Journal of Mathematics Research; Vol. 9, No. 2; April 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

# On the Automorphisms of the Four-dimensional Real Division Algebras

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Received: December 8, 2016 Accepted: January 18, 2017 Online Published: March 21, 2017

doi:10.5539/jmr.v9n2p95 URL: https://doi.org/10.5539/jmr.v9n2p95

**Abstract**

In this paper, we study partially the automorphisms groups of four-dimensional division algebra. We have proved that there is an equivalence between $Der(A) = su(2)$ and $Aut(A) = SO(3)$. For an unitary four-dimensional real division algebra, there is an equivalence between $\dim(Der(A)) = 1$ and $Aut(A) = SO(2)$.

**Keywords:** division algebra, derivations, automorphisms, mutation, isotope.

## 1. Introduction

The finited-dimensional real division algebra A, an actuel problem, takes its origin with the quaternion's discovery $\mathbb{H}$, by Hamilton in 1843. One of the fundementals results of a n-dimensional real division algebra affirms that $n \in \{1, 2, 4, 8\}$(Bott & Milnor, 1958; Kervaire, 1958). For $n \in \{1, 2\}$, the real division algebra $A$ is known (Althoen & Kugler, 1983; Hübner & Peterson, 2004; Dieterich, 2005). However the problem persists for the others cases. One of the method of determining the algebra A is to know its derivations and/or its automorphisms. Benkart and Osborn have classified Lie algebra of derivations $Der(A)$ (Benkart & Osborn, 1981). It's well known that if $A$ is finite dimensional ,then the automorphism group $Aut(A)$ is a group of Lie, whose associated Lie algebra and Lie algebra $Der(A)$ coincide. In dimension 1, the group $Aut(A)$ is trivial. In dimension 2, Dieterich has classified $Aut(A)$, (Dieterich, 2005). However the problem persists for the others cases. This paper is a contribution to the advancement of the determination of the group $Aut(A)$. In the first part, we give some preliminaries results on the automorphism of an algebra A. In the second part, we characterize the 4-dimensional real division algebra $A$ whose $Aut(A) = SO(3)$. Finally, we characterize also an unitary 4-dimensional real division algebra whose $Aut(A) = SO(2)$.

## 2. Preliminary

An algebra is said to be mutation $\alpha$ of $A$ denoted $A^\alpha$, the vector space A which has as product: $x \bullet_\alpha y = \alpha xy + (1-\alpha)yx$, $x$, $y \in A$. If $\lambda, \mu \in \mathbb{R}$ we have $(A^\lambda)^\mu = A^\alpha$ with $\alpha = 2\lambda\mu - \lambda - \mu + 1$. The product of $\mathbb{H}^\lambda$ in the basic $e = 1$, $e_1 = \frac{i}{2\lambda-1}$, $e_2 = \frac{j}{2\lambda-1}$, $e_3 = \frac{k}{2\lambda-1}$, is given by: $ee_n = e_ne = e_n$; $e_n^2 = \dfrac{1}{(2\lambda-1)^2}e$; $e_1e_2 = -e_2e_1 = e_3$; $e_1e_3 = -e_3e_1 = -e_2$; $e_2e_3 = -e_3e_2 = e_1$. Where $\{1, i, j, k\}$ in the canonical basis of the quaternions algebra $\mathbb{H}$. We denote $Aut(A) = \{f : A \longrightarrow A$, linear bijection: $f(xy) = f(x)f(y)$, $\forall x\, y \in A\}$ the automorphism group of $A$. We denote $Der(A) = \{\partial : A \longrightarrow A$, linear mapping: $\partial(xy) = \partial(x)y + x\partial(y)$, $\forall x\, y \in A\}$ the Lie algebra of derivations of $A$. The algebra $A$ is called division if for all $x \in A - \{0\}$ the linears mapping $L_x$ and $R_x$ are bijective. Let $x$, $y \in A$, $[x, y] = xy - yx$ is the commutator of $x$ and $y$. We recall that $I(A) = \{x \in A : \; x^2 = x\}$. Let $\phi$, $\psi$ the linears bijections, we call isotopy of $A$ denoted $A_{\phi,\psi}$, the algebra whose product is: $x \odot y = \phi(x)\psi(y)$, $x, y \in A$.

**Example** *The mutation* $\lambda \in \mathbb{R}$ *of* $\mathbb{C}$, $\mathbb{C}^\lambda$ *is isomorphic to* $\mathbb{C}$. *The mutation* $\frac{1}{2}$ *of* $\mathbb{H}$, $\mathbb{H}^{\frac{1}{2}}$ *is commutative and it's not of division, called the symtrization, one notes it* $\mathbb{H}^+$

**Lemma 1** *Let A be a real algebra, then the following assertions are equivalent:*

1. $f \in Aut(A)$ *and* $[f, \varphi] = [f, \psi] = 0$;

2. $f \in Aut(A_{\phi,\psi})$ *and* $[f, \varphi] = [f, \psi] = 0$.

*Proof.* Let $f \in Aut(A_{\phi,\psi})$, for all $x$ and $y \in A$ we have:

$$\begin{aligned} f(x \odot y) &= f(x) \odot f(y) \\ \Leftrightarrow f(\phi(x) \cdot \psi(y)) &= \varphi(f(x)) \cdot \psi(f(y)) \\ \Leftrightarrow f(\phi(x) \cdot \psi(y)) &= f(\phi(x)) \cdot f(\psi(y)). \quad Then \quad f \in Aut(A). \end{aligned}$$

**Lemma 2** *Let $A$ be an algebra and $\lambda \in \mathbb{R}$, so $Aut(A) \subset Aut(A^{(\lambda)})$. Furthermore if $\lambda \neq \frac{1}{2}$ then $Aut(A) = Aut(A^{(\lambda)})$.*

*Proof.* It's easy to show that $Aut(A) \subset Aut(A^{(\lambda)})$. If $\lambda \neq \frac{1}{2}$, we have $Aut(A^{(\lambda)}) \subset Aut((A^{(\lambda)})^{\frac{\lambda}{2\lambda-1}}) = Aut(A)$

## 3. Characterization of Four-dimensional Real Division Algebra with $SO(3)$ as Its Automorphic Group

In (Benkart & Osborn, $(1981)_2$), we have the following result:

**Theorem 1** *A is an four-dimensional real division algebra with su(2) as its derivation algebra if and only if A has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by* (**1.1**) *for some real numbers $\alpha$, $\beta$, $\gamma$ such that $\alpha\beta\gamma > 0$.*

$$\begin{aligned} &e^2 = e, \quad ee_i = \alpha e_i, \quad e_i e = \beta e_i \quad e_i^2 = -\gamma e \ \ for \ \ all \ \ i \in \{1,2,3\} \\ &e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2 \quad . \end{aligned} \qquad (\mathbf{1.1})$$

**Remark 1** *Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, $y = \lambda'_0 e + \lambda'_1 e_1 + \lambda'_2 e_2 + \lambda'_3 e_3 \in A$, we have:*

$$\begin{aligned} xy &= \Big(\lambda_0\lambda'_0 - \gamma\lambda_1\lambda'_1 - \gamma\lambda_2\lambda'_2 - \gamma\lambda_3\lambda'_3\Big)e + \Big(\alpha\lambda_0\lambda'_1 + \beta\lambda_1\lambda'_0 + \lambda_2\lambda'_3 - \lambda_3\lambda'_2\Big)e_1 \\ &+ \Big(\alpha\lambda_0\lambda'_2 + \beta\lambda_2\lambda'_0 + \lambda_3\lambda'_1 - \lambda_1\lambda'_3\Big)e_2 + \Big(\alpha\lambda_0\lambda'_3 + \beta\lambda_3\lambda'_0 + \lambda_1\lambda'_2 - \lambda_2\lambda'_1.\Big)e_3 \end{aligned}$$

*We defined $\psi_\alpha : A \longrightarrow A$; $\psi_\alpha(\lambda e + u) = \lambda e + \frac{1}{\alpha}u$ with $(\alpha, \lambda) \in \mathbb{R}^* \times \mathbb{R}$ and $u \in lin\{e_1, e_2, e_3\}$.*

**Theorem 2** *Let A be an 4-dimensional real division algebra with su(2) as its derivation algebra, then the isotope $A_{\psi_\alpha,\psi_\beta}$ of A is isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.*

*Proof.* Let $A$ be an algebra of theorem 1. The multiplication of $A_{\psi_\beta,\psi_\alpha}$ in the basis $\{e, e_1, e_2, e_3\}$ is given by (**1.2**)

$$\begin{aligned} &e \odot e = e, \quad e \odot e_i = e_i \odot e = e_i, \quad e_i \odot e_i = -\frac{\gamma}{\alpha\beta}e \ \ for \ \ all \ \ i \in \{1,2,3\} \\ &e_1 \odot e_2 = -e_2 \odot e_1 = \frac{1}{\alpha\beta}e_3, \quad e_2 \odot e_3 = -e_3 \odot e_2 = \frac{1}{\alpha\beta}e_1, \quad e_3 \odot e_1 = -e_1 \odot e_3 = \frac{1}{\alpha\beta}e_2 \quad . \end{aligned} \qquad (\mathbf{1.2})$$

Setting $e' = e$, $e'_1 = \alpha\beta e_1$, $e'_2 = \alpha\beta e_2$ and $e'_3 = \alpha\beta e_3$, we obtain, an algebra isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.

**Corollary 1** *Every four-dimensional real division algebra with su(2) as its derivation algebra is isotope to the algebra $\mathbb{H}^\lambda$.*

**Lemma 3** *Let A be an 4-dimensional real division algebra with su(2) as its derivation algebra. Then A has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by* (**1.1**). *Then we have*

$$\begin{aligned} I(A) &= \Big\{e\Big\} \cup \Big\{\frac{1}{\alpha+\beta}e + \sum_{i=1}^{3}\lambda_i e_i; \sum_{i=1}^{3}\lambda_i^2 = \frac{1-(\alpha+\beta)}{\gamma(\alpha+\beta)^2}\Big\}, \ if \ \ \alpha+\beta \neq 0 \ \ and \ \ \frac{1-(\alpha+\beta)}{\gamma} > 0, \\ I(A) &= \Big\{e\Big\}, \ \ otherwise. \end{aligned}$$

*Proof.* Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:

$$x^2 = x \iff \begin{cases} \lambda_0^2 - \gamma(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \lambda_0 \\ \lambda_i\Big((\alpha+\beta)\lambda_0 - 1\Big) = 0, \quad i \in \{1,2,3\} \end{cases}$$

We obtain $I(A)$ by resolving the system and discussing on $\alpha + \beta$ and $\frac{1-(\alpha+\beta)}{\gamma}$.

**Corollary 2** *Let A be an real algebra of theorem 1. Let u and v ∈ A linearly independent. Then the following assertions are equivalent:*

1. $x \in I(A)$, $u^2 = v^2 = -\gamma x$, $xu = \alpha u$, $ux = \beta u$, $xv = \alpha v$, *and* $vx = \beta v$
2. $x = e$ *and* $u, v \in \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ with\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1\}$.

*Proof.* (1) $\Longrightarrow$ (2) the proof will be reduce in the case $\alpha + \beta \neq 0$ *and* $\frac{1-(\alpha+\beta)}{\gamma} > 0$.
Suppose that $x = \frac{1}{\alpha+\beta} e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in I(A)$ with $\sum_{i=1}^{3} \lambda_i^2 = \frac{1-(\alpha+\beta)}{\gamma(\alpha+\beta)^2}$.
Let $u = \Sigma_{i=0}^{3} \lambda_i' e_i$, and $v = \Sigma_{i=0}^{3} \lambda_i'' e_i \in A$ satisfied the equations of (a). We have:

$u^2 = v^2 = -\gamma x \Longrightarrow \lambda_i = -\frac{\alpha+\beta}{\gamma} \lambda'_0 \lambda'_i = -\frac{\alpha+\beta}{\gamma} \lambda''_0 \lambda''_i \quad i \in \{1, 2, 3\}$. (**E.1**)

And $xu = \alpha u$, $xv = \alpha v \Longrightarrow \lambda'^2_0 = \lambda''^2_0 = \frac{\alpha\gamma\left(1-(\alpha+\beta)\right)}{\beta(\alpha+\beta)^2}$. Consequently $\lambda'_0 = \varepsilon \lambda''_0$ with $\varepsilon^2 = 1$. We have $u = \varepsilon v$ according to (**E.1**), which is adsurd since $u$ and $v$ are linearly independent, then $x = e$. It's easily shown that the equations $u^2 = v^2 = -\gamma e$, $eu = \alpha u$, $ue = \beta u$, $ev = \alpha v$, and $ve = \beta v$ gives $\lambda'_0 = \lambda''_0 = 0$ and $\Sigma_{i=0}^{3} \lambda_i'^2 = \Sigma_{i=0}^{3} \lambda_i''^2 = 1$.

(2) $\Longrightarrow$ (1) the proof is evident.

**Proposition 1** *Let A be a 4-dimensional real division algebra with su(2) as its derivation algebra and* $f \in Aut(A)$, *then* $f(e) = e$ *and* $f(lin\{e_1, e_2, e_3\}) \subseteq lin\{e_1, e_2, e_3\}$. *Moreover* $[f, \psi_\alpha] = 0$.

*Proof.* We notice that $f(e) \in I(A)$ and $f(e_i)$ for all $i \in \{1, 2, 3\}$, satisfy to (a) of corollary 1. Then $f(e) = e$ and $f(e_i) \in lin\{e_1, e_2, e_3\}$. It's easy to show that $[f, \psi_\alpha] = 0$.

**Theorem 3** *Let A be a 4-dimensional real division algebra with su(2) as its derivation algebra, then the following propositions are equivalent:*

1. $Aut(A) \cong SO(3)$;
2. $Der(A) \cong su(2)$;
3. $A_{\psi_\alpha, \psi_\beta}$ *is isomorphic to* $\mathbb{H}^\mu$ *with* $\mu = \frac{1}{2\sqrt{\alpha\beta\gamma}} + \frac{1}{2}$.

*Proof.* (1) $\Longrightarrow$ (2) $Der(A) = Lie(Aut(A)) = Lie(SO(3)) \cong so(3) \cong su(2)$.

(2) $\Longrightarrow$ (3) See the Theorem 2.

(3) $\Longrightarrow$ (1) All automorphisms of $A$ commute with $\psi_\alpha$ and $\psi_\beta$ according to Proposition 1 and also all automorphisms of $A_{\psi_\alpha, \psi_\beta}$ commute with $\psi_\alpha$ and $\psi_\beta$ according to theorem 2, then $Aut(A) = Aut(A_{\psi_\alpha, \psi_\beta})$.
The Lemmas 1 and 2 give $Aut(A) = Aut(A_{\psi_\beta, \psi_\alpha}) = Aut(\mathbb{H}^\mu) = Aut(\mathbb{H}) \cong SO(3)$.

## 4. Characterization Unitary 4-dimensional Real Division Algebra with $SO(2)$ as Its Automorphisms Groups

In (Diabang & all, $(2016)_1$), we have the following result:

**Theorem 4** *Let A be an unital 4-dimensional real division algebra having a non-trivial derivation ∂, then there exists a basis* $\mathcal{B}_1 = \{e, e_1, e_2, e_3\}$ *of A for which the multiplication is given by the table* (**1.3**)*:*

| $\odot$ | $e$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|---|
| $e$ | $e$ | $e_1$ | $e_2$ | $e_3$ |
| $e_1$ | $e_1$ | $-e$ | $\alpha_1 e_2 + \alpha_2 e_3$ | $-\alpha_2 e_2 + \alpha_1 e_3$ |
| $e_2$ | $e_2$ | $\alpha_3 e_2 + \alpha_4 e_3$ | $\alpha_5 e + \alpha_6 e_1$ | $\alpha_7 e + \alpha_8 e_1$ |
| $e_3$ | $e_3$ | $-\alpha_4 e_2 + \alpha_3 e_3$ | $-\alpha_7 e - \alpha_8 e_1$ | $\alpha_5 e + \alpha_6 e_1$ |

(**1.3**)

*for some real numbers* $\alpha_i$, $i \in \{1, \ldots, 7\}$.

**Corollary 3** *Let A be an four-dimensional real unital division algebra A having a non-trivial derivation, then the following propositions are equivalent:*

1. $\alpha_1 = \alpha_3 = \alpha_6 = \alpha_7 = 0$, $\alpha_5 < 0$, $\alpha_2 = -\alpha_4 \neq 0$ *and* $\alpha_8 = -\alpha_2\alpha_5 \neq 0$;
2. *A is quadratic and flexible;*
3. $Der(A) = su(2)$;

4. *$Aut(A) = SO(3)$;*
5. *$A$ is isotope to $\mathbb{H}^\mu$.*

*Proof.* (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) results of Theorem 2 in (Diabang & all, $(2016)_1$).

(3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) results of Theorem 3.

**Lemma 4** *Let $A$ be an unital four-dimensional real division algebra having a non-trivial derivation $\partial$ such that $A$ isn't quadratic or isn't flexible. If $f \in Aut(A)$, then $f(e) = e$ and $f(e_1) = \varepsilon e_1$ with $\varepsilon^2 = 1$.*

*Proof.* $f$ being bijective then for all $y \in A$ there is $x \in A$ such that $f(x) = y$. We have $f(e)y = f(e)f(x) = f(ex) = f(x) = y$ and $yf(e) = f(x)f(e) = f(xe) = f(x) = y$, then $f(e)$ is an unitary element of $A$, therefore $f(e) = e$. The subalgebra of $A$ generated by $f(e_1)$, denoted $< f(e_1) >$, is isomorphic to $B_0 = \ker\partial$. As $\dim(Der(A)) = 1$ then for all $x \in < f(e_1) >$, $\partial(x) = 0$ consequently $f(e_1) \in B_0$. The equation $f(e_1)^2 = -e$ gives $f(e_1) = \varepsilon e_1$.

**Remark 2** *Let $A$ be an unital 4-dimensional real division algebra having a non-trivial derivation. Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:*

$$x^2 = -e \iff \begin{cases} \lambda_0^2 - \lambda_1^2 + \alpha_5(\lambda_2^2 + \lambda_3^2) = -1 & \textbf{(E.2)} \\ 2\lambda_0\lambda_1 + \alpha_6(\lambda_2^2 + \lambda_3^2) = 0, & \textbf{(E.3)} \\ 2\lambda_0\lambda_2 + (\alpha_1 + \alpha_3)\lambda_1\lambda_2 - (\alpha_2 + \alpha_4)\lambda_1\lambda_3 = 0, & \textbf{(E.4)} \\ 2\lambda_0\lambda_3 + (\alpha_2 + \alpha_4)\lambda_1\lambda_2 + (\alpha_1 + \alpha_3)\lambda_1\lambda_3 = 0. & \textbf{(E.5)} \end{cases}$$

$$\begin{aligned} \lambda_2\mathbf{E.4} + \lambda_3\mathbf{E.5} &\Longrightarrow \Big(2\lambda_0 + (\alpha_1 + \alpha_3)\lambda_1\Big)(\lambda_2^2 + \lambda_3^2) = 0 && \textbf{(E.6)} \\ \lambda_3\mathbf{E.4} + \lambda_2\mathbf{E.5} &\Longrightarrow (\alpha_2 + \alpha_4)\lambda_1(\lambda_2^2 + \lambda_3^2) = 0 && \textbf{(E.7)} \end{aligned}$$

*There are four possible cases:*
***Cas 1.*** *If $\alpha_6(\alpha_2 + \alpha_4) \neq 0$, then $x^2 = -e \iff x = \varepsilon e_1$.*
***Cas 2.*** *If $\alpha_6 = 0$ and $\alpha_2 + \alpha_4 \neq 0$, then*

$$x^2 = -e \iff \begin{cases} x \in \{\varepsilon e_1\} \cup \{\lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = -\frac{1}{\alpha_5}\}, & If\ \ \alpha_5 < 0 \\ x = \varepsilon e_1 \quad otherwise, \end{cases}$$

***Cas 3.*** *If $\alpha_6 = \alpha_2 + \alpha_4 = 0$, then*

$$x^2 = -e \iff \begin{cases} x \in \{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ \lambda_1^2 = 1 + \alpha_5\lambda_2^2 + \alpha_5\lambda_3^2\}, & If\ \ \alpha_1 + \alpha_3 = 0 \\ x \in \{\varepsilon e_1\} \cup \{\lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = -\frac{1}{\alpha_5}\}, & If\ \ \alpha_1 + \alpha_3 \neq 0\ \ and\ \ \alpha_5 < 0 \\ x = \varepsilon e_1,\ \ If\ \ \alpha_1 + \alpha_3 \neq 0\ \ and\ \ \alpha_5 \geq 0 \end{cases}$$

***Cas 4.*** *If $\alpha_6 \neq 0$ and $\alpha_2 + \alpha_4 = 0$, then*

$$x^2 = -e \iff \begin{cases} x \in \{\varepsilon e_1\} \cup \{k_o e + \varepsilon\sqrt{k_1} e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ \lambda_2^2 + \lambda_3^2 = \frac{\alpha_1+\alpha_3}{\alpha_6} k_1\},\ \ If\ \ \frac{\alpha_1+\alpha_3}{\alpha_6} > 0, k_1 > 0 \\ x = \varepsilon e_1,\ \ otherwise \end{cases}$$

*with* $k_1 = \dfrac{4\alpha_6}{4\alpha_6 - 4\alpha_5(\alpha_1 + \alpha_3) - \alpha_6(\alpha_1 + \alpha_3)^2}$, $k_0 = -\dfrac{\varepsilon(\alpha_1 + \alpha_3)\sqrt{k_1}}{2}$ *and* $\varepsilon \in \{-1, 1\}$.

**Proposition 2** *Let $A$ be an unital four-dimensional real division algebra having a non-trivial derivation $\partial$ such that $A$ isn't quadratic or isn't flexible. If $f \in Aut(A)$, then*

$$M(f, \mathcal{B}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

*with $\theta \in \mathbb{R}$ so $Aut(A) \cong SO(2)$.*

*Proof.* The lemma 4, gives $f(e) = e$ and $f(e_1) = \varepsilon e_1$. By the definition of the automorphism $f$ and the equations $(E.2), \ldots (E.7)$, we obtains the result.

**Definition 1** *(Unit-duplication process) Let B be an real algebra having an unit element e and let* $\rho, \sigma, \phi, \psi : B \longrightarrow B$ *be linear mappings such that* $\phi(e) = \psi(e) = e$. *We define on the space* $B \times B$ *the produit:*

$$(x, y) \odot (x', y') = (xx' + \rho(\sigma(y')y); y\phi(x') + y'\psi(x)) \qquad \textbf{(2.1)}$$

*The algebra resulting has an unit element* $(e, 0)$ *and contains* $B \times \{0\}$ *as sub-algebra. It is said to be obtained from B and* $\varrho$, *by unit-duplication process and is denoted by* $UDP_B(\rho, \sigma, \phi, \psi)$. *This generalizes the classical Cayley-Dickson process as-well as the process given.*

**Theorem 5** *Let A be an unital 4-dimensional real division algebra having a non-trivial derivation such that A isn't quadratic or isn't flexible, then the following propositions are equivalent:*

1. $Aut(A) \cong SO(2)$;
2. $dim(Der(A)) = 1$;
3. *A is obtained from the unital real algebra* $\mathbb{C}$ *by unit-duplication process.*

*Proof.* (1) $\Longrightarrow$ (2) $Der(A) = Lie(Aut(A)) = Lie(SO(2)) = so(2)$, so $dim(Der(A)) = 1$.

(2) $\Longrightarrow$ (3) See Corollary 1 in (Diabang & all, $(2016)_1$).

(3) $\Longrightarrow$ (1) $A$ admits a nonzero derivation, then $A$ satisfies the hypotheses of the Theoreml 4. The proposition 2 completes the proof.

**Remark 3** *Let A be a finite-dimensional real division algebra, whose Lie algebra of derivations is trivial, then the group* $Aut(A)$ *is finite.*

**Problem 1** *Let A be an four-dimensional real division algebra, whose group* $Aut(A)$ *is finite. Is there an upper limit to the order of the group* $Aut(A)$*?.*

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