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On The Twistor Method for Treating Differential Equations

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Abstract

In this research we utilized complex structure in R^3 to construct geometrical solutions for Laplace equation, wave equation and monopole equation. The complex space used is the so called mini – twistor space and the solutions in all the above cases is given by a contour integral of a twistor function over a bundle space of one – dimensional complex projective space.

Keywords: Laplace equation, wave equation, monopole equation, the complex space, mini–twistor space, a twistor function.

1. Introduction

Twistors were introduced by Sir Roger Penrose and his associates since 1960, as a new way of describing the geometry of space-time where the ordinary space – time concepts can be translated into twistor terms. The primary geometrical object is not a point in Minkowiski space but a null straight line (a twistor) or, more generally, a twisting congruence of null lines. It turns out that twitor algebra has the same type of universality in relation to the Lorentz group. Thus, twistor theory is a applicable to quantum field theory and free fields of zero- rest- mass. It also formulates other fields such as Yang Mills fields. Recently the twistor programme has been utilized in the integrability of differential equations. It was initiated by Atiyah and Ward (Ward, R. S., 1977; Ward, R. & Tabor, M., 1985) and further extended by Nick Woodhouse, Lionel Mason, George, Sparling and others (Murray, M. K., 2002; Hitchin, N., 1982).

In this paper, we discuss the twistor space and some applications for differential equations representing the non Abielian monopole equation. The structure of this paper is as follows. In section (1) we introduced the basic concepts used in this paper, such as complex projective space $\mathrm{CP_n}$ and holomorphic line bundle. Section (2) dealt with a complex structure on R^3. In this section we defined the twistor space to be the space of oriented lines in R^3, it is infact the non- trivial tangent bundle $\mathrm{T}\,S^2$. Differential equations in R^3 in terms of twistor functions have been treated in section (3). In this section we motivated Penrose transform by introducing the solution of the wave equation by a closed contour integral of a twistor function. Similarly integrating of an appropriate twistor function along a closed contour integral delivers a solution of a harmonic equation. The closed contour on both cases is in the one – dimensional complex projective space. The last section provided a twistor solution to the monopole equation. This equation is infact shown to be the itegrability conditions for linear Lax equations that were interpreted geometrically as null 2- planes that correspond to the points of the twistor space T via the incidence relation given by equation (30) that yields two affine coordinates $(\lambda\,,\eta)$ where $\lambda=\pi_0/\pi_1$ and $\eta=\omega/\pi\text{^}2$ correspond to the homogenous coordinates $(\omega\,,\pi_0\,,\pi_1\,)$ on the twistor space T. Thus we constructed holomorphic vector bundle over the twistor space T.

2. Preliminaries

2.1 Complex Projective Space

Consider the set of all complex lines through the origin. It forms a complex differentiable manifold which is the n-dimensional complex projective space denoted by $\mathrm{CP_n}$. The complex line through z is denoted by [z](Barth, W., et al., 2015), and it is in fact

$$[\mathrm{z}]=[\mathrm{z}^0,\dots,\mathrm{z}^n]=[\lambda\mathrm{z}^0,\dots,\lambda\mathrm{z}^n]\ ,\ \lambda\in\mathrm{C}-\{0\} \tag{1}$$

The numbers $(\mathrm{z}^0,\dots,\mathrm{z}^n)$ are called the homogeneous co-ordinates of the line. It can be shown that $\mathrm{CP_n}$ is a complex differentiable manifold.

2.2 Holomorphic Line Bundle (Jacob, A. & Yau, S.-T., 2014)

A holomorphic line bundle is defined by a triple (M, L, π) such that $\pi: L \to M$ satisfies the following properties: (i) $\pi^{-1}(m) = L_m$ is called a fiber over the base manifold. It is one-dimensional complex vector space

(ii) M is covered with open sets U_α, such that there exist a bi-holomorphic maps $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times C$ and $\psi_{\alpha|L_m}$ is a linear isomorphism

If L is holomorphic bundle then we define the holomorphic section of L as a holomorphic map:

$$\psi: M \to L \text{ with } \psi(m) \in L_m \tag{2}$$

for $m \in U_\alpha \cap U_\beta$ and $\psi_\alpha(m)$, $\psi_\beta(m)$ in L_m .

There are holomorphic maps

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to C - \{0\} \tag{3}$$

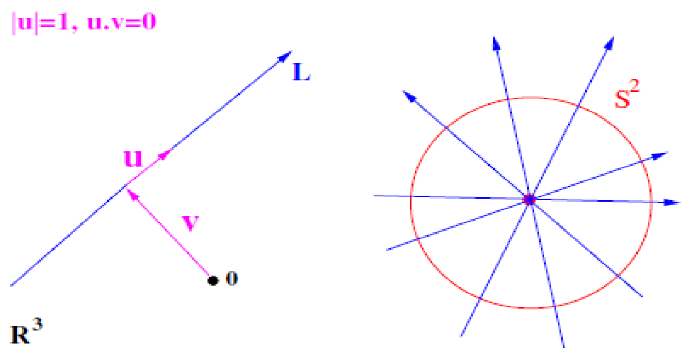
Called the transition functions such that

$$\psi_\beta = g_{\alpha\beta}\psi_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{4}$$

3. Complex Structure on R^3

The three-dimensional Euclidean space R^3 may be represented as a two-dimensional complex manifold which in fact interpreted as a simple twistor space. To see this, consider the space of all oriented lines L in R^3 of the form $L = v + su$ where $s \in R$, u is a unit vector in the direction of L and v is orthogonal to u. Then let

$$T = \{(u, v) \in s^2 \times R^3, u \cdot v = 0\} \tag{5}$$



It is a four –dimensional space which may be regarded as TS^2(Glover, R. & Sawon, J., 2014).

Reversing the orientation of lines induces a map $\tau: T \to T$ given by $\tau(u, v) = (-u, v)$. The points $p = (x, y, z)$ in R^3 correspond to two spheres in T given by τ -invariant maps

$$u \to (u, v(u) = p - (p \cdot u)u) \in T \tag{6}$$

which are sections of the projection $T \to S^2$.

4. Differential Equations and Twistor Functions

On an open set $U \subset T$ not containing the point $(0, 0, 1)$ define a local holomorphic coordinates by

$$\lambda = \frac{u_1 + iu_2}{1 - u_3} \in CP_1 = S^2, \ \eta = \frac{v_1 + iv_2}{1 - u_3} + \frac{u_1 + iu_2}{(1 - u_3)^2} v_3 \tag{7}$$

the corresponding complex coordinates $(\tilde{\lambda}, \tilde{\eta})$ in $\tilde{U}$ containing $(0, 0, 1)$ may also be defined On the overlap region

$$\tilde{\lambda} = 1/\lambda,$$

$$\tilde{\eta} = -\eta/\lambda^2 \tag{8}$$

Then

$$\tau(\lambda,\ \eta) = (-\frac{1}{\lambda}, -\frac{\bar{\eta}}{\bar{\lambda}^2}). \quad (9)$$

From equation (6) we get the τ -invariant holomorphic map

$$\lambda \to (\lambda,\ \eta = (x + iy) + 2\lambda z - \lambda^2(x - iy)). \quad (10)$$

This is map $CP^1 \to T\,CP^1$(Dunajski, M., 2009). For real valued function f on R^3, and an oriented line L in R^3
We define $\phi(L)$ as

$$\phi(L) = \int_L f \quad (11)$$

Equivalently

$$\phi(\alpha_1, \alpha_2, \beta_1, \beta_2) = \int_{-\infty}^{\infty} f(\alpha_1 s + \beta_1, \alpha_2 s + \beta_2, s) ds \quad (12)$$

so we have

$$\frac{\partial^2 \emptyset}{\partial\alpha_1\,\partial\beta_2} - \frac{\partial^2 \emptyset}{\partial\alpha_2\,\partial\beta_1} = 0 \quad (13)$$

We see that smooth solutions to above equation arise from some function on R^3. In twistor theory a twistor function yields solution to a differential equation on space-time. After the change of coordinates

$$\alpha_1 = x + y,$$

$$\alpha_2 = t + z,$$

$$\beta_1 = t - z,$$

$$\beta_2 = x - y \quad (14)$$

Produce the wave equation.

4.1 Penrose Transforms

The following formula for solutions to the wave equation in Minkowski space was provided by penrose

$$\phi(x, y, z, t) = \oint_{\Gamma\subset CP^1} f\,((z + t) + (x + iy)\lambda, (x - iy) - (z - t)\lambda, \lambda)\, d\lambda \quad (15)$$

Here $\Gamma \subset CP^1$ is a closed contour and the function f is holomorphic on CP^1 except some number of poles. Differentiating the RHS verifies that

$$\frac{\partial^2 \emptyset}{\partial t^2} - \frac{\partial^2 \emptyset}{\partial x^2} - \frac{\partial^2 \emptyset}{\partial y^2} - \frac{\partial^2 \emptyset}{\partial z^2} = 0. \quad (16)$$

One could modify a contour and add a holomorphic function inside the contour to f without changing the solution $\emptyset$. The proper description uses sheaf cohomology which considers equivalence classes of functions and contours(Baston, R. J. & M. G., 2015).

4.2 Harmonic Functions(Karp, L., 2016)

To find a harmonic function at $P = (x, y, z)$, restrict a twistor function $f\ (\lambda, \eta)$ defined on $U \cap \tilde{U}$ to a line $\breve{P} = CP^1 = S^2$ and Integrate along a closed contour integral we have

$$\emptyset(x, y, z) = \oint_{\Gamma\subset\breve{P}} f(\lambda, (x + iy) + 2\lambda z - \lambda^2(x - iy))\, d\lambda \quad (17)$$

Then Differentiate under the integral to verify

$$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0 \quad (18)$$

4. 3 Abelian Monopole Equation(Atiyah, M. F. & Hitchin, N., 2014)

We can now consider the A belian monopole equation a function $\emptyset$ and a magnetic potential $A=(A_1,A_2,A_3)$ of the form

$$\nabla\emptyset=\nabla\wedge A \tag{19}$$

This a first order linear equation that is related to our above construction of the twistor contour integral

Geometrically, the one-form $A=A_j dx^j$ is a connection on a U(1)principal bundle over R^3, and $\emptyset$ is a section of the adjoin bundle. Taking the curl of both sides of this equation implies that $\emptyset$ is harmonic, and conversely given a harmonic function $\emptyset$ locally one can always find a one-form A such that the Abelian monopole equation holds.

4.4 Non-abelian Monopoles and Hitchin Correspondence(Shibata, A., et al., 2015)

We can generalize equation (19) using a non Abelian lie group such as SU(n). The generalized equations in R^3 results if we consider the anti- Hermitian $n\times n$ matrices $(A_j,\emptyset)$. The generalized non-abelian monopole equation is given by

$$\frac{\partial\emptyset}{\partial x^j}+\left[A_j,\emptyset\right]=\frac{1}{2}\,\varepsilon_{jkl}F_{kl} \tag{20}$$

where F_{kl} is the non-abelian magnetic field

$$F_{kl}=\frac{\partial A_l}{\partial x^k}-\frac{\partial A_k}{\partial x^l}+[A_k,A_l],\quad k,l=1,2,3 \tag{21}$$

The pair $(A,\emptyset)$ transform as

$$A\to gAg^{-1}-dgg^{-1}\ ,$$

$$\emptyset\to g\emptyset g^{-1}$$

for

$$g=g(x,y,t)\in SU(n) \tag{22}$$

5. Twistor Solution to the Monopole Equation

A brief description of the twistor solution to the monopole equation goes as follows(Shibata, A., et al., 2015):

For the potentials $\left(A_j(X),\emptyset(X)\right)$ we solve the matrix ODE along each line $x(s)=v+su$

$$\frac{dV}{ds}+\left(u^j A_j+i\emptyset\right)V=0 \tag{23}$$

The space of solutions at $p\in R^3$ is a complex vector space C^n, thus giving rise to a complex vector bundle over T with patching matrix $\left(\lambda,\bar{\lambda},\eta,\bar{\eta}\right)\in GL(n,C)$.

The monopole equation (20) on R^3 holds if and only if this vector bundle is holomorphic, i.e. the

Cauchy–Riemann equations

$$\frac{\partial F}{\partial\bar{\lambda},}=0\ ,$$

$$\frac{\partial F}{\partial\bar{\eta}}=0 \tag{24}$$

hold.

We now introduce a metric and a volume forms on $R^{2,1}$

$$h=dx^2-4dudv,$$

$$vol=du\wedge dx\wedge dv \tag{25}$$

where the coordinates $(x, u, v)$are real

With $D_\mu = \partial_\mu + A_\mu$we define $D_\emptyset = d\emptyset + [A, \emptyset]$. The monopole equations become

$$D_x\emptyset = \frac{1}{2} F_{uv}$$

$$D_u\emptyset = F_{ux},$$

$$D_v\emptyset = F_{xv} \tag{26}$$

Where $F_{\mu v} = [D_\mu , D_v]$

We notice that the above equations are the integrability conditions for an overdetermined system of linear Lax equations

$$L_0\Psi = 0,\ L_1\Psi = 0$$

where

$$L_0 = D_u - \lambda(D_x + \emptyset), \quad L_1 = D_x - \emptyset - \lambda D_v \tag{27}$$

And $\Psi = \Psi(x , u , v , \lambda)$ takes values in $GL(n, C)$

For $G = U(n)$, equation (27) provide a gauge $A_v = 0$, and $A_x = -\phi$, with matrix $J : R^{2,1} \to U(n)$ such that

$$A_u = J^{-1}\partial_u J,$$

$$A_x = -\emptyset = \frac{1}{2} J^{-1}\partial_x J \tag{28}$$

The above gauge and (26) yield the integrable chiral model

$$\partial_v(J^{-1}\partial_u J) - \partial_x(J^{-1}\partial_x J) = 0 \tag{29}$$

The Lax representation (27) can be interpreted geometrically: given a pair of real numbers (η, λ) the plane

$$\eta = v + x\lambda + u\lambda^2 \tag{30}$$

is null with respect to the Minkowski metric on $R^{2,1}$, infact all null planes are of this form with $\lambda = \infty$.

We see that M is the two-dimensional complex twistor space $T = TCP^1$ in which points of T are the 2-planes in M via the incidence relation

$$x^{AB}\pi_A\pi_B = \omega \tag{31}$$

Here (ω , π_0 , π_1) are homogeneous coordinates on T as $(\omega, \pi_A) \sim (c^2 w, c\pi_A)$, where $c \in C^*$. In the affine coordinates $\lambda := \frac{\pi 0}{\pi 1}$, $\eta := \omega/(\pi)^2$ equation (31) gives (30).

The homogeneous coordinates are denoted by $\pi_A = (\pi_0 , \pi_1)$, and the two-set covering of CP^1 lifts to a covering of the twistor space T

$$U = \{(w, \pi_A), \pi_1 \neq 0\} ,$$

$$\widetilde{U} = \{(w, \pi_A), \pi_0 \neq 0\} \tag{32}$$

The functions $\lambda = \pi_0/\pi_1$, $\tilde{\lambda} = 1/\lambda$ are the inhomogeneous coordinates in U and $\widetilde{U}$, respectively. It then follows that $\lambda = -\pi^1/\pi^0$

Conversely for a holomorphic vector bundle we can construct a monopole. The construction is as in the following theorm.

5.1 Theorem

There exists a one-to-one correspondence between the gauge equivalence classes of complex solutions to (26) in the complexified Minkowski space M with the gauge group $GL(n, C)$ and holomorphic rank n vector bundles E over the twistor space T which are trivial on the holomorphic sections of $TCP^1 \to CP^1$(Dunajski, M., 2009).

Proof

We first outline how a holomorphic rank n vector bundle with connection (A, ∅) can be constructed. Have (A, ∅) is a solution to (26). Integrating the pair of linear PDEs $L_0V = L_1 V = 0$, where L_0, L_1 are given by (27), we get an n-dimensional vector space to each null plane Z in a complexified Minkowski space. The null plane Z corresponds to a point in T which is a fiber of a holomorphic vector bundle μ: E → T. The fibres of $E \mid L_p$ at Z_0, Z_1 can be identified and therefore E is trivial on each section.

Conversely if E is a vector bundle over T which is trivial on each sectional $L_p \cong CP_1$ we can utilize Birkhoff–Grothendieck theorem to get

$$E \mid L_p = O \oplus O \dots\dots\dots\dots\dots \oplus O \tag{33}$$

where the space of sections restricted to L_p is C^n. Let us now construct a pair (A, ∅) on this bundle that satisfies (26). First cover the twistor space with two open sets U and $\widehat{U}$ so that we have local trivializations

$$\aleph : \mu^{-1}(U) \to U \times C^n,$$

$$\widetilde{\aleph} : \mu^{-1}(\widetilde{U}) \to \widetilde{U} \times C^n \tag{34}$$

The holomorphic patching is simply $F = \widetilde{\aleph} \circ \aleph : C^n \to C^n$ on U $\cap \widehat{U}$. F can be split:

$$F = \widetilde{H}\, H^{-1}, \tag{35}$$

Then $\delta_A F = 0$ implies that

$$H^{-1}\delta_A H = \widetilde{H}^{-1}\delta_A \widetilde{H} = \pi^B \Phi_{AB} \tag{36}$$

Since both sides of the above equation are homogenous of degree 1 in π^A and holomorphic around $\lambda = 0$ and $\lambda = \infty$, respectively we see that the decomposing of Φ_{AB} as

$$\Phi_{AB} = \Phi_{(AB)} + \varepsilon_{AB}\emptyset \tag{37}$$

gives a one-form $\Phi_{AB}dx^{AB}$ and a scalar field $\phi = (1/2)\varepsilon^{AB}\Phi_{AB}$ on the complexified Minkowski space, i.e.

$$\Phi_{AB} = \begin{pmatrix} A_u & A_x + \emptyset \\ A_x - \emptyset & A_v \end{pmatrix} \tag{38}$$

The Lax pair (27) becomes

$$\mathrm{L_A} = \delta_\mathrm{A} + \mathrm{H}^{-1}\delta_\mathrm{A} \tag{39}$$

Where $\delta_A = \pi^B\partial_{AB}$, so that

$$\mathrm{L_A}(\mathrm{H}^{-1}) = -\mathrm{H}^{-1}(\delta_\mathrm{A}\mathrm{H})\mathrm{H}^{-1} + \mathrm{H}^{-1}(\delta_\mathrm{A}\mathrm{H})\mathrm{H}^{-1} = 0 \tag{40}$$

and $\Psi = H^{-1}$ is a solution to the Lax equations regular around $\lambda = 0$. Let us show explicitly that (26) holds. Differentiating (36) with respect to δ_A yields

$$\delta^A(\mathrm{H}^{-1}\delta_\mathrm{A}\mathrm{H}) = -\,(\mathrm{H}^{-1}\delta^A H)(\mathrm{H}^{-1}\delta_\mathrm{A}\mathrm{H}) \tag{41}$$

which holds for all π^A if

$$D_A(C\Phi^A_B) = 0 \tag{42}$$

Where $D_{AC} = \partial_{AC} + \Phi_{AC}$. Equation (42) is the Yang Mills spinor form equation.

The vector bundle E need to be compatible with (10) and therefore $\det F = 1$, this a mounts to Euclidean reality conditions for non abelian monopole. We shoud also have

$$F^*(Z) = F(\tau(Z)) \tag{43}$$

Where $Z \in T$ and * denotes the Hermitian conjugation.

To determent the Lorentzian reality conditions, the bundle must be invariant under the involution (41). Below we shall demonstrate how the gauge choices leading to the integrable chiral model (29) can be made at the twistor level.

Let

$$h = H(x^\mu, \pi^A = O^A)\ ,$$

$$\tilde{h} = \widetilde{H}(x^\mu, \pi^A = l^A) \tag{44}$$

So that

$$\Phi_{A0} = h^{-1} \partial_{A0} h \ ,$$
$$\Phi_{A1} = \tilde{h}^{-1} \partial_{A1} \tilde{h} \tag{45}$$

H is defined up to a multiple by an inverse of a non-singular matrix $g = g(x^{\mu})$ independent of π^A

$$H = Hg^{-1} \ ,$$
$$\widetilde{H} = \widetilde{H} g^{-1} \tag{46}$$

We choose g such that $\widetilde{h} = 1$ so

$$\Phi_{A1} = l^A \, \Phi_{AB} = 0 \tag{47}$$

And

$$\Phi_{AB} = -l_B O^C h^{-1} \partial_{AC} h \tag{48}$$

i.e.

$$A_x + \emptyset = A_v = 0 \tag{49}$$

giving the Ward gauge with $J(x^{\mu}) = h$. With respect to this gauge, the system (42) becomes

$$\partial_1^A \Phi_{A0} = 0 \tag{50}$$

which is (29). The solution is given by

$$J\,(x^{\mu}) = \Psi^{-1}\,(x^{\mu}, \lambda = 0) \tag{51}$$

Where $\Psi = H^{-1}$ is a solution of the Lax pair.

Setting $\mathrm{F} = \exp(\mathrm{f})$ for some f, the nonlinear splitting (35) reduces to the additive splitting of f. This can be done using Cauchy integral formula, taking Γ as a real contour in a rational curve $w = x^{AB} \pi_A \, \pi_B$. The Higgs field satisfies the wave equation and given by

$$\phi = \int_{\Gamma} \frac{\partial f}{\partial w} \, \rho \, . \, d\rho \tag{52}$$

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