3^2}\{u_3,a_1\}\{u_3,u_2\}\frac{du_1}{u_1}$

7. $\frac{a_2}{u_2}\{\frac{a_1}{u_1}\{a_3,u_1\},u_2\}\frac{du_3}{u_3} = \frac{a_2a_1}{u_2u_1}\{\{a_3,u_1\},u_2\}\frac{du_3}{u_3} +$
$\frac{a_2}{u_2u_1}\{a_3,u_1\}\{a_1,u_2\}\frac{du_3}{u_3} - \frac{a_2a_1}{u_2u_1^2}\{a_3,u_1\}\{u_1,u_2\}\frac{du_3}{u_3}$

8. $a_3a_1\{\frac{1}{u_3u_1}\{u_3,u_1\},a_2\}\frac{du_2}{u_2} = \frac{a_3a_1}{u_3u_1}\{\{,u_3,u_1\}a_2\}\frac{du_2}{u_2} -$
$\frac{a_3a_1}{u_1u_3^2}\{u_3,u_1\}\{u_3,a_2\}\frac{du_2}{u_2} - \frac{a_3a_1}{u_3u_1^2}\{u_3,u_1\}\{u_1,a_2\}\frac{du_2}{u_2}$

9. $a_2a_3\{\frac{1}{u_2u_3}\{u_2,u_3\},a_1\}\frac{du_1}{u_1} = \frac{a_2a_3}{u_2u_3}\{\{,u_2,u_3\},a_1\}\frac{du_1}{u_1} -$
$\frac{a_2a_3}{u_2u_3^2}\{u_2,u_3\}\{u_3,a_1\}\frac{du_1}{u_1} - \frac{a_2a_3}{u_3u_2^2}\{u_2,u_3\}\{u_2,a_1\}\frac{du_1}{u_1}$

10. $a_1a_3\{\frac{1}{u_1u_2}\{u_1,u_2\},a_3\}\frac{du_3}{u_3} = \frac{a_1a_2}{u_1u_2}\{\{u_1,u_2\},a_3\}\frac{du_3}{u_3} -$
$\frac{a_1a_2}{u_1u_2^2}\{u_1,u_2\}\{u_2,a_3\}\frac{du_3}{u_3} - \frac{a_1a_2}{u_2u_1^2}\{u_1,u_2\}\{u_1,a_3\}\frac{du_3}{u_3}$

Proof. Let $a_1;a_2;a_3;u_1;u_2;u_3$ as in the lemma.

1. $d(\frac{1}{u_3}\{\frac{1}{u_1u_2}\{u_1,u_2\},u_3\}+\frac{1}{u_1}\{\frac{1}{u_2u_3}\{u_2,u_3\},u_1\}+\frac{1}{u_2}\{\frac{1}{u_3u_1}\{u_3,u_1\},u_2\}) = d[\frac{1}{u_1u_2u_3}\{\{u_1,u_2\},u_3\}-\frac{1}{u_3u_1u_2^2}\{u_1,u_2\}\{u_2,u_3\}-$
$\frac{1}{u_3u_1^2u_2}\{u_1,u_2\}\{u_1,u_3\}+\frac{1}{u_1u_2u_3}\{\{u_2,u_3\},u_1\}-\frac{1}{u_3u_1u_2^2}\{u_2,u_3\}\{u_2,u_1\}-\frac{1}{u_3^2u_1u_2}\{u_2,u_3\}\{u_3,u_1\}+\frac{1}{u_1u_2u_3}\{\{u_3,u_1\},u_2\}-$
$\frac{1}{u_3u_1^2u_2}\{u_3,u_1\}\{u_1,u_2\} - \frac{1}{u_3^2u_1u_2}\{u_3,u_1\}\{u_3,u_2\}]$
This equal to zero since $\{-,-\}$ satisfies the Jacobian identity.

2. Applying Lemma [5.10], we obtain:
$\frac{a_1}{u_1}\{\frac{a_2}{u_2}\{u_2,a_3\},u_1\}\frac{du_3}{u_3}$
$= \frac{a_1}{u_1u_2}\{a_2\{u_2,a_3\},u_1\}\frac{du_3}{u_3} - \frac{a_1a_2}{u_1u_2^2}\{u_2,a_3\}\{u_2,u_1\}\frac{du_3}{u_3}$
$= \frac{a_1a_2}{u_1u_2}\{\{u_2,a_3\},u_1\}\frac{du_3}{u_3} + \frac{a_1}{u_1u_2}\{u_2,a_3\}\{a_2,u_1\}\frac{du_3}{u_3} - \frac{a_1a_2}{u_1u_2^2}\{u_2,a_3\}\{u_2,u_1\}\frac{du_3}{u_3}$ In the same manner, we prove the rest.

Therefore, from Jacobian identity and relation E_1 we have:
$\left[\left[a_1\frac{du_1}{u_1},a_2\frac{du_2}{u_2}\right],a_3\frac{du_3}{u_3}\right] + \circlearrowleft = 0$ □

We need the following

Lemma 5.18. *With the same hypothesis, we have:*

1. $\left[\left[a_2\frac{du_2}{u_2}, a_3\frac{du_3}{u_3}\right], b_1 dv_1\right] = \frac{a_2}{u_2u_3}\{u_2,a_3\}\{u_3,b_1\}dv_1 + b_1\{\frac{a_2}{u_2}\{u_2,a_3\},v_1\}\frac{du_3}{u_3} + \frac{a_2b_1}{u_2}\{u_2,a_3\}d(\frac{1}{u_3}\{u_3,v_1\}) + \frac{a_3}{u_3u_2}\{a_2,u_3\}\{u_2,b_1\}dv_1 + b_1\{\frac{a_3}{u_3}\{a_2,u_3\},v_1\}\frac{u_2}{u_2} + \frac{b_1a_3}{u_3}\{a_2,a_3\}d(\frac{1}{u_2}\{u_2,v_1\}) + a_2a_3\{\frac{1}{u_2u_3},b_1\}dv_1 + b_1\{a_2a_3,v_1\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) + a_2a_3b_1d(\{\frac{1}{u_2u_3}\{u_2,u_3\},v_1\})$

2. $\left[\left[a_3\frac{du_3}{u_3}, b_1dv_1\right], a_2\frac{du_2}{u_2}\right] = \frac{a_3}{u_3}\{u_3,b_1\}\{v_1,a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{\frac{a_3}{u_3}\{u_3,b_1\}u_2\}dv_1 + \frac{a_3a_2}{u_3}\{u_3,b_1\}d(\frac{1}{u_2}\{v_1,u_2\}) + \frac{b_1}{u_3}\{a_3,v_1\}\{u_3,a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{b_1\{a_3,v_1\},u_2\}\frac{du_3}{u_3} + b_1a_2\{a_3,v_1\}d(\frac{1}{u_3u_2}\{u_3,u_2\}) + a_3b_1\{\frac{1}{u_3}\{u_3,v_1\},a_2\}\frac{du_2}{u_2} + \frac{a_2}{u_2}\{a_3b_1,u_2\}d(\frac{1}{u_3}\{u_3,v_1\}) + a_3a_2b_1d(\frac{1}{u_2}\{\frac{1}{u_3}\{u_3,v_1\},u_2\})$

3. $\left[\left[b_1dv_1, a_2\frac{du_2}{u_2}\right], a_3\frac{du_3}{u_3}\right] = \frac{b_1}{u_2}\{v_1,a_2\}\{u_2,a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{b_1\{v_1,a_2\},u_3\}\frac{du_2}{u_2} + b_1a_3\{v_1,a_2\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) + \frac{a_2}{u_2}\{b_1,u_2\}\{v_1,a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{\frac{a_2}{u_2}\{b_1,u_2\},u_3\}dv_1 + \frac{a_2a_3}{u_2}\{b_1,u_2\}d(\frac{1}{u_3}\{v_1,u_3\}) + b_1a_2\{\frac{1}{u_2}\{v_1,u_2\},a_3\}\frac{du_3}{u_3} + \frac{a_3}{u_3}\{b_1a_2,u_3\}d(\frac{1}{u_2}\{v_1,u_2\}) + a_2a_3b_1d(\frac{1}{u_3}\{\frac{1}{u_2}\{v_1,u_2\},u_3\})$

Proof. It follow from lemma 5.10 and above properties of $[-,-]$. □

From Lemma 5.10 we have:

$$b_1\{\frac{a_2}{u_2}\{u_2,a_3\},v_1\} = \frac{b_1}{u_2}\{u_2,a_3\}\{a_2,v_1\} + b_1a_2\{\frac{1}{u_2}\{u_2,a_3\},v_1\}$$
$$\frac{b_1}{u_2}\{u_2,a_3\}\{a_2,v_1\} + \frac{b_1a_2}{u_2}\{\{u_2,a_3\},v_1\} - \frac{b_1a_2}{u_2^2}\{u_2,a_3\}\{u_2,v_1\}$$

$$\begin{aligned}\frac{a_2}{u_2}\{b_1\{a_3,v_1\},u_2\} &= \frac{a_2b_1}{u_2}\{\{a_3,v_1\},u_2\} + \frac{a_2}{u_2}\{a_3,v_1\}\{b_1,u_2\}\\ b_1a_2\{\frac{1}{u_2}\{v_1,u_2\},a_3\} &= \frac{b_1a_2}{u_2}\{\{v_1,u_2\},a_3\} - \frac{b_1a_2}{u_2^2}\{v_1,u_2\}\{u_2,a_3\}\end{aligned}$$

Applying Leibniz rule on the right hand side of the above equations, we obtain:

$$\frac{a_2}{u_2}\{a_3b_1,u_2\}d(\frac{1}{u_3}\{u_3,v_1\}) = \frac{a_2a_3}{u_2}\{b_1,u_2\}d(\frac{1}{u_3}\{u_3,v_1\}) + \frac{a_2b_1}{u_2}\{a_3,u_2\}d(\frac{1}{u_3}\{u_3,v_1\})$$

$$\begin{aligned}&b_1\{a_1a_3,v_1\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) + b_1a_2\{a_3,v_1\}d(\frac{1}{u_2u_3})\{u_3,u_2\} + b_1a_3\{v_1,a_2\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) =\\ &= b_1a_1\{a_3,v_1\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) + b_1a_3d(\frac{1}{u_2u_3}\{u_2,u_3\}) -\\ &-b_1a_2\{a_3,v_1\}d(\frac{1}{u_2u_3}\{u_2,u_3\}) - b_1a_3\{a_2,v_1\}d(\frac{1}{u_2u_3}\{u_2,u_3\})\\ &= 0\end{aligned}$$

The same,
$$b_1\{\frac{a_3}{u_3}\{a_2,u_3\},v_1\} = \frac{b_1}{u_3}\{a_2,u_3\}\{a_3,v_1\} + b_1a_3\{\frac{1}{u_3}\{a_2,u_3\},v_1\} = \frac{b_1}{u_3}\{a_2,u_3\}\{a_3,v_1\} + \frac{b_1a_3}{u_3}\{\{a_2,u_3\},v_1\} - \frac{b_1a_3}{u_3^2}\{a_2,u_3\}\{u_3,v_1\}$$
$$\frac{a_3}{u_3}\{b_1\{v_1,a_2\},u_3\} = \frac{a_3b_1}{u_3}\{\{v_1,a_2\},u_3\} + \frac{a_3}{u_3}\{v_1,a_2\}\{b_1,u_3\}$$
$$a_3b_1\{\frac{1}{u_3}\{u_3,v_1\},a_2\} = \frac{a_3b_1}{u_3}\{\{u_3,v_1\},a_2\} - \frac{a_3b_1}{u_3^2}\{u_3,v_1\}\{u_3,a_2\}$$
then

$b_1\{\frac{a_3}{u_3}\{a_2,u_3\},v_1\}+\frac{a_3}{u_3}\{b_1\{v_1,a_2\},u_3\}+a_3b_1\{\frac{1}{u_3}\{u_3,v_1\},a_2\}=$
$=\frac{a_3b_1}{u_3}(\{\{a_2,u_3\},v_1\}+\{\{v_1,a_2\},u_3\}+\{\{u_3,v_1\},a_2\})=0$
the same
$b_1\{\frac{a_2}{u_2}\{u_2,a_3\},v_1\}+\frac{a_2}{u_2}\{b_1\{a_3,v_1\},u_2\}+b_1a_2\{\frac{1}{u_2}\{v_1,u_2\},a_3\}=\frac{b_1a_2}{u_2}\{\{v_1,u_2\},a_3\}+\frac{a_2b_1}{u_2}\{\{a_3,v_1\},u_2\}+\frac{b_1a_2}{u_2}\{\{u_2,a_3\},v_1\}=0$

This end the prove of the following corollary

Corollary 5.19. $\left[\left[a_2\frac{du_2}{u_2},a_3\frac{du_3}{u_3}\right],b_1dv_1\right]+\circlearrowleft=0$

Using the same methods, we prove that

1. $\left[\left[a_1\frac{du_1}{u_1},b_2dv_2\right],a_3\frac{du_3}{u_3}\right]+\circlearrowleft=0$
2. $\left[[b_1dv_1,b_2dv_2],a_3\frac{du_3}{u_3}\right]+\circlearrowleft=0$
3. $\left[[b_1dv_1,b_2dv_2],a_2\frac{du_2}{u_2}\right]+\circlearrowleft=0$

we can end now the prove of proposition 5.16.

Indeed

$[[\omega_1,\omega_2],\omega_3]+[[\omega_2,\omega_3],\omega_1]+[[\omega_3,\omega_1],\omega_2]$
$=\left[\left[a_1\frac{du_1}{u_1},a_2\frac{du_2}{u_2}\right],a_3\frac{du_3}{u_3}\right]+\left[\left[b_1dv_1,a_2\frac{du_2}{u_2}\right],a_3\frac{du_3}{u_3}\right]+\left[\left[a_1\frac{du_1}{u_1},b_2dv_2\right],a_3\frac{du_3}{u_3}\right]+\left[[b_1du_1,b_2dv_2],a_3\frac{du_3}{u_3}\right]+$
$\left[\left[a_1\frac{du_1}{u_1},a_2\frac{du_2}{u_2}\right],b_3du_3\right]+[[b_1dv_1,b_2dv_2],b_3dv_3]+\left[\left[a_1\frac{du_1}{u_1},b_dv_2\right],a_3\frac{du_3}{u_3}\right]+\left[\left[b_1dv_1,a_2\frac{du_2}{u_2}\right],b_3dv_3\right]+$
$\left[\left[a_2\frac{du_2}{u_2},a_3\frac{du_3}{u_3}\right],a_1\frac{du_1}{u_1}\right]+\left[\left[b_2dv_2,a_3\frac{du_3}{u_3}\right],a_1\frac{du_1}{u_1}\right]+\left[[b_2dv_2,b_3dv_3],a_1\frac{du_1}{u_1}\right]+\left[\left[a_2\frac{du_2}{u_2},b_3dv_3\right],a_1\frac{du_1}{u_1}\right]+$
$\left[\left[a_2\frac{du_2}{u_2},b_3dv_3\right],a_1\frac{du_1}{u_1}\right]+\left[\left[a_2\frac{du_2}{u_2},a_3\frac{du_3}{u_3}\right],b_1dv_1\right]+[[b_2dv_2,b_3dv_3],b_1dv_1]+\left[\left[a_2\frac{du_2}{u_2},b_3dv_3\right],b_1dv_1\right]+$
$\left[\left[b_2dv_2,a_2\frac{du_2}{u_2}\right],b_1dv_1\right]+\left[\left[a_3\frac{du_3}{u_3},a_1\frac{du_1}{u_1}\right],a_2\frac{du_2}{u_2}\right]+\left[\left[b_3dv_3,a_1\frac{du_1}{u_1}\right],a_2\frac{du_2}{u_2}\right]+\left[\left[a_3\frac{du_3}{u_3},b_1dv_1\right],a_2\frac{du_2}{u_2}\right]+$
$\left[[b_3dv_3,b_1dv_1],a_2\frac{du_2}{u_2}\right]+\left[\left[a_3\frac{du_3}{u_3},a_1\frac{du_1}{u_1}\right],b_2dv_2\right]+[[b_3dv_3,b_1dv_1],b_2dv_2]+\left[\left[a_3\frac{du_3}{u_3},b_1dv_1\right],b_2dv_2\right]+$
$\left[\left[b_3dv_3,a_1\frac{du_1}{u_1}\right],b_2dv_2\right]$

Regrouping properly the terms we prove the Jacobian identity. □

More generally, we have $\left[\left[\omega_i,\omega_j\right],\omega_k\right]+\circlearrowleft=0$. (taking $i=1$; $j=2$; $k=3$ in Proposition 5.16).

With all those results we can complete the prove of theorem 5.9

Proof. Let $\omega_i=\sum_{i=1}^{p}a_i\frac{du_i}{u_i}+b_idv_i;\omega_j=\sum_{i=1}^{p}a_j\frac{du_j}{u_j}+b_jdv_j;\omega_k=\sum_{k=1}^{p}a_k\frac{du_k}{u_k}+b_kdv_k$ in $\Omega_K(\log\mathcal{I})$. We have:

$\left[\left[\omega_i,\omega_j\right],\omega_k\right]=\sum_{i=1}^{p}\sum_{i=1}^{p}\sum_{i=1}^{p}\left[\left[a_i\frac{du_i}{u_i}+b_idv_i,a_j\frac{du_j}{u_j}+b_jdv_j\right],a_k\frac{du_k}{u_k}+b_kdv_k\right]=$

$\sum_{i,j,k=1}\left[\left[a_i\frac{du_i}{u_i}+b_idv_i,a_j\frac{du_j}{u_j}+b_jdv_j\right],a_k\frac{du_k}{u_k}+b_kdv_k\right]$. Then, from Proposition [5.16], we have

$\sum_{i,j,k=1}\left[\left[a_i\frac{du_i}{u_i}+b_idv_i,a_j\frac{du_j}{u_j}+b_jdv_j\right],a_k\frac{du_k}{u_k}+b_kdv_k\right]+\circlearrowleft=0$ □

5.2 Lie-Rinehart Structure on $\Omega_K(\log \mathcal{I})$

We will in this subsection put the Lie-Rinehart structure on $\Omega_K(\log \mathcal{I})$ when $\mathcal{A}$ is a log-Poisson algebra.

Of course, we recall that the log-Poisson structure on $\mathcal{A}$ induce a 2-form ω_0 on $\Omega_K(\log \mathcal{I})$

Let $x \in \Omega_K(\log \mathcal{I})$ be a fixed element of $\Omega_K(\log \mathcal{I})$. The map $\rho_{\omega_0}(x) : \mathcal{A} \to \mathcal{A}$ defined by $\rho_{\omega_0}(x)(a) = \omega_0(x, d(a))$ is a k-derivation of $\mathcal{A}$. Precisely, ρ_{ω_0} is an element of $\widehat{Der_K(\log \mathcal{I})}$. Indeed, for all $a \in \mathcal{A}$,

$$\begin{aligned}\rho_{\omega_0}(x)(a) &= \sum_{i=1}^{p} x_i\rho_{\omega_0}(\frac{du_i}{u_i})(a) + \sum_{p+1}^{n} x_i\rho_{\omega_0}(dvi)\\ &= \sum_{i=1}^{p}\frac{x_i}{u_i}\{u_i, a\} + \sum_{p+1}^{n} x_i\{v_i, a\}.\end{aligned}$$

Then $\rho_{\omega_0}(x) = \sum_{i=1}^{p}\frac{x_i}{u_i}\{u_i, -\} + \sum_{p+1}^{n} x_i\{v_i, -\} \in \widehat{Der_K(\log \mathcal{I})}$. The map $\rho_{\omega_0} : \Omega_K(\log \mathcal{I}) \to \widehat{Der_K(\log \mathcal{I})}; x : x \mapsto \rho_{\omega_0}(x)$. is $\mathcal{A}$-linear. Indeed

$$\begin{aligned}[\rho_{\omega_0}(ax)](b) &= \omega_0(ax, db)\\ &= a\omega_0(x, db)\\ &= [a\rho_{\omega_0}(x)](b)\end{aligned}$$

From above properties,
$\omega_0(a\frac{du}{u} \otimes db) = \frac{a}{u}\{u, b\}$ and $\rho_{\omega_0}(a\frac{du}{u})(b) = \frac{a}{u}\{u, b\} = \frac{a}{u}(ad(u))(b)$ then $\rho_{\omega_0}(a\frac{du}{u}) = \frac{a}{u}\{u, -\}$.
so

$$\begin{aligned}\rho_{\omega_0}[a\frac{du}{u}, b\frac{dv}{v}] &= \rho_{\omega_0}\left(\frac{a}{u}\{u, b\}\frac{dv}{v} + \frac{b}{v}\{a, v\}\frac{du}{u} + abd(\frac{1}{uv}\{u, v\})\right)\\ &= \frac{a}{u}\{u, b\}\rho_{\omega_0}(\frac{dv}{v}) + \frac{b}{v}\{a, v\}\rho_{\omega_0}(\frac{du}{u}) + ab\rho_{\omega_0}(d(\frac{1}{uv}\{u, v\}))\\ &= \frac{a}{uv}\{u, b\}\{v, -\} + \frac{b}{vu}\{a, v\}\{u, -\} + ab\{\{\frac{1}{uv}\{u, v\}, -\}\\ &= \frac{a}{uv}\{u, b\}\{v, -\} + \frac{b}{vu}\{a, v\}\{u, -\} + \frac{ab}{uv}\{\{u, v\}, -\}+\\ &\quad -\frac{ab}{u^2v}\{u, v\}\{u, -\} - \frac{ab}{uv^2}\{u, v\}\{v, -\}\end{aligned}$$

In other hand,

$$\begin{aligned}\rho_{\omega_0}(a\frac{du}{u})\left(\rho_{\omega_0}(b\frac{dv}{v})\right) &= \frac{a}{u}\{u, \frac{b}{v}\{v, -\}\}\\ &= \frac{a}{uv}\{u, b\}\{v, -\} + \frac{ab}{u}\{u, \frac{1}{v}\{v, -\}\\ &= \frac{a}{uv}\{u, b\}\{v, -\} + \frac{ab}{uv}\{u, \{v, -\}\} - \frac{ab}{uv^2}\{u, v\}\{v, -\}\end{aligned}$$

$$\begin{aligned}\rho_{\omega_0}(b\frac{dv}{v})\left(\rho_{\omega_0}(a\frac{du}{u})\right) &= \frac{b}{v}\{v, \frac{a}{u}\{u, -\}\}\\ &= \frac{b}{uv}\{v, a\}\{u, -\} + \frac{ab}{v}\{v, \frac{1}{u}\{u, -\}\\ &= \frac{b}{uv}\{v, a\}\{u, -\} + \frac{ab}{uv}\{v, \{u, -\}\} - \frac{ab}{vu^2}\{v, u\}\{u, -\}\end{aligned}$$

Then $[\rho_{\omega_0}(a\frac{du}{u}), \rho_{\omega_0}(b\frac{dv}{v})] = \frac{a}{uv}\{u, b\}\{v, -\} + \frac{ab}{uv}\{u, \{v, -\}\} - \frac{ab}{uv^2}\{u, v\}\{v, -\} + \frac{b}{uv}\{v, a\}\{u, -\} - \frac{ab}{uv}\{v, \{u, -\}\} + \frac{ab}{vu^2}\{v, u\}\{u, -\}$.

From Jacobian identity, we have $\{u, \{v, -\}\} - \{v, \{u, -\}\} + \{-, \{u, v\} = 0$; and then $[\rho_{\omega_0}(a\frac{du}{u}), \rho_{\omega_0}(b\frac{dv}{v})] = \frac{a}{uv}\{u, b\}\{v, -\} - \frac{ab}{uv^2}\{u, v\}\{v, -\} - \frac{b}{uv}\{v, a\}\{u, -\} + \frac{ab}{vu^2}\{v, u\}\{u, -\} + \frac{ab}{uv}\{\{u, v\}, -\} - \frac{ab}{uv^2}\{u, v\}\{v, -\}$.

Therefore, $\rho_{\omega_0}[a\frac{du}{u}, b\frac{dv}{v}] = [\rho_{\omega_0}(a\frac{du}{u}), \rho_{\omega_0}(b\frac{dv}{v})]$.

In the same manner we have, $\rho_{\omega_0}[a\frac{du}{u}, bdv] = [\rho_{\omega_0}(a\frac{du}{u}), \rho_{\omega_0}(bdv)]$ and $\rho_{\omega_0}[adu, bdv] = [\rho_{\omega_0}(adu), \rho_{\omega_0}(bdv)]$. We can state the following proposition

Proposition 5.20. *Let $(\mathcal{A}; \{-,-\}, \mathcal{I})$ be a log-Poisson algebra. The map $\rho_{\omega_0} : x \mapsto \rho_{\omega_0}(x)$ is an homomorphism of Lie algebra.*

We will show now that ρ_{ω_0} satisfy the Leibniz rule; of course, giving $u, v \in \mathcal{I}; a \in \mathcal{A}$ we have:

$$
\begin{aligned}
&[\frac{du}{u}, a\frac{dv}{v}]\frac{1}{u}\{u,a\}\frac{dv}{v} + ad(\frac{1}{uv}\{u,v\})\\
&= \frac{1}{u}\{u,a\}\frac{dv}{v} + a[\frac{du}{u}, \frac{dv}{v}]\\
&= \left(\frac{1}{u}\{u,-\}\right)(a)\frac{dv}{v} + a[\frac{du}{u}, \frac{dv}{v}]\\
&= \rho_{\omega_0}(\frac{du}{u})(a)\frac{dv}{v} + a[\frac{du}{u}, \frac{dv}{v}]
\end{aligned}
$$

More generally, for $\omega_i = a_i\frac{du_i}{u_i} + b_i dv_i$ and $\omega_j = a_j\frac{du_j}{u_j} + b_j dv_j$, $f \in A$, on $\Omega_K(\log \mathcal{I})$, $\left[\omega_i, f\omega_j\right] = \rho_{\omega_0}(\omega_i)(a)\omega_j + f\left[\omega_i, \omega_j\right]$. Indeed,

$$
\begin{aligned}
&[\omega_i, f\omega_j][a_i\frac{du_i}{u_i}, fa_j\frac{du_j}{u_j}] + [a_i\frac{du_i}{u_i}, fb_jdv_j] + [b_idv_i, fa_j\frac{du_j}{u_j}]\\
&= \frac{a_i}{u_i}\{u_i, fa_j\} + \frac{fa_i}{u_j}\{a_i,u_j\}\frac{du_i}{u_i} + fa_ia_jd(\frac{1}{u_iu_j}\{u_i,u_j\}) + \frac{a_i}{ui}\{u_i, fb_j\}dv_j +\\
&fb_j\{a_i,v_j\}\frac{du_i}{u_i} + fa_id(\frac{1}{u_i}\{u_i,v_j\}) + b_i\{v_i, fa_j\}\frac{du_j}{u_j} + \frac{fa_j}{u_j}\{b_i,u_j\}dv_i +\\
&fb_ia_jd(\frac{1}{u_j}\{v_i,u_j\}) + b_i\{v_i, fb_j\}dv_j + fb_j\{b_i,v_j\}dv_i + fb_ib_jd(\{v_i,v_j\})\\
&= \frac{fa_i}{u_i}\{u_i,a_j\}\frac{du_j}{u_j} + \frac{a_ia_j}{u_i}\{u_i,f\}\frac{du_j}{u_j} + \frac{fa_j}{u_j}\{a_i,u_j\}\frac{du_i}{u_i} + fa_ia_jd(\frac{1}{u_iu_j}\{u_i,u_j\}) +\\
&\frac{fa_i}{u_i}\{u_i,b_j\}dv_j + \frac{a_ib_j}{u_i}\{u_i,f\}dv_j + fb_j\{a_i,v_j\}\frac{du_i}{u_i} + fa_ib_jd(\frac{1}{u_i}\{u_i,v_j\}) +\\
&b_if\{v_i,a_j\}\frac{du_j}{u_j} + fb_i\{v_i,b_j\}dv_j + b_ib_j\{v_i,f\}dv_j + fb_j\{b_i,v_j\}dv_i + fb_ib_jd(\{v_i,v_j\})\\
&= f(\frac{a_i}{u_i}\{u_i,a_j\}\frac{du_j}{u_j} + \frac{a_j}{u_j}\{a_i,u_j\}\frac{du_i}{u_i} + a_ia_jd(\frac{1}{u_iu_j}\{u_i,u_j\})\frac{a_i}{u_i}\{u_i,b_j\}dv_j +\\
&b_j\{a_i,v_j\}\frac{du_i}{u_i} + a_ib_jd(\frac{1}{u_i}\{u_i,v_j\})b_i\{v_i,b_j\}dv_j + \frac{a_i}{u_i}\{b_i,u_j\}dv_i + b_ia_jd(\frac{1}{u_j}\{v_i,u_j\}) +\\
&b_i\{v_i,b_j\}dv_j + b_j\{b_i,v_j\}dv_i + b_ib_jd(\{v_i,v_j\}))\\
&= f(\frac{a_i}{u_i}\{u_i,a_j\}\frac{du_j}{u_j} + \frac{a_j}{u_j}\{a_i,u_j\}\frac{du_i}{u_i} + a_ia_jd(\frac{1}{u_iu_j})\frac{a_i}{u_i}\{u_i,b_j\}dv_j + b_j\{a_i,v_j\}\frac{du_i}{u_i} +\\
&a_ib_jd(\frac{1}{u_i}\{u_i,v_j\})b_i\{v_i,b_j\}dv_j + \frac{a_i}{u_i}\{b_i,u_j\}dv_i + b_ia_jd(\frac{1}{u_j}\{v_i,u_j\}) + b_i\{v_i,b_j\}dv_j +\\
&b_j\{b_i,v_j\}dv_i + b_ib_jd(\{v_i,v_j\})) +\\
&\left[\left(\frac{a_i}{u_i}\{u_i,-\} + \frac{b_j}{u_i}\{u_i,-\}dv_j + b_ia_j\{v_i,-\} + b_j\{v_i,-\}dv_j\right)(f)\right](a_j\frac{du_j}{u_j} + b_jdv_j)\\
&= f[\omega_i,\omega_j] + (\rho_{\omega_0}(\omega_i)(f))\omega_j.
\end{aligned}
$$

Therefore,
$[\omega_i, f\omega_j] = f[\omega_i,\omega_j] + (\rho_{\omega_0}(\omega_i)(f))\omega_j.$

Theorem 5.21. *If $(\mathcal{A}, \{-,-\}, \mathcal{I})$ is a logarithmic Poisson-algebra, then $(\Omega_K(\log \mathcal{I}), \rho_{\omega_0}, [-,-])$ is a logarithmic Lie-Rinehart algebra and ω_0 is a Log Lie Rinehart Cohomology structure.*

This theorem says that each log Poisson algebra induce on $\Omega_K(\log \mathcal{I})$ a logarithmic Lie-Rinehart structure. We are interested to know if the inverse is true. In other words, giving a skew symmetric 2-forme ω such that $d_\rho(\omega) = 0$ and a Lie structure $[-,-]$ on $\Omega_K(\log \mathcal{I})$, does it exist on $\mathcal{A}$ a logarithmic Poisson structure along $\mathcal{I}$?

Since $\Omega_K(\log \mathcal{I})$ have a Lie-Rinehart structure, then apply the usual technic of Palais and Rinehart we construct a chain complex and deduce the notion of logarithmic Poisson cohomology. Of course, Let $Alt_A(\Omega_K(\log \mathcal{I}, M))$ be the set of all multilinear skew symmetric map on $\Omega_K(\log \mathcal{I})$ with value in the $\mathcal{A}$-module of Lie-Rinehart M, $Alt_A(\Omega_K(\log \mathcal{I}), M)$

is commutative graded k-algebra for the product "shifte"defined by: $\alpha \wedge \beta(x_i, ..., x_{p+q}) = \sum_{\sigma}\varepsilon_\sigma \mu(\alpha(x_{\sigma(1)}, ..., x_{\sigma(p)}) \otimes \beta(x_{\sigma(p+1)}, ..., x_{\sigma(p+q)}))$ where $\mu : M \otimes M \to M$ is $\Omega_K(\log \mathcal{I})-$'paire; (a morphism of $\Omega_K(\log \mathcal{I})$-modules). Equipped with Cartan-Chevalley-Eilenberg differential $d_{\rho_{\omega_0}}$ associated to the representation ρ_{ω_0}

$d_{\rho_{\omega_0}}(f)(\alpha_0, ...\alpha_p) = \sum_{i=0}^{p}(-1)^i \rho_{\omega_0}(\alpha_i) f(\alpha_0, ...\hat{\alpha}_i, ...\alpha_p) +$
$\sum_{i,j}(-1)^{i+j} f([\alpha_i, \alpha_j], \alpha_0, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \alpha_p)$

We denoted $H^*_{\log-Poisson}(A, \{-,-\}; M)$ the associated cohomology. Follow G. Rinehart in (Rinehart, G. S. (1963)), $H^*_{\log-Poisson}(A, \{-,-\}; M) \simeq Ext_{(U(A,\Omega_K(\log \mathcal{I})))}(A, M)$ if $\Omega_K(\log \mathcal{I})$; is a projective $\mathcal{A}-$module.

Definition 5.22. *$H^*_{\log-Poisson}(A, \{-,-\}; M)$; is called logarithmic Poisson cohomology with value in M*

$d_{\rho_{\omega_0}}(\omega_0)(\alpha_0, \alpha_1, \alpha_2) = \rho_{\omega_0}(\alpha_0)\omega_0(\alpha_1, \alpha_2) - \rho_{\omega_0}(\alpha_1)\omega_0(\alpha_0, \alpha_2) + \rho_{\omega_0}(\alpha_2)\omega(\alpha_0, \alpha_1) - \omega([\alpha_0, \alpha_1], \alpha_2) + \omega([\alpha_0, \alpha_2], \alpha_1) - \omega([\alpha_1, \alpha_2], \alpha_0.) = 0$ For all $\alpha_0, \alpha_1, \alpha_2 \in Alt^1_A(\Omega_K(\log \mathcal{I}), A)$.

Indeed, it suffice to prove it on generator of $\Omega_K(\log \mathcal{I})$. For all $\alpha_0 = \frac{du_0}{u_0}, \alpha_1 = \frac{du_1}{u_1}, \alpha_2 = \frac{du_2}{u_2}$, we have

$$
\begin{aligned}
& d_{\rho_{\omega_0}}(\omega_0)(\alpha_0, \alpha_1, \alpha_2) \\
&= \frac{1}{u_0}\{u_0, \frac{1}{u_1u_2}\{u_1, u_2\}\} - \frac{1}{u_1}\{u_1, \frac{1}{u_0u_2}\{u_0, u_2\}\} + \frac{1}{u_2}\{u_2, \frac{1}{u_0u_1}\{u_0, u_1\}\} \\
&- \frac{1}{u_2}\{\frac{1}{u_0u_1}\{u_0, u_1\}, u_2\} + \frac{1}{u_1}\{\frac{1}{u_0u_2}\{u_0, u_2\}, u_1\} - \frac{1}{u_0}\{\frac{1}{u_1u_2}\{u_1, u_2\}, u_0\} \\
&= \frac{1}{u_0u_1u_2}\{u_0, \{u_1, u_2\}\} - \frac{1}{u_0u_1u_2^2}\{u_1, u_2\}\{u_0, u_2\} - \frac{1}{u_0u_2u_1^2}\{u_1, u_2\}\{u_0, u_1\} - \\
& \frac{1}{u_1u_0u_2}\{u_1, \{u_0, u_2\}\} + \frac{1}{u_2u_1u_0^2}\{u_0, u_2\}\{u_1, u_0\} + \frac{1}{u_2^2u_1u_0}\{u_0, u_2\}\{u_1, u_2\} + \\
& \frac{1}{u_0u_1u_2}\{u_2, \{u_0, u_1\}\} - \frac{1}{u_2u_1u_0^2}\{u_0, u_1\}\{u_2, u_0\} - \frac{1}{u_2u_0u_1^2}\{u_0, u_1\}\{u_2, u_1\} + \\
& \frac{1}{u_2u_0u_1}\{\{u_0, u_1\}, u_2\} + \frac{1}{u_2u_0^2u_1}\{u_0, u_1\}\{u_0, u_2\} + \frac{1}{u_1u_0u_2}\{\{u_0, u_2\}, u_1\} - \\
& \frac{1}{u_1u_0^2u_2}\{u_0, u_2\}\{u_0, u_1\} - \frac{1}{u_1u_0u_2^2}\{u_0, u_2\}\{u_2, u_1\} - \frac{1}{u_0u_1u_2}\{\{u_1, u_2\}, u_0\} + \\
& \frac{1}{u_0u_1^2u_2}\{u_1, u_2\}\{u_1, u_0\} + \frac{1}{u_0u_1u_2^2}\{u_1, u_2\}\{u_2, u_2\}
\end{aligned}
$$

It follows from Jacobian identity of $\{-,-\}$ that $d_{\rho_{\omega_0}}(\omega_0)(\alpha_0, \alpha_1, \alpha_2) = 0$. For $\alpha_0 = \frac{du_0}{u_0}, \alpha_2 = \frac{du_1}{u_1}, \alpha_2 = du_2$,

$$
\begin{aligned}
& d_{\rho_{\omega_0}}(\omega_0)(\alpha_0, \alpha_1, \alpha_2) \\
&= \frac{1}{u_0}\{u_0, \frac{1}{u_1}\{u_1, u_2\}\} - \frac{1}{u_1}\{u_1, \frac{1}{u_0}\{u_0, u_2\}\} + \{u_2, \frac{1}{u_0u_1}\{u_0, u_1\}\} \\
&= \frac{1}{u_0u_1}\{u_0, \{u_1, u_2\}\} - \frac{1}{u_0u_1^2}\{u_1, u_2\}\{u_0, u_1\} - \frac{1}{u_1u_0}\{u_1, \{u_0, u_2\}\} + \\
& \frac{1}{u_1u_0^2}\{u_0, u_2\}\{u_1, u_0\} + \frac{1}{u_0u_1}\{u_2, \{u_0, u_1\}\} - \frac{1}{u_0u_1^2}\{u_0, u_1\}\{u_2, u_1\} - \\
& \frac{1}{u_0^2u_1}\{u_0, u_1\}\{u_2, u_0\} - \frac{1}{u_0u_1}\{\{u_0, u_1\}, u_2\} + \frac{1}{u_0u_1^2}\{u_0, u_1\}\{u_1, u_2\} + \\
& \frac{1}{u_0^2u_1}\{u_0, u_1\}\{u_0, u_2\} + \frac{1}{u_1u_0}\{\{u_0, u_2\}, u_1\} - \frac{1}{u_1u_0^2}\{u_0, u_2\}\{u_0, u_1\} - \\
& \frac{1}{u_0u_1}\{\{u_1, u_2\}, u_0\} + \frac{1}{u_0u_1^2}\{u_1, u_2\}\{u_1, u_0\} \\
&= 0
\end{aligned}
$$

In the same manner, we prove that $d_{\rho_{\omega_0}}(\omega_0)(\alpha_0, \alpha_1, \alpha_2) = 0$ for $\alpha_0 = \frac{du_0}{u_0}; \alpha_1 = du_1, \alpha_2 = du_2$ and $\alpha_0 = du_0, \alpha_1 = du_1, \alpha_2 = du_2$. This end the proof of the following theorem.

Theorem 5.23. *Let $\{-,-\}$ be log Poisson structure. The associated 2-form ω_0 is a co cycle of $H^*_{\log-Poisson}(A, \{-,-\}; \mathcal{A})$*

We denote $[\omega_{0\{-,-\}}]$ the cohomology class of ω_0

Definition 5.24. $[\omega_{0\{-,-\}}]$ *is called logarithmic Poisson class of* $(A, \{-,-\}; \mathcal{I})$.

6. Prequantization of Logarithmic Poisson Algebra

In the context of geometric quantization the term Prequantization refer to a certain type of representation of Poisson algebra.

6.1 Extension of Lie-Rinehart Algebra and Logarithmic Connection

Let $\mathcal{A}$ be an algebra and L, L', L'' a Lie-Rinehart algebra. An extension L'' of L along L' is a short exact sequences of Lie-Rinehart algebras.

$$0 \longrightarrow L' \xrightarrow{f} L \xrightarrow{g} L'' \longrightarrow 0 \tag{6}$$

For each extension of type (1), there is a linear map $\omega : L'' \to L$ such that $g \circ \omega = id$. ω is $\mathcal{A}$-modules homomorphism when L'' is projectif. Extension (1) is split if ω is homomorphism of Lie-Rinehart algebras. Each extension of type (1) induce two maps

$$\begin{array}{rccl} \alpha : & L'' & \longrightarrow & End_K(L') \\ & x & \mapsto & \alpha_X : y \mapsto [\omega(x), y] \end{array}$$

$$\begin{array}{rccl} \Omega : & \bigwedge^2 L'' & \longrightarrow & L' \\ & (x; y) & \mapsto & [\omega(x), \omega(y)] - \omega([x, y]) \end{array}$$

such that

$$[\alpha_x; \alpha_y] - \alpha_[x, y] \quad = \quad [\omega(x, y), -] \tag{7}$$

$$\sum_{cycler\{x,x,z\}} (\alpha_x \omega(x, y) - \omega([x, y], z)) = 0 \tag{8}$$

Note that $\Omega(x, y) = [\omega(x), \omega(y)] - \omega([x, y])$ look like Maurer-Cartan formula

$$\rho = d\omega + \frac{1}{2}[\omega, \omega]_G \tag{9}$$

for curvature of principal bundles of differential geometry. Where

$$(d\varphi)(X_0, ..., X_p) = \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_p) \tag{10}$$

and

$$\begin{array}{l} [-,-]_G : Alt^*_A(L'', L) \otimes Alt^*_A(L'', L) \to Alt^*_A(L'', L) \\ (\varphi, \psi) \mapsto \dfrac{1}{p!q!} \sum_\sigma \varepsilon_\sigma [\varphi(X_{\sigma(1)}, ..., X_{\sigma(p)}), \psi(X_{\sigma(1)}, ..., X_{\sigma(p)})] \end{array} .$$

$[-,-]_G$ is a Generalization of Gerstenhaber structure extending the Lie structure $[-,-]_{L"}$ on L"

Expression (8) look like Bianchi identity $d^\omega(\Omega) = 0$ where $d^\omega(\varphi) = d(\varphi) + \tilde{d}(\varphi)$ and

$\tilde{d}(\varphi)(X_0, ..., X_p) = \sum_{i=0}^{p} (-1)^i \alpha_{X_i}(\varphi(X_0, ..., \hat{X}_i, ..., X_p))$

We note that $d^\omega \circ d^\omega \varphi = [\omega, \varphi]_G$; therefore, it is not a differential. If we change the linear section ω by $\omega' = \omega + b$ where $b : L" \to L'$ is linear map, then

$$\alpha'_X = \alpha_X + [b(X), -]_{L'} \tag{11}$$

and

$$\Omega' - \Omega = d^\omega b + \frac{1}{2}[b, b]_G \tag{12}$$

Thus, each Lie-Rinehart algebra extension induce a 2-form Ω.

Recall that a Lie-Rinehart algebra L' is Abelian, is an $\mathcal{A}$-module with trivial bracket. It is well know that if in extension (1), L' is Abelian, then adjoint representation of L on L' induce an action of L" on L'. In this case, d^ω is it cohomology operator. We denote $H^2(Alt_A(L", L'))$ to be its second cohomologie "group". Huebschmann in (Huebschmann, J. (2013)) Prove the following Theorem:

Theorem 6.1. *(Huebschmann, J. (2013))[Théorème 2.6] Let L' and L" be two Lie-Rinehart algebras, with L' abelian, and let $\varrho : L" \to End(L')$ be the Lie-Rinehart module structure of L" on L'. Then the correspondence which associate for each isomorphism class of extension of type (1) a 2-cocycle $\Omega \in Alt_A(L", L')$ is a bijection between set of congruence classes of extension of L' by L" with split module extension and $H^2(Alt_A(L", L'))$.*

Since Ω satisfies the Maurer-Cartan formula for connections, there exist a relation between associated connection and linear connection. We referring to (Huebschmann, J. (2013)) for the prove. We recall that giving a Lie-Rinehart L and a $\mathcal{A}$-module M, an L-connection is a k-linear map $\nabla : L \to End(M)$ satisfy the following conditions:

$$\nabla(a\alpha)(m) = a(\nabla(\alpha))(m) \tag{13}$$

$$\nabla(\alpha)(am) = a\nabla(m) + (\rho_L(\alpha))(a)m \tag{14}$$

when $L = Der_K(\log I)$, ∇ is called logarithmic connection along $\mathcal{I}$.

Let L be a Lie-Rinehart logarithmic along $\mathcal{I}$, M an $\mathcal{A}$-module an ∇ an L-connection. Denote

$\tilde{\nabla} : M \to Hom(L; M)$ the adjoint morphism of ∇;

$$\tilde{\nabla}_\alpha(m) := \tilde{\nabla}(m)(\alpha) = (\nabla)(\alpha)(m \tag{15}$$

From equation 13, image of $\tilde{\nabla}$ live in $Hom_A(L, M)$ and then, from 14, we have

$$\tilde{\nabla}_\alpha(am) = ((\rho_L)(a))m + a\tilde{\nabla}_\alpha(m) \tag{16}$$

Each morphism $\triangle : M \to Hom_A(L, M)$ satisfying 16 induce on M an L-connection. $\nabla : L \to End(M)$. Giving an L-connection $\nabla : L \to End(M)$, ∇ induce on $Alt_A(L, M)$ the following operator,

$$\begin{array}{rcl}(d^\nabla f)(\alpha_0, ..., \alpha_p) & = & \sum\limits_{i=0}^{i=p}(-1)^i\tilde{\nabla}_{\alpha_i}f(\alpha_0, ..., \hat{\alpha_i}, ..., \alpha_p) + \\ & & \sum\limits_{i<j}(-1)^{i+j}f([\alpha_i, \alpha_j], \alpha_0, ..., \hat{\alpha_i}, ..., \hat{\alpha_j}, ..., \alpha_p)\end{array} \tag{17}$$

for all $f \in Alt(L, M)$

$$\begin{array}{rcl}(d^\nabla f)(\alpha_0, \alpha_1) & = & \tilde{\nabla}_{\alpha_0}(f(\alpha_1)) - \tilde{\nabla}_{\alpha_1}(f(\alpha_0)) - f([\alpha_0, \alpha_1] \\ & = & (\nabla(\alpha_0))(f(\alpha_1)) - (\nabla(\alpha_1)(f(\alpha_0))) - f([\alpha_0, \alpha_1])\end{array}$$

We denote $f = \tilde{\nabla}(m)$, and we obtain

$$\begin{array}{rcl}(d^\nabla\tilde{\nabla}(m))(\alpha_0, \alpha_1) & = & \tilde{\nabla}_{\alpha_0}(f(\alpha_1)) - \tilde{\nabla}_{\alpha_1}(f(\alpha_0)) - f([\alpha_0, \alpha_1] \\ d^\nabla \circ d^\nabla(m)(\alpha_0, \alpha_1) & = & (\nabla(\alpha_0))(f(\alpha_1)) - (\nabla(\alpha_1)(f(\alpha_0))) - f([\alpha_0, \alpha_1]) \\ & = & (\nabla(\alpha_0))(\tilde{\nabla}(m)(\alpha_1)) - (\nabla(\alpha_1)(\tilde{\nabla}(m)(\alpha_0))) - \tilde{\nabla}(m)([\alpha_0, \alpha_1]) \\ & = & (\nabla(\alpha_0))(\nabla(\alpha_1)(m)) - (\nabla(\alpha_1)(\nabla(\alpha_0)(m))) - \nabla([\alpha_0, \alpha_1])(m) \\ & = & ((\nabla(\alpha_0))(\nabla(\alpha_1)) - (\nabla(\alpha_1)(\nabla(\alpha_0))) - \nabla([\alpha_0, \alpha_1]))(m) \\ & = & ([\nabla(\alpha_0), \nabla(\alpha_1)] - \nabla([\alpha_0, \alpha_1]))(m)\end{array}$$

It follow that $d^\nabla \circ d^\nabla$ is a morphism from M to $Alt^2_A(L, M)$

We put $\Omega(\alpha_0, \alpha_1) = ([\nabla(\alpha_0), \nabla(\alpha_1)] - \nabla([\alpha_0, \alpha_1])$ and we have:

$\Omega : L \otimes L \to End(M)$. More precisely, image by Ω live in $End_A(M)$ it is the logarithmic tensor see (Dongho, J., & Yotcha, S. R. (2016))

In (Huebschmann, J. (2013)), Huebschmann have prove that for a giving $\mathcal{A}$-module M, on a Lie-Rinehart algebra L, M have L-connection if and only if M is L-normal an associate extension is split in the category of $\mathcal{A}$-modules. On the other hand, he prove that each projective $\mathcal{A}$-module relatively free M have L-connection. We denote $Pic(A)$ the group of isomorphism classes of projective modules He prove the following Theorem.

Theorem 6.2. *(Huebschmann, J. (2013)) The map*

$$\begin{array}{rccc}C: & Pic(A) & \to & H^2(Alt_A(L, A)) \\ & M & \mapsto & [\Omega_M]\end{array}$$

is an $\mathcal{A}$-modules homomorphism.

The Theorem state that for all Lie-Rinehart algebra L, in particular for $L = (\Omega_K(\log \mathcal{I}), \rho_\omega, [-,-])$, we have the following morphism of modules

$$\begin{array}{cccc} C: & Pic(A) & \to & H^2(Alt_A(\Omega_K(\log I), A)) \\ & M & \mapsto & [\Omega_M] \end{array} . \tag{18}$$

But from Theorem 5.23, $[\omega_0] \in H^2(Alt_A(\Omega_K(\log \mathcal{I}), A))$ it is possible that it live in ImC.

6.2 Linear Representation of Logarithmic Poisson Algebra

Let $(A, \{-,-\}; \mathcal{I})$ be a Logarithmic Poisson algebra. We should link the Poisson class of $(A, \{-,-\}; \mathcal{I})$, to the representation of the underline Lie algebra and this according to Dirac principe of quantization. We shall construct a k-linear representation for which k act by scalar multiplication. In ((Huebschmann, J. (2013))), follow Markenzie, Huebschmann, prove that for all $\mathcal{A}$-module M with L-connection, there exist an Lie-Rinehart algebra $DO(A,\nabla)$ having L as extension along $End_A(M)$. In other words, the following short exact sequence split in the category of $\mathcal{A}$-modules.

$$0 \longrightarrow End_A(M) \longrightarrow DO(A,\nabla) \longrightarrow L \longrightarrow 0 \tag{19}$$

where $DO(A\nabla) = End_A(M) \oplus L$. It is well know that each rank one module M, $End_A(M)$ is isomorphic to $\mathcal{A}$. The for all rank one projective module, 19, become

$$0 \longrightarrow A \longrightarrow A \oplus L \longrightarrow L \longrightarrow 0 \tag{20}$$

in particular for $L = \Omega_K(\log \mathcal{I})$ the proposition 4.2 in ((Huebschmann, J. (2013))) prove the commutativity of the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & Ham_{\{,\}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & \Omega_K\widetilde{(\log I)} \arrow[r] & \Omega_K(\log I) \arrow[r] & 0 \end{tikzcd}$$

Definition 6.3. $(A, \{,\})$ *is prequantizable if there exist a projective rank 1 module M with* $\Omega_K\widetilde{(\log \mathcal{I})}$*-connection with curvature* ω_0.

It follow from Theorem 6.2 that

Proposition 6.4. *A logarithmic Poisson structure;* $\{-,-\}$*, along* $\mathcal{I}$ *on* $\mathcal{A}$ *is prequantizable if and only if* $[\omega_0] \in ImC$.

Acknowledgement

We will like to thank PEKA MINGA SARIELLE who carry out the necessary English translation.

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