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Over Absolute Valued Algebras with Central Element not Necassary Idempotent

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Abstract

We study the absolute valued algebras containing a central element non necessary idempotent. We determine the absolute valued algebras containing a central element if we add some requirements. Also we gives a classification of finite-dimensional absolute valued algebras containing a generalized left unit and central element.

Keywords: Absolute valued algebra, central element, left unit and generalized left unit.

Mathematics Subject Classification: 17A35, 17A36

1. Introduction

The absolute valued algebras are introduced by Ostrowski in 1918. It's the normed algebra A such that $\|xy\| = \|x\|\|y\|$ for all x, y in A. An algebra is called division if and only if R_x and L_x are bijective for all x in A. The category of finite-dimensional absolute valued algebra is a full subcategory of the category of division algebra. If A is a finite dimensional absolute valued algebra, then A has dimension 1, 2, 4 or 8 (Bott, et al., 1958; Kervaire, 1958), A is isotopic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ and the norm of A comes from an inner product(Albert, 1947). We have in (Beslimane & Moutassim, 2011; Diankha, et al., 2013) a classification of absolute valued algebras with left unit and containing a central element. The norm of absolute valued algebra containing a central idempotent c, comes from to an inner product and the isometric map $x \mapsto x^\star := 2(x|c)c - x$ is an involution (El-Mallah, 1990). For $\|u\| = 1$, we recall the following notations $\mathbb{H}_u := \mathbb{H}_{T_{u,\bar{u}}}$, and $\mathbb{O}_u := \mathbb{O}_{T_{u,\bar{u}}}$. Let $a, b \in \mathbb{H}$ such that $\|a\| = \|b\| = 1$, we recall that $\mathbb{H}(a, b) := (\mathbb{H}, \star_1)$, with $x \star_1 y = axyb$ and ${}^\star\mathbb{H}(a, b) := (\mathbb{H}, \star_2)$, with $x \star_2 y = \bar{x}ayb$ (Ramirez, 1999). Let A be an algebra, we note that $Z(A) = \{a \in A : ax = xa \; for \; all \; x \in A\}$. In this work we give a characterization of finite dimensional absolute valued algebra containing a central element. We determine the finite-dimensional absolute algebra containing a genaralized left unit and central element. We classify the absolute valued algebra containing a central element if we add some conditions.

2. Preliminary

Let f, g, f', g' be linear isometries of euclidean space $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ fixing 1, and let $\Phi : \mathbb{A} \to \mathbb{A}$ be a linear mapping. Then it is easy to see that $\Phi : \mathbb{A}_{f,g} \to \mathbb{A}_{f',g'}$ is an algebra isomorphism fixing 1 if and only if $\Phi : \mathbb{A} \to \mathbb{A}$ is an algebra automorphism and $(f', g') = (\Phi \circ f \circ \Phi^{-1}, \Phi \circ g \circ \Phi^{-1})$ (Calderon, et al., 2011).

Let $\mathbb{A}$ be one of the unital absolute valued algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of dimension m. Consider the caley dickson product $\odot$ in $\mathbb{A} \times \mathbb{A}$, we define on the space $\mathbb{A} \times \mathbb{A}$ the product

$$(x, y) \star (x', y') = (f_1(x), f(x)) \odot (g_1(x'), g(y')).$$

With f_1, g_1, f, g be linear isometries of $\mathbb{A}$ and $f_1(1) = g_1(1) = 1$. We obtain a $2m$-dimensional absolute valued real algebra $\mathbb{A} \times \mathbb{A}_{(f_1,f),(g_1,g)}$. The process is called the duplication process (Calderon, & et al., 2011). Note that the algebra is left unit if $g_1 = g = I_{\mathbb{A}}$ and this case we note the algebra by $\mathbb{A} \times \mathbb{A}_{(f_1,f)}$ (Rochdi, 2003).

Theorem 1 *The finite-dimensional absolute valued real algebras with left unit are precisely those of the form $\mathbb{A}_\varphi$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and φ is an isometric of the euclidien espace $\mathbb{A}$ fixed* 1*, and $\mathbb{A}_\varphi$ denotes the absolute valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : \mathbb{A} \to \mathbb{A}$ fixing* 1*, the algebras $\mathbb{A}_\varphi$ and $\mathbb{A}_\phi$ are isomorphic if and only if there exists an algebra automorphism ψ of $\mathbb{A}$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$* ((Rochdi, 2003)).

3. Finite Dimensional Absolute Valued Algebra Containing a Central Element

An element c in A is called central if $L_c = R_c$. In this paragraph, the central element is non necessary idempotent. As A isalternative, Artin's theorem (Schafer, 1996) shows that for any $x, y \in A$, the set $\{x, y, \overline{x}, \overline{y}\}$ is contained in an associative subalgebra of A.

Theorem 2 *Let A be an finite dimensional absolute valued algebra with nonzero central element c. Then A is precisely* $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$ *or of the form* $\mathbb{A}_{\varphi,\psi}$, *with* $\mathbb{A} = \{\mathbb{H}, \mathbb{O}\}$, φ *a linear isometry of the euclidien space* $\mathbb{A}$ *fixing* 1 *and* $\psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi$. *Moreover for* $dim(A) \geq 4$, *if* $\psi = I_A$, *then A is isomorphic to* $\mathbb{H}(c, 1)$ *or* $\mathbb{O}_c$.

Proof. If $dim(A) \leq 2$, the result is clear. Assume now $dim(A) \geq 4$. Then the algebra A is of the form $\mathbb{A}_{\varphi,\psi}$, where ψ, φ are the linear isometries of the euclidien space $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$ such that $\psi(1) = \varphi(1) = 1$ (Calderon, at al., 2011). Using now $x \odot c = c \odot x$, for all x in $\mathbb{A} \Leftrightarrow \varphi(x)\psi(c) = \varphi(c)\psi(x)$, for all x in $\mathbb{A}$.
For $x = 1$, we have $\psi(c) = \varphi(c)$.

$$\begin{aligned}
\varphi(x)\psi(c) = \varphi(c)\psi(x),\ for\ all\ x\ in\ \mathbb{A} \quad &\Rightarrow \quad \varphi(x)\varphi(c) = \varphi(c)\psi(x),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi(x) = \overline{\varphi(c)}\varphi(x)\varphi(c),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi(x) = (L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi)(x),\ for\ all\ x\ in\ \mathbb{A} \\
&\Rightarrow \quad \psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi.
\end{aligned}$$

Moreover if $\psi = I_A$, then A is left unit and $\varphi = L_c \circ R_{\overline{c}}$ (Diankha, et al., 2013). For the algebra $\mathbb{H}_c$, we have the following isomorphism of algebra $\Phi : \mathbb{H}(c, 1) \to \mathbb{H}_c \ \ x \mapsto xc$.▪

Theorem 3 *Let A be an finite-dimensional absolute valued algebra containing a central idempotent c. Then* $c \in \{1\} \cup \{-\frac{1}{2} + u\colon u \in Im(\mathbb{A})$ *and* $\|u\| = \frac{\sqrt{3}}{2}\}$.

Proof. Using Theorem 3.3., A is of the form $\mathbb{A}_{\varphi,\psi}$, where φ is a linear isometric of $\mathbb{A}$ fixing 1 and $\psi = L_{\overline{\varphi(c)}} \circ R_{\varphi(c)} \circ \varphi$. We remark also $\varphi(c) = \psi(c)$, hence $c \odot c = \varphi(c)\psi(c) = \varphi(c)^2$. Assume now $c = \alpha + u \in S(\mathbb{A})$ (with $\mathbb{A} = \mathbb{R} \oplus Im(\mathbb{A})$: Frobenius decomposition). We note that if $u \in 1^{\perp} = Im(\mathbb{A})$, $< 1, \varphi(u) >=< \varphi(1), \varphi(u) >=< 1, u >= 0$. Hence we have $\varphi^n(1^{\perp}) \subseteq 1^{\perp}$, with $n \in \mathbb{N}$ and $\varphi(1^{\perp})^n \in \mathbb{R}$ if and only if $n \in 2\mathbb{N}$.

Hence $c \odot c = c$ and $\|c\| = 1$ are equivalent to $\begin{cases} \alpha^2 + \varphi(u)^2 = & \alpha \quad (1) \\ 2\alpha\varphi(u) = & u \quad (2) \\ \|\varphi(c)\| = & 1 \quad (3) \end{cases}$

The assertions (2) and (3) imply that $\frac{1}{4\alpha^2}u^2 = \alpha^2 - 1$ (4). Otherwise the assertions (1) and (2) implies that $\alpha^2 + (\frac{1}{2\alpha}u)^2 = \alpha$ (5). The equality between (4) and (5) gives $\alpha = 1$ or $-\frac{1}{2}$.
If $\alpha = 1$, this is equivalent to $c = 1$.
Assume now $\alpha = -\frac{1}{2}$, hence $c = -\frac{1}{2} + u$. Then

$$\begin{aligned}
\|c\|^2 &= < -\frac{1}{2} + u, -\frac{1}{2} + u > \\
&= < -\frac{1}{2}, -\frac{1}{2} > + < u, u > \\
&= \frac{1}{4} + \|u\|^2 \\
&= 1
\end{aligned}$$

This implies that $\|u\| = \frac{\sqrt{3}}{2}$.▪

Lemma 1 *Let A be an absolute valued algebras containing a nonzero central element c. The following assertions are equivalent:*

1. $x^2c = x^2$, *for all* $x \in A$
2. *A is finite dimensional and is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *or* $\mathbb{O}$.

Proof. 2) ⇒ 1) is clear.
Now assume 1), Using the equality $(x+c)^2c = (x+c)^2$ for all x in A, we have $(xc-x)c = 0$ for all x in A. Then $L_c = R_c = I_A$ and A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (Urbanik & Wright, 1960).

The group G_2 acts transitively on the sphere $S(Im(\mathbb{O})) := S^6$, that is the mapping $G_2 \to S^6\ \Phi \mapsto \Phi(i)$ is surjective (Postnikov, 1985).

Definition 1 *An element $e \in A$ is called strongly left unit, if it's left unit and square root of right unit: $L_e = R_e^2 = I_A$* (Diouf, 2017).

Theorem 4 *Let A be an absolute valued algebra with strongly left unit and containing a central element c. Then $c \in S(\mathbb{R}) \cup 1^{\perp}$ and A is finite dimensional and isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}(i, 1)$, $\mathbb{O}$ or $\mathbb{O}_i$.*

Proof. It's clear that A is of finite dimensional. If $dim(A) \leq 2$, the result is clear that A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Assume now $dim(A) \geq 4$ and A contains a central element c (Diankha, et al., 2013) proves that A is of the form $\mathbb{A}_{\varphi}$, where $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$, $c \in S(\mathbb{A})$ and $\varphi = L_c \circ R_{\bar{c}}$.
Otherwise we have $R_e^2 = I_A \Leftrightarrow (x \odot 1) \odot 1 = x$. Hence

$$\begin{aligned} x &= (x \odot 1) \odot 1 \\ &= c(cx\overline{c})\overline{c} \\ &= c^2 x\overline{c}^2 \ Artin's\ theorem \end{aligned}$$

The equality $c^2 x = xc^2$, implies that $c^2 \in S(\mathbb{R}) = \{-1, 1\}$.
If $c^2 = 1$, then $c = \pm 1$ and A is isomorphic to $\mathbb{H}$ or $\mathbb{O}$.
If $c^2 = -1$, then $c \in S(Im(\mathbb{A}))$.
There exists $u \in S(Im(\mathbb{A}))$ such that $uc\overline{u} = i$ and let the automorphism $\Phi := T_{u,\overline{u}}$ of $\mathbb{A} = \{\mathbb{H}, \mathbb{O}\}$, with $\Phi^{-1} = T_{\overline{u},u}$. We have

$$\begin{aligned} \Phi \circ T_{c,\bar{c}} \circ \Phi^{-1} &= T_{u,\overline{u}} \circ T_{c,\bar{c}} \circ T_{\overline{u},u} \\ &= T_{uc\overline{u},uc\overline{u}} \\ &= T_{i,\bar{i}} \end{aligned}$$

Then $\mathbb{A}_{T_{c,\bar{c}}}$ and $\mathbb{A}_{T_{i,\bar{i}}}$ are isomorphic (Theorem 2.1) and (Diouf, 2017), we have $\mathbb{H}_{T_{i,\bar{i}}}$ is isomorphic to $\mathbb{H}(i, 1)$.
I's clear that if $dim(A) \geq 2$, theirs algebras can be obtained by using the duplication process.

Corollary 1 *Let A be an absolute valued algebra containing two elements e and c. The following assertions are equivalent:*

1. *e is left unit and c central orthogonal to e,*
2. *A is isomorphic to $\mathbb{C}$, $\mathbb{H}_i$ or $\mathbb{O}_i$.*

Definition 2 *An element e is called generalyzed left unit if it satisfies to $[L_e, L_x] = 0$, for all x in A* (Chandid & Rochdi, 2008).

We give a generalisation of the papier (Diankha, et al., 2013).

Theorem 5 *Let A be an finite dimensional absolute valued algebra contains generalized left unit e and central element c. Then A is precisely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}(a, b)$ or $\mathbb{O}_c$.*

Proof. If $dim(A) \in \{1, 2, 8\}$, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{O}_c$ ((Diankha & all, 2013), (Chandid & Rochdi, 2008)). The algebras $\mathbb{H}(a, b)$ and $^{\star}\mathbb{H}(a, b)$ are the unique four-dimensional absolute valued algebras containing a generalized left unit (Diouf, 2014). Without loss of generality, assume that $\|c\| = 1$.
For the algebra $\mathbb{H}(a, b)$,

$$\begin{aligned} c\ is\ central &\Leftrightarrow x \star_1 c = c \star_1 x,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow axcb = acxb,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow xc = cx,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow c \in Z(\mathbb{H}) \cap S(\mathbb{R}) = \{-1, 1\}. \end{aligned}$$

Then the algebra $\mathbb{H}(a, b)$ contains a central element.
For the algebra $^{\star}\mathbb{H}(a, b)$,

$$\begin{aligned} c\ is\ central &\Leftrightarrow x \star_2 c = c \star_2 x,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow \overline{x}acb = \overline{c}axb,\ for\ all\ x\ in\ \mathbb{H} \\ &\Leftrightarrow \overline{x}ac = \overline{c}ax\ (*),\ for\ all\ x\ in\ \mathbb{H} \end{aligned}$$

For $x = 1$, we have $ac = \overline{c}a$ and $(*)$ imply $\overline{x}ac = acx$ $(**)$, $for\ all\ x\ in\ \mathbb{H}$.
New put $x = ac$, we have $(ac)^2 = ||ac||^2 = ||a||^2||c||^2 = 1$. Hence $ac = \pm 1$ and $(**)$ imply $\overline{x} = x$, for all x in $\mathbb{H}$, which is absurd. Then the algebra ${}^{\star}\mathbb{H}(a, b)$ does not contain a central element..

Proposition 1 *Let A be an absolute valued algebra containing a generalized left unit e and a central idempotent element c such that* $e \in c^{\perp}$. *Then* $A(e, c)$ *is finite dimensional and isomorphic to* $\mathbb{C}$.

Proof. The norm $||.||$ of A comes from an inner product and $x \mapsto x^{\star} := 2 < c, x > c - x$ is an involution (El-Mallah, 1990). Without loss of generality, assume that $||e|| = 1$. We have $ce = ec = ec^2 = c(ec)$. This implies $ec = ce = e$. The element e is orthogonal to c, then $e^2 = -||e||^2 c = -c$ (El-Mallah, 1990).

Problem 1 Let A be an absolute valued algebra containing a generalized left unit e and a central element c. Is A a finite dimensional? This problem is solved partialy if e is idempotent (Calderon, et al., Preprint)..

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