

Nonexistence of Nontrivial Stationary Solutions with Decay Order for Some Nonlinear Evolution Equations

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Abstract

We show that there are no nontrivial stationary solutions of certain decay order for some applied nonlinear evolution equations which include the thin epitaxial film model with slope selection and the square phase field crystal (SPFC) equation. The method is to use the Morawetz multiplier and the Gagliardo-Nirenberg inequality.

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1. Introduction

Consider the thin epitaxial film model with slope selection (Li & Liu, 2004; Shen, Wang, Wang, & Wise, 2012; Wang, Wang, & Wise, 2010; Xu & Tang 2006),

$$\frac{\partial u}{\partial t} = -\varepsilon^2 \Delta^2 u + \nabla \cdot (|\nabla u|^2 \nabla u) - \Delta u \quad (1.1)$$

and the square phase field crystal equation (Cross & Hohenberg, 1993; Hernández, Castañeda, & Cadenas, 2014; Hoyle, 1995; Lloyd, Sandstede, Avitabile, & Champneys, 2008),

$$\frac{\partial u}{\partial t} = \Delta(\gamma_0 u + \gamma_1 \Delta u + \varepsilon^2 \Delta^2 u - \nabla \cdot (|\nabla u|^2 \nabla u)) \quad (1.2)$$

The stationary equation of (1.1) is

$$-\varepsilon^2 \Delta^2 u + \nabla \cdot (|\nabla u|^2 \nabla u) - \Delta u = 0,$$

that is,

$$\varepsilon^2 \Delta^2 u - \nabla \cdot (|\nabla u|^2 \nabla u) + \Delta u = 0. \quad (1.3)$$

The stationary equation of (1.2) is

$$\Delta(\gamma_0 u + \gamma_1 \Delta u + \varepsilon^2 \Delta^2 u - \nabla \cdot (|\nabla u|^2 \nabla u)) = 0.$$

Thus

$$\gamma_0 u + \gamma_1 \Delta u + \varepsilon^2 \Delta^2 u - \nabla \cdot (|\nabla u|^2 \nabla u) = 0, \quad (1.4)$$

if the solution decays at infinity.

In this paper, we will study the equation that is more general than the above equations (1.3) and (1.4),

$$\Delta^2 u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + a\Delta u + bu + f(x, u) = 0, \tag{1.5}$$

where a and b are constants and f is a continuous function. We will show that under certain conditions of a , b , and $f(x, u)$, there is no nontrivial solution with certain decay order for (1.5). The method is to use the Morawetz multiplier (Levandosky & Strauss, 2000; Morawetz 1968) and the Gagliardo-Nirenberg inequality (Bellazzini, Frank, & Visciglia, 2014).

Nonlinear fourth-order elliptic equations related to (1.5) have attracted a lot of attentions lately. For example, for the equation $-\Delta^2 u = u^p$, it is known (Lin 1998; Wei and Xu 1999; Fazly, Wei, & Xu 2015) that it doesn't have a nontrivial nonnegative solution when $1 < p < \frac{n+4}{n-4}$ with $n \geq 5$, and $p > 1$ with $n \leq 4$. Also for the case of $p = \frac{n+4}{n-4}$, all solutions are classified. Thus it is natural to consider the case of equation (1.5) which combines the effect of nonlinearity, fourth order, and the p -Laplacian operator (Drabek 2007).

As usual, $x = (x_1, x_2, \dots, x_n)$, ∇u denotes the gradient of u , $\nabla \cdot u$ denotes the divergence of u , and $r = |x|$. Also the subscript denotes the partial derivative, thus $u_s = \frac{\partial u}{\partial s}$. We also use the notation $u_r = \frac{\partial u}{\partial r} = \frac{x}{r} \cdot \nabla u$ and

$$\partial_j = \frac{\partial}{\partial x_j} \cdot F_r(x, s) \text{ denotes } \frac{\partial F(x, s)}{\partial r} = \frac{x}{r} \cdot \nabla_x F(x, s). C^k(\mathbb{R}^n) \text{ is the space of functions whose partial derivatives of order up to and including } k \text{ are continuously differentiable.}$$

Define the following four sets of functions $D_{h,k}(\mathbb{R}^n)$, $E_m(\mathbb{R}^n)$, $F_n(\mathbb{R}^n)$, and $NF(\mathbb{R}^n)$, which we will use in this article:

$$D_{h,k}(\mathbb{R}^n) = \{ u \mid u \in C^k(\mathbb{R}^n), \lim_{R \rightarrow \infty} (\sup_{|x| \leq R} |x^\alpha| |D^\beta u(x)|) = 0, \text{ for all multi-indices } \alpha \text{ and } \beta \in \mathbb{N}_0^n \text{ such that}$$

$$|\alpha| \leq h \text{ and } |\beta| \leq k, \text{ where } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n), |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \text{ and } |\beta| = \beta_1 + \beta_2 + \dots + \beta_n \},$$

$$E_m(\mathbb{R}^n) = \{ a \mid a \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^{-m} |a(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0 \},$$

$$F_n(\mathbb{R}^n) = \{ u \mid \lim_{R \rightarrow \infty} (R^n \sup_{|x|=R} |F(\mathbf{x}, u(\mathbf{x}))|) = 0 \},$$

And

$$NF(\mathbb{R}^n) = \{ u \mid \int_{\mathbb{R}^n} [nF(\mathbf{x}, u(\mathbf{x})) + rF_r(\mathbf{x}, u(\mathbf{x})) - ((n-1)/2)f(\mathbf{x}, u(\mathbf{x}))u(\mathbf{x})] \, d\mathbf{x} \leq 0 \},$$

where $F(\mathbf{x}, u)$ is the antiderivative of $f(\mathbf{x}, u)$ with respect to u such that $F(\mathbf{x}, 0) = 0$.

Remark 1. A function u is said to be of decay order (h, k) if and only if $u \in D_{h,k}(\mathbb{R}^n)$. All the functions are assumed to be real-valued.

2. Morawetz Multiplier

Multiplying both sides of Equation (1.5) by Morawetz multiplier $\zeta(u_r + ((n-1)u/(2r)))$, where $\zeta \in C^3(\mathbb{R}^n)$ and $\zeta(\mathbf{x}) = \zeta(|\mathbf{x}|) = \zeta(r)$, we get

$$0 = [\Delta^2 u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + a\Delta u + bu + f(x, u)] \zeta(u_r + \frac{(n-1)u}{2r}) = \nabla \cdot Y + Z, \tag{2.1}$$

where Y depends on ζ and u as well as their partial derivatives up to and including the third order and $F(\mathbf{x}, u)$, and

$$\begin{aligned}
 Z &= \frac{3\zeta'}{2} (\Delta u)^2 + A(u_r)^2 + B(|\nabla u|^2 - |u_r|^2) + Cu^2 + (\zeta - r\zeta')P \\
 &- \left\{ \left(\frac{\zeta'}{r} - \zeta \right) |\nabla u|^{p-2} \left[(u_r)^2 + \frac{n-1}{2r} u_r u \right] + \left[\frac{\zeta'}{p} - \left(\frac{\zeta}{2pr} \right) ((n+1)p - 2(n-1)) \right] |\nabla u|^p \right\} \\
 &- a \left\{ \frac{1}{2} \zeta_r |\nabla u|^2 + (\nabla u \cdot \nabla \zeta) u_r - \zeta_r |u_r|^2 + \left(\frac{1}{r} \zeta - \zeta_r \right) (|\nabla u|^2 - |u_r|^2) \right. \\
 &+ \left. \frac{n-1}{2r} \left[\frac{1}{r} \zeta_r - \frac{1}{2} \Delta \zeta + \frac{n-3}{2r^2} \zeta \right] u^2 \right\} \\
 &+ b \left\{ \frac{\zeta'}{2} u^2 - \left(\zeta_r + \frac{n-1}{r} \zeta \right) F(x, u) + \frac{n-1}{2r} \zeta f(x, u) u - \zeta F_r(x, u) \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 A &= -\frac{7\zeta'''}{2} - \frac{(n-1)(n-3)(\zeta' - \frac{\zeta}{r})}{2r^2}, \\
 B &= \frac{-3\zeta'''}{2} + \frac{(n-5)\zeta''}{r} - \frac{(n^2 + 2n - 19)(\zeta' - \frac{\zeta}{r})}{2r^2}, \\
 C &= \frac{n-1}{2} \left[\frac{\zeta'''}{2r} + \frac{(n-3)\zeta'''}{r^2} + \frac{(n-3)(n-7)\zeta''}{2r^3} - \frac{3(n-3)(n-5)(\zeta' - \frac{\zeta}{r})}{2r^4} \right],
 \end{aligned}$$

and

$$P = \frac{2}{r} \left[\sum_{i,j} (S_{i,j} u)^2 - \sum_i \left(\sum_j \frac{x_j}{r} S_{i,j} u \right)^2 \right] \geq 0,$$

Where $S_{i,j} u = \frac{x_i}{r^3} \sum_k [x_k (x_k \partial_j - x_j \partial_k) u_r] + \partial_j \sum_k \left[\frac{x_k}{r^2} (x_k \partial_i - x_i \partial_k) u \right]$.

3. Nonexistence of Nontrivial Solutions

Theorem Assume $p \geq (2n)/(n+1)$ and $u \in D_{n,3}(\mathbf{R}^n) \cap F_n(\mathbf{R}^n) \cap NF(\mathbf{R}^n)$.

(a) If $a > 0$ and $b > a^2 C_n^2 / 12$, then $u \equiv 0$, where C_n is the constant, depending only on the dimension n , for the Gagliardo-Nirenberg inequality.

(b) If $a \leq 0$ and $b \geq 0$, then $u \equiv 0$.

Proof:

Let $R > 0$. Integrating both sides of (2.1) in $|\mathbf{x}| \leq R$ and using the Divergence theorem, we get

$$0 = \int_{|\mathbf{x}|=R} Y \cdot (x/R) ds + \int_{R^n} Z dx$$

Let $R \rightarrow \infty$. We get $0 = \lim_{R \rightarrow \infty} \int_{|\mathbf{x}|=R} Y \cdot (x/R) ds + \int_{R^n} Z dx$

Now we let $\zeta(\mathbf{x}) = \zeta(|\mathbf{x}|) = \zeta(r) = r$ in (2.1). Then

$$\begin{aligned}
 0 &= \int_{R^n} Z dx \\
 &= \int_{R^n} \left\{ \frac{3}{2}(\Delta u)^2 - \frac{2n - (n+1)p}{2p} |\nabla u|^p - a \frac{1}{2} |\nabla u|^2 + b \frac{1}{2} u^2 \right. \\
 &\quad \left. - nF(x, u) - \frac{n-1}{2} f(x, u)u - rF_r(x, u) \right\} dx
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\int_{R^n} \left\{ \frac{3}{2}(\Delta u)^2 - \frac{2n - (n+1)p}{2p} |\nabla u|^p - a \frac{1}{2} |\nabla u|^2 + b \frac{1}{2} u^2 \right\} dx \\
 &= \int_{R^n} \left\{ nF(x, u) - \frac{n-1}{2} f(x, u)u + rF_r(x, u) \right\} dx
 \end{aligned} \tag{3.1}$$

To show the assertion (a), we recall the Gagliardo-Nirenberg inequality (Bellazzini, Frank, & Visciglia, 2014)

$$\int_{R^n} |\nabla u|^2 dx \leq C_n \left(\int_{R^n} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{R^n} |\Delta u|^2 dx \right)^{\frac{1}{2}} \tag{3.2}$$

where C_n , depending only on n , is the least positive constant such that the above inequality (3.2) holds for all u in $H^2(R^n)$.

Therefore, since $a > 0$,

$$\begin{aligned}
 \frac{a}{2} \int_{R^n} |\nabla u|^2 dx &\leq \frac{aC_n}{2} \left(\int_{R^n} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{R^n} |\Delta u|^2 dx \right)^{\frac{1}{2}} \\
 &\leq \frac{a^2 C_n^2}{24} \int_{R^2} |u|^2 dx + \frac{3}{2} \int_{R^n} |\Delta u|^2 dx
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\int_{R^n} \left\{ -\frac{2n - (n+1)p}{2p} |\nabla u|^p + \left[\frac{b}{2} - \frac{a^2 C_n^2}{24} \right] u^2 \right\} dx \\
 &\leq \int_{R^n} \left\{ \frac{3}{2}(\Delta u)^2 - \frac{2n - (n+1)p}{2p} |\nabla u|^p - a \frac{1}{2} |\nabla u|^2 + b \frac{1}{2} u^2 \right\} dx \\
 &= \int_{R^n} \left\{ nF(x, u) - \frac{n-1}{2} f(x, u)u + rF_r(x, u) \right\} dx
 \end{aligned}$$

Since $b > \frac{a^2 C_n^2}{12}$ and $p \geq \frac{2n}{n+1}$.

we have

$$\begin{aligned} 0 &\leq \int_{R^n} \left\{ -\frac{2n-(n+1)p}{2p} |\nabla u|^p + \left[\frac{b}{2} - \frac{a^2 C_n^2}{24} \right] u^2 \right\} dx \\ &\leq \int_{R^n} \left\{ nF(x, u) - \frac{n-1}{2} f(x, u)u + rF_r(x, u) \right\} dx \\ &\leq 0, \end{aligned}$$

since $u \in NF(\mathbf{R}^n)$.

Thus

$$\int_{R^n} \left\{ -\frac{2n-(n+1)p}{2p} |\nabla u|^p + \left[\frac{b}{2} - \frac{a^2 C_n^2}{24} \right] u^2 \right\} dx = 0$$

Since $u \in D_{n,3}(\mathbf{R}^n)$, $u \equiv 0$.

For the assertion (b), since $a \leq 0$, $b \geq 0$, and $p \geq \frac{2n}{n+1}$ we have from (3.1),

$$\begin{aligned} 0 &\leq \int_{R^n} \left\{ \frac{3}{2} (\Delta u)^2 - \frac{2n-(n+1)p}{2p} |\nabla u|^p - a \frac{1}{2} |\nabla u|^2 + b \frac{1}{2} u^2 \right\} dx \\ &= \int_{R^n} \left\{ nF(x, u) - \frac{n-1}{2} f(x, u)u + rF_r(x, u) \right\} dx \\ &\leq 0, \end{aligned}$$

since $u \in NF(\mathbf{R}^n)$.

Thus,

$$\int_{R^n} \left\{ \frac{3}{2} (\Delta u)^2 - \frac{2n-(n+1)p}{2p} |\nabla u|^p - a \frac{1}{2} |\nabla u|^2 + b \frac{1}{2} u^2 \right\} dx = 0$$

Since $u \in D_{n,3}(\mathbf{R}^n)$, $u \equiv 0$.

Remark 2. As an example for $f(\mathbf{x}, u)$, let $f(\mathbf{x}, u) = q(\mathbf{x})|u|^{s-1}u$, $s \geq 1$. Then

$$F(x, u) = \frac{1}{s+1} q(x) |u|^{s+1},$$

where

$$q(x) = q(|x|) = q(r).$$

Assume $q \in E_m(\mathbf{R}^n)$.

For u to be in $F_n(\mathbf{R}^n)$, we need

$$\lim_{R \rightarrow \infty} (R^n \sup_{|x|=R} |F(x, u(x))|) = 0$$

that is,

$$\lim_{R \rightarrow \infty} (R^n \sup_{|x|=R} (q(|x|) |u(x)|^{s+1})) = 0$$

This would be satisfied if

$$\lim_{R \rightarrow \infty} (R^n \sup_{|x|=R} (|x|^m |u(x)|^{s+1})) = 0 \tag{3.3}$$

The above condition (3.3) would be satisfied if u is of decay order (n+m, 0).

As for u to be in $NF(\mathbf{R}^n)$, since

$$nF(x, u) + rF_r(x, u) - \left(\frac{n-1}{2}\right) f(x, u)u = \left[\frac{n}{s+1}q + \frac{r}{s+1}q_r - \left(\frac{n-1}{2}\right)q\right] |u|^{s+1}$$

u would be in $NF(\mathbf{R}^n)$, if $\left(\frac{n}{s+1}q + \frac{r}{s+1}q_r - \frac{n-1}{2}q\right) \leq 0$

Thus, if

$$rq_r \leq \left(\frac{ns - n - s - 1}{2}\right)q, \tag{3.4}$$

then u is in $NF(\mathbf{R}^n)$.

Therefore, if u is of decay order (n + m, 1) and $rq_r \leq \left(\frac{ns - n - s - 1}{2}\right)q$, u satisfies the assumptions of the

theorem on u. In particular, s must be $\geq (n+1)/(n-1)$ if q is a positive constant and s must be $\leq (n+1)/(n-1)$ if q is a negative constant, in order for the inequality (3.4) to hold.

Remark 3. A similar conclusion can be obtained for $f(x, u) = q_1(x)|u|^{\alpha-1}u + q_2(x)|u|^{\beta-1}u$, where $\alpha > \beta \geq 1$.

4. Conclusion

The classification of entire solutions of nonlinear elliptic equations is one of the intensively investigated topics. Our goal is to completely classify all the solutions of the relatively new equation (1.5). The work in this paper is only a first step in this direction. We will carry out more research in this direction in the future.

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