A Pilot Included Column Mean Vanishing Matrix

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Abstract

In this paper, we study a special matrix used in OFDM technology including the pilot vector. This is based on the property of 'column mean vanishing' and orthogonal columns. We study the spectral decomposition. Using this, we suggest a new method of generating such matrices. Numerical examples are included.

Keywords: pilot, orthogonal columns, column mean vanishing property, SVD

1. Introduction

Recently, there have arisen a large necessity of developing a new technology in wireless communications. An OFDM or its generalization is a big trend (O. Edfors, et al., 1998; Frederiksen, F. B. & Prasad, R., 2002; Myungsup, K. & Kwak, D. Y.). In this paper, we review the algorithm developed in (Myungsup, K. & Kwak, D. Y.) and study some properties of the OFDM matrix. Based on this we propose a simple method to generate the matrix. In the resulting matrix, we see the pilot column has only two nonzero entries which correspond to zero rows, so that the pilot does not interfere with other data.

Definition 1.1. *We say a matrix A has a column mean vanishing (CMV) property if the sum of each column is zero.*

2. Generation of CMV Matrix Having Orthonormal Columns

Let $L = n \gg N$. Recall the scheme introduced in (Myungsup, K. & Kwak, D. Y.):

Algorithm Orth-pilot

1. Given a $N \times (N-2)$ initial matrix K_p with orthonormal columns.

2. Multiply by $L \times N$ matrix $\mathbf{P}$ obtaining $A = \mathbf{P}K_p$.

3. Perform IFFT to obtain $\mathbf{F}^{-1}(\mathbf{P}K_p)$.

4. Subtract the first row from all the rows, the result is $\mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.

5. Perform FFT to get $\mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.

6. Multiply $\mathbf{P}^T$ to obtain $\hat{K} := \mathbf{P}^T \circ \mathbf{F} \circ \mathbf{\Phi} \circ \mathbf{F}^{-1}(\mathbf{P}K_p)$.

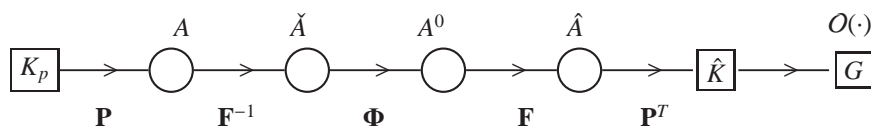7. Let $G = UV^H$ where $U\Sigma V^H$ is the SVD of $\hat{K}$.

Figure 1. Signal flow diagram for matrix generation. $O(\cdot)$ is orthonormalization operator.

Now we will explain more details of the algorithm:

Step (2). Permute and Pad Zeros

Assume $N = 2m + 1$ is odd. Set $M = N - 2$, $L = n \gg N$. Starting from an $N \times M$ initial matrix, we construct an $L \times M$ matrix as follows: Move the last $m + 1$ rows of K_p to the first $m + 1$ rows of K_p. Next fill it with pad with $L - M$ zero rows (called zero padding). This process can be expressed as $\mathbf{P}K_p$ where

$$
\mathbf{P} = \begin{bmatrix} \mathbf{0}_{(m+1)\times m} & \mathbf{I}_{m+1} \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline \mathbf{I}_m & \mathbf{0}_{m\times(m+1)} \end{bmatrix}. \tag{1}
$$

Steps (3) and (4) : IFFT Followed by Subtraction of the First Row

Let us use the notation $K = (k_{ij})$ and $K_1 = (k_{ij}^1) := \mathbf{P}K_p$. Let $\check{K}_1 = \mathbf{F}^{-1}(\mathbf{P}K_p)$ be the inverse FFT of $\mathbf{P}K_p$. By definition of IFFT the first row of $\check{K}_1$ is

$$
\check{\mathbf{k}}_1 = \left[\check{k}_{11}, \check{k}_{12}, \cdots, \check{k}_{1M} \right] = \frac{1}{n} \left[\sum_{i=0}^{n-1} k_{i1}^1, \sum_{i=0}^{n-1} k_{i2}^1, \cdots, \sum_{i=0}^{n-1} k_{iM}^1 \right]. \tag{2}
$$

Hence the matrix after step (4) is

$$
\check{K}_1{}' = \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{21} & \check{k}_{22} & \cdots & \check{k}_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{n,1} & \check{k}_{n,2} & \cdots & \check{k}_{n,M} \end{bmatrix} - \begin{bmatrix} \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \check{k}_{11} & \check{k}_{12} & \cdots & \check{k}_{1M} \end{bmatrix} \equiv \check{K}_1 - \check{K}_1^*. \tag{3}
$$

Here $\check{K}_1^*$ is the matrix all of whose rows are the vector $\check{\mathbf{k}}_1$.

Lemma 2.1. *The sum of all entries of each column of the matrix $\hat{K}$ is zero.*

Proof. Clear from (2) and (3). □

Step (7) - Nearest Orthogonal Matrix

The scheme to find the nearest orthogonal matrix (Higham, N. J. 1986; R. -C. Li., 1995; Ji-Guang, S., 1995; Banerjee, 2014) is given by

$$
G = UV^H = \hat{K} \left(\hat{K}^H \hat{K} \right)^{-1/2}, \tag{4}
$$

where $U\Sigma V^H$ is the SVD of $\hat{K}$.

Theorem 2.1. *The matrix G obtained in step (7) satisfies CMV property:*

Proof. Let $\vec{\mathbf{1}} = [1, \cdots, 1]$. Then by Lemma 2.1, we have

$$
\vec{\mathbf{1}} \cdot \hat{K} = [0, 0, \cdots, 0].
$$

Hence by (4) we see

$$
\vec{\mathbf{1}} \cdot G = \vec{\mathbf{1}} \cdot \hat{K} \left(\hat{K}^H \hat{K} \right)^{-1/2} = [0, 0, \cdots, 0].
$$

□

3. Simplification of the Algorithm

In this paper we simplify the algorithm above. We apply the algorithm to an initial matrix having CMV property. First consider the case $N = 2m + 1$ is odd. We will explain with $m = 2$, general case follows easily from this. Consider the following $N \times (N − 1)$ initial matrix.

$$K^* = \frac{1}{\sqrt{2}} \left[\begin{array}{cc|c|ccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{5}$$

Using this matrix we will generate a pilot included matrix having the desired properties. We remove a specified column (3rd, say) consisting of two 1's and move the next column to the first to get $N \times (N − 2)$ matrix

$$K_{t1} = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \Rightarrow K_{t2} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{6}$$

Then subtract the row vector $\begin{bmatrix} 0 & 2 & 0 & 2 & 0 \end{bmatrix}$ from the center row, to get

$$K_{p,e} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|cccc} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{7}$$

In general it looks like this:

$$\mathbf{K}_{p,e} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccccc} & & & & & & 1 & 1 \\ 1 & & & & & \cdot^{\cdot^{\cdot}} & & \\ \hline & & 1 & 1 & & & & \\ & 1 & 1 & & & & & \\ & -2 & 0 & & & & -2 & \\ & 1 & -1 & & & & & \\ & & & 1 & -1 & & & \\ -1 & & & & & \ddots & & \\ \hline & & & & & & 1 & -1 \end{array} \right] \tag{8}$$

Next, we see the case $N = 2m$ (even). We start from $(N + 1) \times N$ matrix. For example, when we want 8×6 matrix, we start from a 9×8 matrix

$$\frac{1}{\sqrt{2}} \left[\begin{array}{cccc|c|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right] \tag{9}$$

Remove the zero row in the center from (9) and remove first column. Then we remove a specified column (5-th, say) and move the next one to first to get $N \times (N-2)$ matrix

$$
K_{t1} = \frac{1}{\sqrt{2}}
\left[
\begin{array}{ccc|c|ccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}
\right]
\Rightarrow
K_{t2} = \frac{1}{\sqrt{2}}
\left[
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{array}
\right]
\tag{10}
$$

Then subtract the row vector $[0, 0, 1, 0, 1, 0]$ from 4-th and 5-th rows. The resulting matrix is the initial for pilot included matrix.

$$
K_{p,e} = \frac{1}{\sqrt{2}}
\left[
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 \\
0 & -1 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{array}
\right]
\tag{11}
$$

Lemma 3.1. *If the initial matrix K_p ($K_{p,e}$ or $K_{p,o}$) satisfies CMV property, then steps (1)-(7) is simplified as*

Algorithm Orth-pilot-CMV

1. *Given a $N \times (N-2)$ initial CMV matrix K_p with orthonormal columns.*

2. *Let $G = UV^H$ where $U\Sigma V^H$ is the SVD of $\hat{K} = K_p$.*

4. Property of Odd Columns

We assume N is odd. The case of even is similar. Let $\mathbf{k}_i$ and $\mathbf{g}_i$ and denote the i-th column of the matrix K_p and G respectively.

Lemma 4.1. *The odd columns of K_p are orthogonal to all other columns of K_p. As a consequence, for all odd j, the vector $\mathbf{e}_j = [0, \cdots, 1, \cdots, 0]^T$ is an eigenvector of $K_p^H K_p$ corresponding to the eigenvalue 1.*

Proof. The orthogonality of odd columns of K_p comes from that of K^* of (5) since during the transformation of K^* to K_p in (5), (7), the odd columns did not change essentially(only the orders are permuted). Let $K_p = [\mathbf{k}_1, \cdots, \mathbf{k}_{N-2}]$. Then $K_p \mathbf{e}_j = \mathbf{k}_j$ and hence the j-th column of $\hat{K}_p^H \hat{K}_p$ satisfies

$$
\hat{K}_p^H \hat{K}_p \mathbf{e}_j = K_p^H K_p \mathbf{e}_j =
\begin{bmatrix}
\mathbf{k}_1^T \\
\vdots \\
\mathbf{k}_{N-1}^T
\end{bmatrix}
\mathbf{k}_j =
\begin{bmatrix}
\mathbf{k}_1^T \cdot \mathbf{k}_j \\
\vdots \\
\mathbf{k}_{N-1}^T \cdot \mathbf{k}_j
\end{bmatrix}
= \mathbf{e}_j.
\tag{12}
$$

This means that when j is odd, the j-th columns of K_p are orthogonal to all other columns of K_p. Clearly (12) implies the second assertion of the lemma.   □

Example 4.1. *For $N = 5$ we see*

$$
K_p^H K_p = \frac{1}{2}
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -2 & 1 & 0 \\
0 & 1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
\boxed{0} & -2 & \boxed{0} \\
0 & 1 & -1 \\
-1 & 0 & 0
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
2 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 2
\end{bmatrix}
\tag{13}
$$

Note that the zeros in the box keep the odd columns of K_p orthogonal to other columns. In view of (12), $K_p^H K_p$ has two eigenvectors $\mathbf{e}_j$, $j = 1, 3$ corresponding to the eigenvalue 1.

Theorem 4.2. *The odd columns of $G = K_p(K_p^H K_p)^{-1/2}$ are the same as those of K_p.*

Proof. From the spectral decomposition of $K_p = U\Sigma V^H$ we have that of $K_p^H K_p$:

$$K_p^H K_p = V\Sigma^H \Sigma V^H := V\Lambda V^H (V^H = V^{-1}), \tag{14}$$

where by (12) Λ and V have the following form:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdots & & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \Lambda^{-1/2} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdots & & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad V = [\mathbf{e}_1, \mathbf{v}_2, \cdots, \mathbf{e}_{M-1}, \mathbf{v}_M]. \tag{15}$$

Note that for j odd $V\mathbf{e}_j = \mathbf{e}_j$ and for each odd j,

$$\begin{aligned} K_p(K_p^H K_p)^{-1/2}\mathbf{e}_j &= K_p V\Lambda^{-1/2}V^{-1}\mathbf{e}_j \\ &= K_p V\Lambda^{-1/2}\mathbf{e}_j \\ &= K_p V\mathbf{e}_j \\ &= K_p\mathbf{e}_j. \end{aligned}$$

This is the same as j-th column of K_p (normalization does not change even columns). □

5. Numerical Example

Example 5.1. *Let $N = 7$. We have, from initial matrix (7)*

$$G_p = \begin{bmatrix} 0.0000 & -0.1954 & 0.0000 & 0.5117 & 0.7071 \\ 0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5117 & 0.7071 & -0.1954 & 0.0000 \\ 0.0000 & -0.6325 & 0.0000 & -0.6325 & 0.0000 \\ 0.0000 & 0.5117 & -0.7071 & -0.1954 & 0.0000 \\ -0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.1954 & 0.0000 & 0.5117 & -0.7071 \end{bmatrix}$$

Example 5.2 (Even N). *When $N = 8, M = 6$ we start from the initial matrix*

$$K_p = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

to get

$$G = \begin{bmatrix} 0.0000 & 0.0000 & -0.1494 & 0.0000 & 0.5577 & 0.7071 \\ 0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.5577 & 0.7071 & -0.1494 & 0.0000 \\ 0.0000 & 0.7071 & -0.4082 & 0.0000 & -0.4082 & 0.0000 \\ 0.0000 & -0.7071 & -0.4082 & 0.0000 & -0.4082 & 0.0000 \\ 0.0000 & 0.0000 & 0.5577 & -0.7071 & -0.1494 & 0.0000 \\ -0.7071 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.1494 & 0.0000 & 0.5577 & -0.7071 \end{bmatrix}$$

In these matrices, we observe that rows corresponding to the nonzero entries of pilot vector (blue) are zeros (red) except the first entry.

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