

On the Existence of Global Weak Solutions to 1D Pollutant Transport Model

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Abstract

We consider a one-dimensionnal bilayer model coupling shallow water and Reynolds lubrication equations with a molecular interactions between molecules. These molecular interactions give rise to intermolecular forces, namely the long-range van der Waals forces and short-range Born intermolecular forces. In this paper, an expression will be used to take into account all these intermolecular forces. Our model is a similar model studied in (Roamba, Zabsonré & Zongo, 2017). The model considered is represented by the two superposed immiscible fluids. A similar model was studied in (Zabsonré Lucas & Fernandez-Nieto, 2009) but the authors do not take into account the intermolecular forces. Without hypothesis about the unknowns as in (Roamba, Zabsonré & Zongo, 2017), we show the existence of global weak solution in time in a periodic domain.

Keywords: shallow water equations, bilayer models, viscosity, friction, capillarity, intermolecular forces

1. Introduction

The purpose of this paper is devoted to the existence of global weak solutions to 1D pollutant transport model. This model can be used to simulate for instance the evolution of a pollutant fluid over water. It is a bilayer model of two immiscible fluids where the upper layer can be represented by a Reynolds lubrications model and the lower layer by a shallow water model.

This work follows the work done in (Roamba, Zabsonré & Zongo, 2017). In (Roamba, Zabsonré & Zongo, 2017) as in this present work, we use a model of transport of pollutant in 1D formally derived in (Fernandez-Nieto, Narbona-Reina & Zabsonré 2013). In (Roamba, Zabsonré & Zongo, 2017), the authors showed the existence of global weak solutions of similar model derived in (Fernandez-Nieto, Narbona-Reina & Zabsonré 2013). To lead well this result, the authors considered the condition according to which $h_2 \leq h_1$ (the water layer is more important than the layer of the pollutant). We suppose in this paper the existence of molecular interactions between molecules and this leads us to use the force of Van Der Waals which is given by $V(h_1) = \frac{1}{h_1^3} - \frac{\alpha}{h_1^4}$ ($\alpha > 0$), see (Kitavtsev, Laurençot & Niethammer, 2011; Seemann, Herminghaus & Jacobs, 2001). This force of Van Der Waals allows us to lower the height of water which allows us to get around hypothesis made in (Roamba, Zabsonré & Zongo, 2017). The model studied is read as follow:

$$\partial_t h_1 + \partial_x(h_1 u_1) = 0, \quad (1)$$

$$\begin{aligned} \partial_t(h_1 u_1) + \partial_x(h_1 u_1^2) + \frac{1}{2} g \partial_x h_1^2 - 4\nu_1 \partial_x(h_1 \partial_x u_1) + \frac{u_1}{\beta} - h_1 \partial_x(\sigma \partial_x^2 h_1 - V(h_1)) \\ rgh_1 \partial_x h_2 + rgh_2 \partial_x(h_1 + h_2) + r_1 h_1 |u_1|^2 u_1 = 0. \end{aligned} \quad (2)$$

$$\partial_t h_2 + \partial_x(h_2 u_1) - \varepsilon \partial_x^2 h_2 - \partial_x((ah_2^2 + bh_2^3) \partial_x p_2) = 0. \quad (3)$$

with

$$\partial_x p_2 = \rho_2 g \partial_x(h_1 + h_2) \quad \text{and} \quad V(h_1) = \frac{1}{h_1^3} - \frac{\alpha}{h_1^4} \quad (\alpha > 0), \quad (4)$$

where $(t, x) \in (0, T) \times [0, 1]$.

These equations represent a system composed of two layers of immiscible fluids.

We denote h_1, h_2 respectively, the water and the pollutant heights, u_1 is the water velocity. We introduce the ratio of densities $r = \frac{\rho_2}{\rho_1}$ where ρ_1 and ρ_2 denoted respectively the densities of the water and the pollutant. ν_1 is the kinematic viscosity; g is the constant gravity.

The coefficients σ, r_1 and β are respectively the coefficients of the interfaz tension, quadratic friction and positive slip length parameters. a and b are respectively depending on the friction at the interfaz and coefficient of the viscosity of the pollutant. α and ε are positive constants.

From a theoretical point of view, several studies have been carried out on the existence of global weak solutions of shallow-water equations model. We cite among others some results concerning the existence of global weak solutions of the transport models.

The existence of weak solutions for a viscous sedimentation model is obtained by assuming smallness of the data in (Toumbou, Le Roux & Sene, 2007) in which the viscosity term is the form $-\nu \Delta u$. The authors in (Zabsonré Lucas & Fernandez-Nieto, 2009) studied the stability of global weak solutions for a sediment transport model in two-dimensional with the term $\nu \operatorname{div}(h \nabla u)$.

The stability result is obtained without any restriction on the data and by using a mathematical entropy introduced first in (Bresch & Desjardins, 2003) namely BD entropy. The BD entropy allows us to get regularity on $\partial_x \sqrt{h}$, see **lemma 2**. The authors in (Bresch, Desjardins & Grad-Varet, 2007; Bresch, Desjardins & Lin, 2003; Marche, 2007; Mellet & Vasseur, 2007) use the BD entropy inequality to get the existence results of global weak solutions for Shallow-Water and viscous compressible Navier-Stokes equations. In (Kitavtsev, Laurençot & Niethammer, 2011) the authors proved the existence of global weak solutions for one-dimensional lubrication models that describe the dewetting process of nanoscopic thin polymer films on hydrophobized substrates and take into account a large slippage at the polymer-substrate interface. In their work, the authors have used an intermolecular force that is very crucial in ours.

In the analysis, our contribution is to show the existence of global weak solutions for a similar model studied in (Roamba, Zabsonré & Zongo, 2017) taking into account the presence of intermolecular forces, namely long-range attractive van der Waals and short-range Born repulsive intermolecular forces without any hypothesis on unknown.

We complete the system studied with the initial conditions

$$h_1(0, x) = h_{10}(x), \quad h_2(0, x) = h_{20}(x), \quad (h_1 u_1)(0, x) = \mathbf{m}_0(x) \quad \text{in } (0, 1). \quad (5)$$

$$h_{10} \in L^2(0, 1), \quad h_{10} + h_{20} \in L^2(0, 1), \quad \partial_x(h_{10}) \in L^2(0, 1),$$

$$\partial_x \mathbf{m}_0 \in L^1(0, 1), \quad \mathbf{m}_0 = 0 \quad \text{if } h_{10} = 0, \quad (6)$$

$$\frac{|\mathbf{m}_0|^2}{h_{10}} \in L^1(0, 1), \quad \varphi(h_{10}) \in L^1(0, 1).$$

where φ will be defined later on (see (13)).

Our paper is organized as follows. First of all, we will give in the Section 2, the definition of global weak solutions, secondly, we will establish a classical energy equality and the "mathematical BD entropy", which entails some regularities on the unknown. Thirdly, we will give a proposition allowing us to limit inferiorly the height of water which is very fundamental for the continuation since this limit study gives us additional regularities on the unknowns. Fourthly, we will give an existence theorem of global weak solutions. And finally, we will give the proof of existence Theorem including the limits passage.

2. Main Results

Definition 1. We say that (h_1, h_2, u_1) is a weak solution of (1)-(3), with the initial condition verifying the entropy inequality (12) for all smooth test functions $\phi = \phi(t, x)$ with $\phi(T, \cdot) = 0$, we have:

$$h_{01} \phi(0, \cdot) - \int_0^T \int_0^1 h_1 \partial_t \phi - \int_0^T \int_0^1 h_1 u_1 \partial_x \phi = 0 \quad (7)$$

$$-h_{02} \phi(0, \cdot) - \int_0^T \int_0^1 h_2 \partial_t \phi - \int_0^T \int_0^1 h_2 u_1 \partial_x \phi + \varepsilon \int_0^T \int_0^1 \partial_x h_2 \partial_x \phi$$

$$+ \int_0^T \int_0^1 \left((a h_2^2 + b h_2^3) \partial_x p_2 \right) \partial_x \phi = 0 \quad (8)$$

$$\begin{aligned}
& h_{01}u_{01}\phi(0, \cdot) - \int_0^T \int_0^1 h_1 u_1 \partial_t \phi - \int_0^T \int_0^1 h_1 u_1^2 \partial_x \phi + 4v_1 \int_0^T \int_0^1 h_1 \partial_x u_1 \partial_x \phi \\
& + \frac{1}{\beta} \int_0^T \int_0^1 u_1 \phi + \int_0^T \int_0^1 (\sigma \partial_x^2 h_1 - V(h_1)) \phi \partial_x h_1 + \int_0^T \int_0^1 (\sigma \partial_x^2 h_1 - V(h_1)) h_1 \partial_x \phi \\
& - \frac{1}{2} g \int_0^T \int_0^1 h_1^2 \partial_x \phi - r g \int_0^T \int_0^1 h_2 h_1 \partial_x \phi + r_1 \int_0^T \int_0^1 h_1 |u_1|^2 u_1 \phi \\
& - r g \int_0^T \int_0^1 \phi h_2 \partial_x h_1 - r g \int_0^T \int_0^1 (h_1 + h_2) h_2 \partial_x \phi - r g \int_0^T \int_0^1 (h_1 + h_2) \partial_x h_2 \phi = 0
\end{aligned} \quad (9)$$

Lemma 1. (Energy inequality) For classical solutions of the system (1)-(3), the following inequality holds

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 \left[\frac{1}{2} h_1 |u_1|^2 + U(h_1) + \frac{1}{2} g (1-r) |h_1|^2 + \frac{1}{2} r g |h_1 + h_2|^2 + \frac{1}{2} \sigma |\partial_x h_1|^2 \right] \\
& + 4v_1 \int_0^1 h_1 |\partial_x u_1|^2 + \frac{1}{\beta} \int_0^1 |u_1|^2 + \frac{1}{2} g r \varepsilon \int_0^1 |\partial_x h_2|^2 \\
& + r_1 \int_0^T \int_0^1 h_1 |u_1|^4 + \rho_2 r g^2 \int_0^1 h_2^2 |\partial_x (h_1 + h_2)|^2 (a + b h_2) \leq \frac{1}{2} r g \varepsilon \int_0^1 |\partial_x h_1|^2
\end{aligned} \quad (10)$$

where the potential function U is the indefinite integral of V defined by $U(h_1) = -\frac{1}{2h_1^2} + \frac{\alpha}{3h_1^3}$, $h_1 > 0$.

Proof: First, we multiply the momentum equation by u_1 and we integrate from 0 to 1. We use the mass conservation equation of the first layer for simplification. Then, we obtain

$$\begin{aligned}
& \int_0^1 \frac{1}{2} \partial_t (h_1 u_1^2) + \frac{1}{2} \int_0^1 g \partial_x h_1^2 u_1 - 4 \int_0^1 \partial_x (v_1 h_1 \partial_x u_1) u_1 - \int_0^1 h_1 u_1 \partial_x (\sigma \partial_x^2 h_1 - V(h_1)) + \int_0^1 \frac{u_1^2}{\beta} \\
& + r g \int_0^1 h_1 \partial_x h_2 u_1 + r g \int_0^1 h_2 \partial_x (h_1 + h_2) u_1 + r_1 \int_0^T \int_0^1 h_1 |u_1|^4 = 0
\end{aligned} \quad (11)$$

Now, let us simplify each term:

- $-4 \int_0^1 \partial_x (v_1 h_1 \partial_x u_1) u_1 = 4v_1 \int_0^1 h_1 (\partial_x u_1)^2$
- $-\int_0^1 h_1 u_1 \partial_x (\sigma \partial_x^2 h_1 - V(h_1)) = \int_0^1 \partial_x (h u) (\sigma \partial_x^2 h_1 - V(h_1))$
 $= -\int_0^1 \partial_t h_1 (\sigma \partial_x^2 h_1 - V(h_1))$
 $= \int_0^1 \sigma \partial_{xt} h_1 \partial_x h_1 + \int_0^1 \partial_t (U(h_1))$
 $= \int_0^1 \partial_t \left(\frac{1}{2} \sigma |\partial_x h_1|^2 + U(h_1) \right)$
- $r g \int_0^1 h_1 \partial_x h_2 u_1 = -r g \int_0^1 h_2 \partial_x (h_1 u_1) = r g \int_0^1 h_2 \partial_t h_1$
- $\frac{1}{2} g \int_0^1 \partial_x h_1^2 u_1 = \frac{1}{2} g \frac{d}{dt} \int_0^1 |h_1|^2$
- $r g \int_0^1 h_2 \partial_x (h_1 + h_2) u_1 = -r g \int_0^1 (h_1 + h_2) \partial_x (h_2 u_1)$

The equation for the thin film flow gives us : $\partial_x (h_2 u_1) = -\partial_t h_2 + \varepsilon \partial_x^2 h_2 + \partial_x ((a h_2^2 + b h_2^3) \partial_x p_2)$ and we have:

- $r g \int_0^1 h_2 \partial_x (h_1 + h_2) u_1 = r g \varepsilon \int_0^1 \partial_x h_1 \partial_x h_2 + r g \varepsilon \int_0^1 |\partial_x h_2|^2 + \frac{1}{2} r g \frac{d}{dt} \int_0^1 |h_2|^2$
 $+ \rho_2 r g^2 \int_0^1 h_2^2 |\partial_x (h_1 + h_2)|^2 (a + b h_2) + r g \int_0^1 h_1 \partial_t h_2$

Substituting all these terms in (11), we get (10) by integrating under 0 to T .

Remark 1. 1. Notice that the term in the right of (10) can be controlled using Gronwall's lemma.

2. The term $\int_0^1 \int_0^T U(h_1) |\partial_x h_1|^2$ can be absorbed by $\int_0^1 \int_0^T (\frac{\alpha}{6h_1^3} - \frac{2}{3\alpha^2}) |\partial_x h_1|^2$ thanks to the work done in (Kitavtsev, Laurençot & Niethammer, 2011).

Corollary 1. Let (h_1, h_2, u_1) be a solution of model (1) – (2). Then, thanks to Lemma 1 we have:

$$\begin{aligned} h_1 & \text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ \partial_x h_1 & \text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ (h_1 + h_2) & \text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ \sqrt{h_1} \partial_x u_1 & \text{ is bounded in } L^2(0, T; L^2(0, 1)), \\ u_1 & \text{ is bounded in } L^2(0, T; L^2(0, 1)), \\ h_2 \sqrt{a + bh_2} (\partial_x (h_1 + h_2)) & \text{ is bounded in } L^2(0, T; L^2(0, 1)), \\ h_1^{-1} & \text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ h_1^{-\frac{3}{2}} & \text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ \partial_x h_2 & \text{ is bounded in } L^2(0, T; L^2(0, 1)), \\ \sqrt{h_1} |u_1|^2 & \text{ is bounded in } L^2(0, T; L^2(0, 1)). \end{aligned}$$

We will need furthermore, some additional regularities on h_1 and this will be achieved through an additional BD entropy inequality presented in the next **lemma**.

Lemma 2. For smooth solutions (h_1, h_2, u_1) of model (1) – (3) satisfying the classical energy equality of the **Lemma 1**, we have the following mathematical BD entropy inequality:

$$\begin{aligned} & \frac{d}{dt} \int_0^1 \left[\frac{1}{2} h_1 |u_1 + \partial_x \varphi(h_1)|^2 - \frac{1}{\beta} \varphi(h_1) + \frac{1}{2} g(1-r) |h_1|^2 + \frac{1}{2} r g |h_1 + h_2|^2 + \frac{1}{2} \sigma |\partial_x h_1|^2 + U(h_1) \right] \\ & + \frac{1}{\beta} \int_0^1 |u_1|^2 + 4\nu_1 \int_0^1 (g + gr \frac{h_2}{h_1} + V'(h_1)) |\partial_x h_1|^2 + r g \int_0^1 (\varepsilon + 4\nu_1 \frac{h_2}{h_1}) \partial_x h_1 \partial_x h_2 + 4\nu_1 \sigma \int_0^1 |\partial_x^2 h_1|^2 \\ & + 4\nu_1 r_1 \int_0^T \int_0^1 |u_1|^2 u_1 \partial_x h_1 + gr\varepsilon \int_0^1 |\partial_x h_2|^2 + rg^2 \int_0^1 h_2^2 (a + bh_2) (\partial_x (h_1 + h_2))^2 \leq \frac{1}{2} r g \varepsilon \int_0^1 |\partial_x h_1|^2 \end{aligned} \quad (12)$$

where

$$\varphi(h_1) = 4\nu_1 \log h_1. \quad (13)$$

We need these results to prove the above **lemma**.

Proposition 1. If h_1 has the regularities established in **corollary 2.1**, then there exists constants c_1 and c_2 such as $0 < c_1 < h_1 < c_2$.

Proof: We follow the lines performed in (Kitavtsev, Laurençot & Niethammer, 2011). Using the bound on $\partial_x h_1$ we obtain:

$$h(x, t) - h(y, t) \leq \left| \int_x^y \partial_x h_1(z, t) dz \right| \leq \|x - y\|^{1/2} \|\partial_x h_1(t)\|_2 \leq \frac{C_1}{\sqrt{\sigma}} |x - y|^{1/2}$$

for all $(x, y) \in (0, 1) \times (0, 1)$ and $t \in (0, T)$. Next we integrate the above inequality with respect to $y \in (0, 1)$, readily give the upper bound. To establish the lower bound for h_1 , we combine the $L^\infty(0, T; L^2(0, 1))$ -estimates on $h_1^{-3/2}$ and $\sqrt{\sigma} \partial_x h_1$ just established to obtain a bound on the norm of $1/\sqrt{h_1}$ in $L^\infty(0, T; W^{1,1}(0, 1))$ since

$$\int_0^1 |\partial_x (h_1^{-1/2})| = \frac{1}{2} \int_0^1 \frac{|\partial_x h_1|}{h_1^{3/2}} \leq \frac{1}{2\sqrt{\sigma}} \|\sqrt{\sigma} \partial_x h_1\|_2 \|h_1^{-3/2}\|_2.$$

Due to the continuous embedding of $W^{1,1}(0, 1)$ in $L^\infty(0, 1)$, we get the positive lower bound.

Proof: (**Lemma 2**) Let us multiply the equation (2) by $\partial_x \varphi(h_1)$, integrate with respect to x and use an integration by parts, and using (1), we have:

$$4\nu_1 \int_0^1 (\partial_t u_1 + u_1 \partial_x u_1) \partial_x h_1 + 4\nu_1 g \int_0^1 |\partial_x h_1|^2 + 16\nu_1^2 \int_0^1 h_1 \partial_x u_1 \partial_x \left(\frac{\partial_x h_1}{h_1} \right) + 4\nu_1 \int_0^1 \frac{u_1 \partial_x h_1}{\beta h_1} + 4\nu_1 \sigma \int_0^1 |\partial_x^2 h_1|^2 + 4\nu_1 \int_0^1 V'(h_1) |\partial_x h_1|^2 + 4\nu_1 r g \int_0^1 \partial_x h_2 \partial_x h_1 + 4\nu_1 r g \int_0^1 \frac{h_2}{h_1} |\partial_x h_1|^2 + 4\nu_1 r g \int_0^1 \frac{h_2}{h_1} \partial_x h_2 \partial_x h_1 = 0. \quad (14)$$

On the one hand, a further integration by parts of the first integral of (14), equation (1), and the energy equality (10) give

$$\begin{aligned} & 4\nu_1 \int_0^1 (\partial_t u_1 + u_1 \partial_x u_1) \partial_x h_1 \\ &= 4\nu_1 \left(\frac{d}{dt} \int_0^1 u_1 \partial_x h_1 - \int_0^1 u_1 \partial_{xx}^2 h_1 + \int_0^1 u_1 \partial_x u_1 \partial_x h_1 \right) \\ &= 4\nu_1 \left(\frac{d}{dt} \int_0^1 u_1 \partial_x h_1 - \int_0^1 \partial_x u_1 \partial_x (h_1 u_1) + \int_0^1 u_1 \partial_x u_1 \partial_x h_1 \right) \\ &= 4\nu_1 \left(\frac{d}{dt} \int_0^1 u_1 \partial_x h_1 - \int_0^1 h_1 (\partial_x u_1)^2 \right) \\ &= \frac{d}{dt} \int_0^1 \left[4\nu_1 u_1 \partial_x h_1 + \frac{1}{2} h_1 |u_1|^2 + U(h_1) + \frac{1}{2} g(1-r) |h_1|^2 + \frac{1}{2} r g |h_1 + h_2|^2 + \frac{1}{2} \sigma |\partial_x h_1|^2 \right] \\ &\quad + \frac{1}{\beta} \int_0^1 |u_1|^2 + r g \varepsilon \int_0^1 \partial_x h_1 \partial_x h_2 + r g \varepsilon \int_0^1 |\partial_x h_2|^2 + r g^2 \int_0^1 h_2^2 (a + b h_2) (\partial_x (h_1 + h_2))^2 \end{aligned} \quad (15)$$

On the other hand, we can write the third and the fourth integrals of (14) as

$$\begin{aligned} & \bullet \quad 16\nu_1^2 \int_0^1 \partial_x \left(\frac{\partial_x h_1}{h_1} \right) \partial_x u_1 h_1 = \frac{1}{2} \frac{d}{dt} \int_0^1 h_1 |\varphi(h_1)|^2 \\ & \bullet \quad 4\nu_1 \int_0^1 \frac{u_1 \partial_x h_1}{\beta h_1} = -4\nu_1 \int_0^1 \frac{\partial_x (u_1 h_1)}{\beta h_1} + 4\nu_1 \int_0^1 \frac{\partial_x u_1}{\beta} \\ & \quad = -\frac{1}{\beta} \frac{d}{dt} \int_0^1 \varphi(h_1). \end{aligned}$$

(See (Kitavtsev, Laurençot & Niethammer, 2011)).

Substituting finally the last three identities into (14), we obtain (13).

Remark 2.

In **Lemma 2** all the terms, excepted

$$\int_0^T \int_0^1 |u_1|^2 u_1 \partial_x h_1, \quad \int_0^T \int_0^1 (\varepsilon + 4\nu_1 \frac{h_2}{h_1}) \partial_x h_1 \partial_x h_2 \quad \text{and} \quad \int_0^T \int_0^1 V'(h_1) |\partial_x h_1|^2$$

are controlled since they have the good sign. The control of the term $\int_0^T \int_0^1 |u_1|^2 u_1 \partial_x h_1$ takes inspiration in (Roamba, Zabsonr? & Traor? 2016). The term $\int_0^T \int_0^1 V'(h_1) |\partial_x h_1|^2$ can be absorbed thanks to the work done in (Kitavtsev, Laurençot & Niethammer, 2011). It remains for us to control the term $\int_0^T \int_0^1 (\varepsilon + 4\nu_1 \frac{h_2}{h_1}) \partial_x h_1 \partial_x h_2$.

Proposition 2. There exists a constant C such as $\| (\varepsilon + 4\nu_1 \frac{h_2}{h_1}) \partial_x h_1 \partial_x h_2 \|_{L^2(0,T;L^2(0,1))} \leq C$.

Proof: We have:

$$\int_0^T \int_0^1 \partial_x h_1 \partial_x h_2 \leq \frac{1}{2} \int_0^T \int_0^1 (|\partial_x h_1|^2 + |\partial_x h_2|^2) \leq C' \text{ because } \partial_x h_1, \partial_x h_2 \text{ are in } L^\infty(0, T; L^2(0, 1)). \text{ It remains for us to}$$

limit $\frac{h_2}{h_1} \partial_x h_1 \partial_x h_2$.

Since $\frac{h_2}{h_1} \partial_x h_1 \partial_x h_2 = \partial_x \varphi(h_1) \partial_x h_1 \partial_x h_2$, we can write:

$$\int_0^T \int_0^1 \left| \frac{h_2}{h_1} \partial_x h_1 \partial_x h_2 \right| \leq \frac{1}{2} \int_0^T \int_0^1 |\partial_x \varphi(h_1)|^2 + \frac{1}{2} \int_0^T \int_0^1 |h_2 \partial_x h_2|^2.$$

We will now look at the two terms to the right of the above inequality separately. For the first one, we have:

$$\int_0^T \int_0^1 |\partial_x \varphi(h_1)|^2 = \int_0^T \int_0^1 \frac{|\partial_x h_1|^2}{h_1^2} \leq \int_0^T \int_0^1 |\partial_x h_1|^2 \leq C (h_1 \text{ being bounded inferiorly}) \text{ and } \partial_x h_2 \text{ is in } L^2(0, T; L^2(0, 1)),$$

so $\partial_x \varphi(h_1)$ is in $L^2(0, T; L^2(0, 1))$. For the second one, since $h_2 \in L^\infty(0, T; H^1(0, 1))$ and $\partial_x h_2 \in L^\infty(0, T; L^2(0, 1))$ then, $h_2 \partial_x h_2 \in L^\infty(0, T; L^2(0, 1))$ see (Marche, 2007) for instance. Which completes the proof.

Lemma 3. For classical solutions of the system (1) – (3) with a first component h_1 , we have

$$\frac{1}{4} \int_0^1 h_1 |\partial_x \varphi(h_1)|^2 \leq \frac{1}{2} \int_0^1 h_1 (u + \partial_x \varphi(h_1))^2 + 2E(h_1, h_2, u_1) + \frac{1}{3\alpha^2} \quad (16)$$

with

$$E(h_1, h_2, u_1) = \int_0^1 \left[\frac{1}{2} h_1 |u_1|^2 + U(h_1) + \frac{1}{2} r g |h_1 + \frac{1}{2} g (1-r) |h_1|^2 + h_2|^2 + \frac{1}{2} \sigma |\partial_x h_1|^2 \right].$$

Proof: Using the elementary inequality

$$(y + z)^2 \geq \frac{y^2}{2} - z^2,$$

the fact that

$$U(h_1) \geq \frac{-1}{6\alpha^2} \quad (17)$$

for $h_1 > 0$, and the definition of E , we obtain:

$$\begin{aligned} \int_0^1 h_1 (u_1 + \partial_x \varphi(h_1))^2 &\geq \frac{1}{2} \int_0^1 h_1 |\partial_x \varphi(h_1)|^2 - \int_0^1 h_1 u_1^2 \\ &\geq \frac{1}{2} \int_0^1 h_1 |\partial_x \varphi(h_1)|^2 - 2 \left[E(h_1, h_2, u_1) - \int_0^1 \left(U(h_1) + \frac{1}{2} g (1-r) |h_1|^2 + \frac{1}{2} r g |h_1 + h_2|^2 + \frac{1}{2} \sigma |\partial_x h_1|^2 \right) \right] \\ &\geq \frac{1}{2} \int_0^1 h_1 |\partial_x \varphi(h_1)|^2 - 2E(h_1, h_2, u_1) - \frac{1}{3\alpha^2} \end{aligned}$$

from that the statement of the lemma follows.

Corollary 2. Let (h_1, h_2, u_1) be a solution of model (1) – (3).

Then, thanks to **Lemma 2** and **Lemma 3** we have:

$$\begin{aligned} \sqrt{h_1} &\text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ \partial_x \sqrt{h_1} &\text{ is bounded in } L^\infty(0, T; L^2(0, 1)), \\ \partial_x^2 h_1 &\text{ is bounded in } L^2(0, T; L^2(0, 1)). \end{aligned}$$

Remark 3. 1. In the **Corollary 1**, the estimate

$$\sqrt{h_1} u_1 \text{ is bounded in } L^\infty(0, T; L^2(0, 1))$$

implies,

$$h_1 u_1 \text{ is bounded in } L^\infty(0, T; L^2(0, 1))$$

this leads us

$$\partial_t h_1 \text{ is bounded in } L^\infty(0, T; W^{-1,2}(0, 1)).$$

2. We have the additional regularities thanks to **Corollary 1**:

$$(a) \ h_1 \text{ is bounded in } L^2(0, T; H^1(0, 1)),$$

(b) $h_1 u_1$ is bounded in $L^3(0, T; L^3(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; W^{1,1}(0, 1))$.

Remark 4. We have the following additional regularities:

1. h_1 and h_2 are bounded in $L^\infty(0, T; L^2(0, 1))$.

2. $\sqrt{h_1}$ is bounded in $L^2(0, T; H^1(0, 1))$.

Indeed,

by integrating the mass equation, we obtain directly $\sqrt{h_1^k}$ in $L^\infty(0, T; L^2(\Omega))$. As **Corollary 2** gives us $\partial_x \sqrt{h_1^k}$ in $L^\infty(0, T; L^2(\Omega))$, so $\sqrt{h_1^k}$ is bounded in $L^\infty(0, T; H^1(\Omega))$.

Theorem 1. There exists a global weak solutions to the system (1)-(3) with initial data (5), (6) and satisfying energy equality (10) and energy inequality (12).

3. Convergences

The objective of this section is to prove the **Theorem 1**. Let (h_1^k, h_2^k, u_1^k) be a sequence of weak solutions with initial data

$$h_{1|t=0}^k = h_{1_0}^k, \quad h_{2|t=0}^k = h_{2_0}^k, \quad (h_1^k u_1^k)|_{t=0} = m_0^k$$

such as

$$h_{1_0}^k \longrightarrow h_{1_0} \text{ in } L^1(\Omega), \quad h_{2_0}^k \longrightarrow h_{2_0} \text{ in } L^1(\Omega), \quad m_0^k \longrightarrow m_0 \text{ in } (L^1(\Omega))^2,$$

and satisfies

$$-\frac{1}{\beta} \int_0^1 \varphi(h_{1_0}^k) + \int_0^1 \left[h_{1_0}^k |u_{1_0}^k|^2 + 64v_1^2 |\partial_x \sqrt{h_{1_0}^k}|^2 + \frac{1}{2} g(1-r) |h_{1_0}^k|^2 + \frac{1}{2} r g |h_{1_0}^k + h_{2_0}^k|^2 + \frac{1}{2} \sigma |\partial_x h_{1_0}^k|^2 \right] \leq C.$$

Such approximate solutions can be built by a regularization of capillary effect.

3.1 Strong Convergence of $\sqrt{h_1^k}$, h_1^k and h_2^k

From the **remark 4**:

$$\sqrt{h_1^k} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)). \quad (18)$$

Moreover, using the mass equation, we obtain the following equality:

$$\partial_t \sqrt{h_1^k} = \frac{1}{2} \sqrt{h_1^k} \partial_x u_1^k - \partial_x (\sqrt{h_1^k} u_1^k),$$

which gives that $\partial_t \sqrt{h_1^k}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$.

Applying Aubin-Simon lemma (see (Lions, 1969), (Simon, 1987)), we can extract a subsequence, still denoted $(h_1^k)_{1 \leq k}$, such as

$$\sqrt{h_1^k} \text{ converges strongly to } \sqrt{h_1} \text{ in } C^0(0, T; L^2(0, 1)).$$

According to the **Proposition 1**, we show that

$$\left| h_1^k - h_1 \right| \leq \sqrt{c_2} \left| \sqrt{h_1^k} - \sqrt{h_1} \right| \Rightarrow \left| h_1^k - h_1 \right|^2 \leq c_2 \left| \sqrt{h_1^k} - \sqrt{h_1} \right|^2.$$

This ensures

$$h_1^k \text{ converges strongly to } h_1 \text{ in } L^2(0, T; L^2(0, 1)).$$

We have h_2^k bounded in $L^2(0, T; H^1(0, 1))$. Moreover, we have $\partial_t h_2^k = -\partial_x (h_2^k u_1^k) + \varepsilon \partial_x^2 h_2$.

Since h_2^k is in $L^\infty(0, T; L^2(0, 1))$ and u_1^k is in $L^2(0, T; L^2(0, 1))$, we show that the first term is in $L^2(0, T; W^{-1,1}(0, 1))$. By analogy we prove that the last term is in the same space and we also get $\partial_t h_2^k$ in this space. According to the **Aubin-Simon** lemma, we have:

$$h_2^k \text{ converges strongly to } h_2 \text{ in } L^2(0, T; W^{-1,1}(0, 1)).$$

3.2 Strong Convergence of $h_1^k u_1^k$

We have $h_1^k u_1^k = \sqrt{h_1^k} \sqrt{h_1^k} u_1^k$. Since $\sqrt{h_1^k}$ is bounded in $L^\infty(0, T; L^4(0, 1))$ and $\sqrt{h_1^k} u_1^k$ is bounded in $L^\infty(0, T; L^2(0, 1))$ so $h_1^k u_1^k$ bounded in $L^\infty(0, T; L^{\frac{4}{3}}(0, 1))$. Writing the gradient as follows:

$$\partial_x(h_1^k u_1^k) = h_1^k \partial_x u_1^k + u_1^k \partial_x h_1^k = \sqrt{h_1^k} \sqrt{h_1^k} \partial_x u_1^k + u_1^k \partial_x h_1^k,$$

since the first term is in $L^2(0, T; L^{\frac{4}{3}}(0, 1))$ and thanks to the **Corollary 1**, the second one belongs to $L^\infty(0, T; W^{-1, \frac{4}{3}}(0, 1)) \cap L^2(0, T; L^1(0, 1))$, we have

$$(h_1^k u_1^k)_k \text{ bounded in } L^2(0, T; W^{1,1}(0, 1)).$$

Moreover, the momentum equation (2) enables us to write the time derivation of the water discharge:

$$\begin{aligned} \partial_t(h_1^k u_1^k) = & -\partial_x(h_1^k u_1^{k^2}) - \frac{1}{2} g \partial_x h_1^{k^2} + 4\nu_1 \partial_x(h_1^k \partial_x u_1) + \frac{u_1^k}{\beta} + h_1^k \partial_x(\sigma \partial_x^2 h_1^k - V(h_1^k)) \\ & - r g h_1^k \partial_x h_2^k - r g h_2^k \partial_x(h_1^k + h_2^k) - r_1 h_1^k |u_1^k|^2 u_1^k. \end{aligned}$$

We then study each term:

- $\partial_x(h_1^k (u_1^k)^2) = \partial_x(\sqrt{h_1^k} \sqrt{h_1^k} (u_1^k)^2)$ which is bounded in $L^2(0, T; W^{-1, \frac{4}{3}}(0, 1))$.
- As h_1^k is in $L^\infty(0, T; L^2(0, 1))$, we have:

$$\partial_x[(h_1^k)^2] \text{ is in } L^\infty(0, T; W^{-1,1}(0, 1)).$$

- $\partial_x(h_1^k \partial_x u_1^k)$ is bounded in $L^2(0, T; W^{-1, \frac{4}{3}}(0, 1))$.
- Let us write $h_1^k u_1^k (u_1^k)^2 = \sqrt{h_1^k} u_1^k \sqrt{h_1^k} (u_1^k)^2$, which is in $L^2(0, T; W^{-1,1}(0, 1))$.
- $r g h_1^k \partial_x h_2^k$ is bounded in $L^2(0, T; W^{-1,1}(0, 1))$.
- $r g h_2^k \partial_x(h_1^k + h_2^k)$ is bounded in $L^2(0, T; W^{-1,1}(0, 1))$.
- $(h_1^k \partial_x(\sigma \partial_x^2 h_1^k - V(h_1^k)))_k$ is bounded in $L^2(0, T; H^{-3}(0, 1))$ see (Kitavtsev, Laurençot & Niethammer, 2011).

Thus the sequence $\partial_t(h_1^k u_1^k)$ is bounded in $L^2(0, T; H^{-3}(0, 1))$. Then, applying Aubin-Simon lemma, we obtain,

$$(h_1^k u_1^k)_k \text{ converges strongly to } \mathbf{m} \text{ in } L^2(0, T; L^2(0, 1)).$$

3.3 Strong Convergence of $\sqrt{h_1^k} u_1^k$

Setting $\mathbf{m}^k = h_1^k u_1^k$, we have $\sqrt{h_1^k} u_1^k = \frac{\mathbf{m}^k}{\sqrt{h_1^k}}$. We want to prove the strong convergence for this term. We know that $\frac{\mathbf{m}^k}{\sqrt{h_1^k}}$

is bounded in $L^\infty(0, T; (L^2(\Omega))^2)$; consequently, Fatou lemma reads:

$$\int_{\Omega} \liminf_k \frac{(\mathbf{m}^k)^2}{h_1^k} \leq \liminf_k \int_{\Omega} \frac{(\mathbf{m}^k)^2}{h_1^k} < +\infty.$$

In particular, \mathbf{m} is equal to zero for almost every x where $h_1(t, x)$ vanishes. Then, we can define the limit velocity taking $u_1^k(t, x) = \frac{\mathbf{m}(t, x)}{h_1(t, x)}$ since $h_1(t, x) \neq 0$. So we have a link between the limits $\mathbf{m}(t, x) = h_1(t, x) u_1(t, x)$ and:

$$\int_{\Omega} \frac{(\mathbf{m})^2}{h_1} = \int_0^1 \int_{\Omega} h_1 |u_1|^2 < +\infty.$$

Moreover, we can use Fatou lemma again to write

$$\begin{aligned} \int_0^T \int_{\Omega} h_1 |u_1|^4 & \leq \int_0^T \int_{\Omega} \liminf h_1 |u_1|^4 \leq \liminf \int_0^T \int_{\Omega} h_1 |u_1|^4 \\ & = \liminf \int_0^T \int_{\Omega} \sqrt{h_1} |u_1|^2 \sqrt{h_1} |u_1|^2, \end{aligned}$$

which gives $\sqrt{h_1}|u_1|^2$ in $L^2(0, T; L^2(\Omega))$.

As \mathbf{m}^k and h_1^k converge almost everywhere, the sequence of $\sqrt{h_1^k}u_1^k = \frac{\mathbf{m}^k}{\sqrt{h_1^k}}$ converges almost everywhere to $\sqrt{h_1}u_1 = \frac{\mathbf{m}}{\sqrt{h_1}}$.

Moreover, for all M positive $\sqrt{h_1^k}u_1^k 1_{|u_1^k| \leq M}$ converges to $\sqrt{h_1}u_1 1_{|u_1| \leq M}$ (still assuming that h_1^k does not vanish). Then, almost everywhere, we obtain the convergence of $(\sqrt{h_1^k}u_1^k 1_{|u_1^k| \leq M})_k$

Finally, let us consider the following norm: $\int_0^T \int_0^1 \left| \sqrt{h_1^k}u_1^k - \sqrt{h_1}u_1 \right|^2 \leq$

$$\int_0^T \int_0^1 \left(\left| \sqrt{h_1^k}u_1^k 1_{|u_1^k| \leq M} - \sqrt{h_1}u_1 1_{|u_1| \leq M} \right|^2 + \left| \sqrt{h_1^k}u_1^k 1_{|u_1^k| > M} \right|^2 + \left| \sqrt{h_1}u_1 1_{|u_1| > M} \right|^2 \right)$$

$$\leq 3 \int_0^T \int_0^1 \left| \sqrt{h_1^k}u_1^k 1_{|u_1^k| \leq M} - \sqrt{h_1}u_1 1_{|u_1| \leq M} \right|^2 + 3 \int_0^T \int_0^1 \left| \sqrt{h_1^k}u_1^k 1_{|u_1^k| > M} \right|^2 + 3 \int_0^T \int_0^1 \left| \sqrt{h_1}u_1 1_{|u_1| > M} \right|^2.$$

Since $\sqrt{h_1^k}$ is in $L^2(0, T; L^4(0, 1))$, $\sqrt{h_1^k}u_1^k 1_{|u_1^k| \leq M}$ is bounded in this space. So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:

$$\int_0^1 \left| \sqrt{h_1^k}u_1^k 1_{|u_1^k| > M} \right|^2 \leq \frac{1}{M^2} \int_0^1 h_1^k (u_1^k)^4 \leq \frac{c}{M^2} \quad \text{and} \quad \int_0^1 \left| \sqrt{h_1}u_1 1_{|u_1| > M} \right|^2 \leq \frac{1}{M^2} \int_0^1 h_1 u_1^4 \leq \frac{c}{M^2},$$

for all $M > 0$. When M tends to the infinity, our two integrals tend to zero. Then

$$\sqrt{h_1^k}u_1^k \text{ converges strongly to } \sqrt{h_1}u_1 \text{ in } L^2(0, T; (L^2(\Omega))^2).$$

3.4 Strong Convergence of $\partial_x h_1^k$, $h_2^k \partial_x h_1^k$, $\partial_x^2 h_1^k$, $h_1^k \partial_x^2 h_1$ and $\partial_x h_1^k \partial_x^2 h_1^k$

• We have $\partial_x h_1$ bounded in $L^2(0, T; H^1(0, 1))$ and $\partial_t \partial_x h_1$ is bounded in $L^\infty(0, T; H^{-1}(0, 1))$ since $\partial_t h_1$ is bounded in $L^\infty(0, T; H^{-1}(0, 1))$. Thanks to compact injection of $H^1(0, 1)$ in $L^2(0, 1)$ in one dimension, we have:

$$\partial_x h_1^k \text{ converges strongly to } \partial_x h_1 \text{ in } L^2(0, T; L^2(0, 1)).$$

• The bound of $\partial_x^2 h_1^k$ in $L^2(0, T; L^2(0, 1))$ and $\partial_x h_2^k$ in $L^2(0, T; L^2(0, 1))$ gives us:

$$\partial_x^2 h_1^k \text{ converges weakly to } \partial_x^2 h_1 \text{ in } L^2(0, T; L^2(0, 1)),$$

$$\partial_x h_2^k \text{ converges weakly to } \partial_x h_2 \text{ in } L^2(0, T; L^2(0, 1)).$$

• Thanks to the strong convergence of h_1^k , h_2^k , $\partial_x h_1^k$ and the weak convergence of $\partial_x^2 h_1^k$, we have:

$$(h_2^k \partial_x h_1^k)_k \text{ converges strongly to } h_2 \partial_x h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_1^k \partial_x^2 h_1^k)_k \text{ converges weakly to } h_1 \partial_x^2 h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(\partial_x h_1^k \partial_x^2 h_1^k)_k \text{ converges weakly to } \partial_x h_1 \partial_x^2 h_1 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_1^k \partial_x h_2^k)_k \text{ converges strongly to } h_1 \partial_x h_2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_2^k \partial_x h_2^k)_k \text{ converges strongly to } h_2 \partial_x h_2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$((h_1^k)^2)_k \text{ converges strongly to } h_1^2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$((h_2^k)^2)_k \text{ converges strongly to } h_2^2 \text{ in } L^1(0, T; L^1(0, 1)),$$

$$(h_1^k h_2^k)_k \text{ converges strongly to } h_1 h_2 \text{ in } L^1(0, T; L^1(0, 1)).$$

3.5 Convergences of $h_1^k \partial_x u_1^k$, u_1^k and $h_1^k |u_1^k|^2 u_1^k$

As u_1^k is bounded in $L^2(0, T; L^2(0, 1))$, then $\partial_x u_1^k$ is bounded in $L^2(0, T; W^{-1,2}(0, 1))$.

Then,

$$(u_1^k)_k \text{ converges weakly to } u_1 \text{ in } L^2(0, T; L^2(0, 1)).$$

However, the function $(h_1^k, \partial_x u_1^k) \mapsto h_1^k \partial_x u_1^k$ is a continuous in $L^\infty(0, T; H^1(0, 1)) \times L^2(0, T; W^{-1,2}(0, 1))$ to $L^2(0, T; W^{-1,2}(0, 1))$.

So,

$$(h_1^k \partial_x u_1^k)_k \text{ converges weakly to } h_1 \partial_x u_1 \text{ in } L^2(0, T; H^{-1}(0, 1)).$$

At last, thanks to the strong convergence of $\sqrt{h_1^k} u_1^k$ in $L^2(0, T; L^2(0, 1))$ and above the weak convergence of u_1^k , gives us:

$$(h_1^k |u_1^k|^2 u_1^k)_k \text{ converges weakly to } h_1 |u_1|^2 u_1 \text{ in } L^1(0, T; L^1(0, 1)).$$

3.6 Convergences of $(h_2^k u_1^k)_k$ and $(\partial_x^2 h_2^k)_k$

We know that $\partial_x h_2^k$ is bounded in $L^2(0, T; L^2(0, 1))$ this implies $\partial_x^2 h_2^k$ is in $L^1(0, T; W^{-1,2}(0, 1))$.

So,

$$(\partial_x^2 h_2^k)_k \text{ converges weakly to } \partial_x^2 h_2 \in L^1(0, T; W^{-1,2}(0, 1))$$

To end we have $(u_1^k)_k$ converges weakly to u_1 in $L^2(0, T; L^2(0, 1))$ and the strong convergence of $(h_2^k)_k$ to h_2 , gives us:

$$(h_2^k u_1^k)_k \text{ converges weakly to } h_2 u_1 \text{ in } L^1(0, T; L^1(0, 1)).$$

3.7 Convergence of $\left[\left(a(h_2^k)^2 + b(h_2^k)^3 \right) \partial_x (h_1^k + h_2^k) \right]_k$

We know that $(\partial_x (h_1^k + h_2^k))_k$ converges weakly to $\partial_x (h_1 + h_2)$ in $L^2(0, T; L^2(0, 1))$ and $((a(h_2^k)^2 + b(h_2^k)^3))_k$ converges strongly to $ah_2^2 + bh_2^3$ in $L^1(0, T; L^1(0, 1))$.

So,

$$\left[\left(a(h_2^k)^2 + b(h_2^k)^3 \right) \partial_x (h_1^k + h_2^k) \right]_k \text{ converges weakly to } (ah_2^2 + bh_2^3) \partial_x (h_1 + h_2) \text{ in } L^1(0, T; L^1(0, 1))$$

3.8 Convergences of $(h_1 V(h_1^k))_k$ and $(V(h_1^k) \partial_x h_1^k)_k$

We will begin by studying the convergence of the term $h_1 V(h_1^k)$. We have $h_1 V(h_1^k) = \frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3}$ and

$$\begin{aligned} \left| \frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right| &\leq \left| \frac{1}{(h_1^k)^2} - \frac{1}{h_1^2} \right| + \left| \frac{1}{(h_1^k)^3} - \frac{1}{h_1^3} \right| \\ \left| \frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right| &\leq \frac{|h_1^k - h_1| |h_1^k + h_1|}{(h_1^k)^2 h_1^2} + \frac{|h_1^k - h_1| ((h_1^k)^2 + h_1^k h_1 + h_1^2)}{(h_1^k)^3 h_1^3}. \end{aligned}$$

We use the **Proposition 1** to find two constants η_1 and η_2 such as

$$\left| \frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right| \leq \eta_1 |h_1^k - h_1| + \eta_2 |h_1^k - h_1|.$$

So

$$\left| \frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3} - \left(\frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \right) \right|^2 \leq \eta_3^2 |h_1^k - h_1|^2 \rightarrow 0. \quad \text{with } \eta_3 = 2\max(\eta_1, \eta_2).$$

We have

$$\left(\frac{1}{(h_1^k)^2} - \frac{\alpha}{(h_1^k)^3} \right)_k \text{ converges strongly to } \frac{1}{h_1^2} - \frac{\alpha}{h_1^3} \text{ in } L^2(0, T; L^2(0, 1)).$$

A similar reasoning ensures the strong convergence of $\left(\frac{1}{(h_1^k)^3} - \frac{\alpha}{(h_1^k)^4} \right)_k$ to $\frac{1}{h_1^3} - \frac{\alpha}{h_1^4}$ in $L^2(0, T; L^2(0, 1))$.

The strong convergence of $(\partial_x h_1^k)_k$ in $L^2(0, T; L^2(0, 1))$ gives us

$$(V(h_1^k) \partial_x h_1^k)_k \text{ converges weakly to } V(h_1) \partial_x h_1 \text{ in } L^1(0, T; L^1(0, 1)).$$

4. Conclusion

In this paper, we shown the existence of global weak solutions to 1D pollutant transport model. We have considered the existence of molecular interactions between the molecules and we added the term $\varepsilon \partial_x^2 h_2$ to the level of the transport equation. This has led us to the use of a molecular interaction force in this work. We obtained a result of existence of

weak solutions. For our future works, we intend to tend the coefficient ε (of $\varepsilon \partial_x^2 h_2$) to 0. We will also show the existence of strong global solutions of our model.

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