On the Automorphisms of the Four-dimensional Real Division Algebras

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Abstract

In this paper, we study partially the automorphisms groups of four-dimensional division algebra. We have proved that there is an equivalence between \( \text{Der}(A) = su(2) \) and \( \text{Aut}(A) = SO(3) \). For an unitary four-dimensional real division algebra, there is an equivalence between \( \dim(\text{Der}(A)) = 1 \) and \( \text{Aut}(A) = SO(2) \).

Keywords: division algebra, derivations, automorphisms, mutation, isotope.

1. Introduction

The finitedimensional real division algebra \( A \), an actual problem, takes its origin with the quaternion’s discovery \( \mathbb{H} \), by Hamilton in 1843. One of the fundaments results of a \( n \)-dimensional real division algebra affirms that \( n \in \{1, 2, 4, 8\} \) (Bott & Milnor, 1958; Kervaire, 1958). For \( n \in \{1, 2\} \), the real division algebra \( A \) is known (Althoen & Kugler, 1983; Hübner & Peterson, 2004; Dieterich, 2005). However the problem persists for the others cases. One of the method of determining the algebra \( A \) is to know its derivations and/or its automorphisms. Benkart and Osborn have classified Lie algebra of derivations \( \text{Der}(A) \) (Benkart & Osborn, 1981). It’s well known that if \( A \) is finite dimensional, then the automorphism group \( \text{Aut}(A) \) is a group of Lie algebra and Lie algebra \( \text{Der}(A) \) coincides. In dimension 1, the group \( \text{Aut}(A) \) is trivial. In dimension 2, Dieterich has classified \( \text{Aut}(A) \) (Dieterich, 2005). However the problem persists for the others cases. This paper is a contribution to the advancement of the determination of the group \( \text{Aut}(A) \). In the first part, we give some preliminaries results on the automorphism of an algebra \( A \). In the second part, we characterize the 4-dimensional real division algebra \( A \) whose \( \text{Aut}(A) = SO(3) \). Finally, we characterize also an unitary 4-dimensional real division algebra whose \( \text{Aut}(A) = SO(2) \).

2. Preliminary

An algebra is said to be mutation \( \alpha \) of \( A \) denoted \( A^\alpha \), the vector space \( A \) which has as product: \( x \bullet y = \alpha xy + (1-\alpha)yx \), \( x, y \in A \). If \( \lambda, \mu \in \mathbb{R} \) we have \( A^\lambda^\mu = A^\mu \) with \( \alpha = 2\lambda \mu - \lambda - \mu + 1 \). The product of \( \mathbb{H}^4 \) in the basic \( e = 1, e_1 = \frac{1}{\sqrt{2}} \), \( e_2 = \frac{-i}{\sqrt{2}} \), \( e_3 = \frac{k}{\sqrt{2}} \), is given by: \( ee_n = e_n e = e_n, e_n^2 = \frac{1}{(2\lambda - 1)} \), \( e_1 e_2 = -e_2 e_1 = e_3, e_1 e_3 = -e_3 e_1 = -e_2, e_2 e_3 = -e_3 e_2 = e_1 \).

Where \( 1, i, j, k \) in the canonical basis of the quaternions algebra \( \mathbb{H} \). We denote \( \text{Aut}(A) = \{ f : A \rightarrow A \} \), linear bijection: \( f(xy) = f(x)f(y), \forall x, y \in A \) the automorphism group of \( A \). We denote \( \text{Der}(A) = \{ \partial : A \rightarrow A \} \), linear mapping: \( \partial(xy) = \partial(x)y + x \partial(y), \forall x, y \in A \) the Lie algebra of derivations of \( A \). The algebra \( A \) is called division if for all \( x \in A \) \( xy = yx \).

Example The mutation \( \lambda \in \mathbb{R} \), \( \mathbb{C}^4 \) is isomorphic to \( \mathbb{C} \). The mutation \( \frac{1}{2} \) of \( \mathbb{H} \), \( \mathbb{H}^2 \) is commutative and it’s not of division, called the symtrization, one notes it \( \mathbb{H}^* \).

Lemma 1 Let \( A \) be a real algebra, then the following assertions are equivalent:

1. \( f \in \text{Aut}(A) \) and \( [f, \varphi] = [f, \psi] = 0; \)

2. \( f \in \text{Aut}(A_{\varphi, \psi}) \) and \( [f, \varphi] = [f, \psi] = 0. \)
Proof. Let $f \in \text{Aut}(A_{\phi, \psi})$, for all $x$ and $y \in A$ we have:

$$f(x \odot y) = f(x) \odot f(y)$$

$$f(\phi(x) \odot \psi(y)) = \phi(f(x)) \cdot \psi(f(y))$$

Then $f \in \text{Aut}(A)$.

**Lemma 2** Let $A$ be an algebra and $\lambda \in \mathbb{R}$, so $\text{Aut}(A) \subset \text{Aut}(A^{(\lambda)})$. Furthermore if $\lambda \neq \frac{1}{2}$ then $\text{Aut}(A) = \text{Aut}(A^{(\lambda)})$.

Proof. It’s easy to show that $\text{Aut}(A) \subset \text{Aut}(A^{(\lambda)})$. If $\lambda \neq \frac{1}{2}$, we have $\text{Aut}(A^{(\lambda)}) \subset \text{Aut}(\lambda Aut(A^{(\lambda)})) = \text{Aut}(A)$

3. Characterization of Four-dimensional Real Division Algebra with $SO(3)$ as Its Automorphic Group

In (Benkart & Osborn, (1981)), we have the following result:

**Theorem 1** A is a four-dimensional real division algebra with $su(2)$ as its derivation algebra if and only if $A$ has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by (1.1) for some real numbers $\alpha, \beta, \gamma$ such that $a\beta \gamma > 0$.

$$
\begin{align*}
  e^2 &= e, \quad ee_i = ae_i, \quad e_i e = \beta e_i, \quad e_i^2 &= -\gamma e_i \\
  e_1 e_2 &= -e_2 e_1 = e_3, \quad e_2 e_3 &= -e_3 e_2 = e_1, \quad e_3 e_1 &= -e_1 e_3 = e_2.
\end{align*}
$$

(1.1)

**Remark 1** Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, $y = \lambda'_0 e + \lambda'_1 e_1 + \lambda'_2 e_2 + \lambda'_3 e_3 \in A$, we have:

$$
\begin{align*}
  xy &= \left(\lambda_0 \lambda'_0 - \gamma \lambda_1 \lambda'_1 - \gamma \lambda_2 \lambda'_2 - \gamma \lambda_3 \lambda'_3\right)e + \left(\alpha \lambda_0 \lambda'_1 + \beta \lambda_1 \lambda'_0 + \lambda_3 \lambda'_2 - \lambda_2 \lambda'_3\right)e_1 \\
  &+ \left(\alpha \lambda_0 \lambda'_2 + \beta \lambda_2 \lambda'_0 + \lambda_3 \lambda'_1 - \lambda_1 \lambda'_3\right)e_2 + \left(\alpha \lambda_0 \lambda'_3 + \beta \lambda_3 \lambda'_0 + \lambda_1 \lambda'_2 - \lambda_2 \lambda'_1\right)e_3.
\end{align*}
$$

We defined $\psi: A \to A$; $x \mapsto (x, \lambda) \in \mathbb{R}^+ \times \mathbb{R}$ and $u \in \text{lin}(e_1, e_2, e_3)$.

**Theorem 2** Let $A$ be an 4-dimensional real division algebra with $su(2)$ as its derivation algebra, then the isotope $A_{\psi, \phi}$ of $A$ is isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{\sqrt{\gamma \alpha \beta}} + \frac{1}{2}$.

Proof. Let $A$ be an algebra of theorem 1. The multiplication of $A_{\psi, \phi}$ in the basis $\{e, e_1, e_2, e_3\}$ is given by (1.2)

$$
\begin{align*}
  e \odot e &= e, \quad e \odot e_i = e_i \odot e = e_i, \quad e_i \odot e_i &= -\frac{\gamma}{\alpha \beta} e_i \quad \text{for all } i \in \{1, 2, 3\} \\
  e_1 \odot e_2 &= -e_2 \odot e_1 = e_3, \quad e_2 \odot e_3 &= -e_3 \odot e_2 = e_1, \quad e_3 \odot e_1 &= -e_1 \odot e_3 = e_2.
\end{align*}
$$

(1.2)

Setting $e' = e$, $e_1' = \alpha \beta e_1$, $e_2' = \alpha \beta e_2$ and $e_3' = \alpha \beta e_3$, we obtain, an algebra isomorphic to $\mathbb{H}^\mu$ with $\mu = \frac{1}{\sqrt{\gamma \alpha \beta}} + \frac{1}{2}$.

**Corollary 1** Every four-dimensional real division algebra with $su(2)$ as its derivation algebra is isotope to the algebra $\mathbb{H}^\mu$.

**Lemma 3** Let $A$ be an 4-dimensional real division algebra with $su(2)$ as its derivation algebra. Then $A$ has a basis $\{e, e_1, e_2, e_3\}$ with multiplication given by (1.1). Then we have:

$$
\begin{align*}
  I(A) &= \left\{e\right\} \cup \left\{\frac{1}{\alpha + \beta} e + \sum_{i=1}^{3} \lambda_i e_i; \sum_{i=1}^{3} \lambda_i^2 = 1 - \frac{\alpha + \beta}{\gamma (\alpha + \beta)^2}\right\}, \text{ if } \alpha + \beta \neq 0 \text{ and } \frac{1 - (\alpha + \beta)}{\gamma} > 0, \\
  I(A) &= \left\{e\right\}, \text{ otherwise.}
\end{align*}
$$

Proof. Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:

$$x^2 = x \iff \begin{cases}
\lambda_0^2 - \gamma (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \lambda_0 \\
\lambda_0 \left((\alpha + \beta) \lambda_0 - 1\right) = 0, \quad i \in \{1, 2, 3\}
\end{cases}$$

We obtain $I(A)$ by resolving the system and discussing on $\alpha + \beta$ and $\frac{1 - (\alpha + \beta)}{\gamma}$.

Corollary 2 Let A be an real algebra of theorem 1. Let u and v ∈ A linearly independent. Then the following assertions are equivalent:

1. $x ∈ I(A)$, $u^2 = v^2 = −γx$, $ux = αu$, $ux = βu$, $v = αv$, and $v = βv$
2. $x = e$ and $u, v ∈ \{A_1e_1 + A_2e_2 + A_3e_3; \text{ with } A_1^2 + A_2^2 + A_3^2 = 1\}$.

Proof. (1) ⇒ (2) the proof will be reduce in the case $α + β ≠ 0$ and $ω = (α + β) / (αβ) > 0$.

Suppose that $x = (α + β) / (αβ) = e + A_1e_1 + A_2e_2 + A_3e_3 ∈ I(A)$ with $\sum_{i=1}^{3} A_i^2 = (1 - (α + β) / (αβ)) / (1 - (α + β) / (αβ))$.

Let $u = \sum_{i=1}^{3} A_i e_i$ and $v = \sum_{i=1}^{3} A_i e_i ∈ A$ satisfied the equations of (a). We have:

$u^2 = v^2 = −γx \Rightarrow λ_i = −αβ / γ λ_i 0 λ_i = −αβ / γ \lambda_i + λ_i + λ_i = 1$. Consequently $λ_i' = e_i₀$ with $e_i = 1$. We have $u = ev$ according to (E.1), which is absurd since $u$ and $v$ are linearly independent, then $x = e$. It’s easily shown that the equations $u^2 = v^2 = −γx$, $ux = αu$, $ue = βu$, $v = αv$, and $v = βv$ gives $λ_i' = λ_i 0 = 0$ and $\sum_{i=0}^{3} λ_i^2 = \sum_{i=0}^{3} λ_i^2 = 1$.

(2) ⇒ (1) the proof is evident.

Proposition 1 Let A be a 4-dimensional real division algebra with $su(2)$ as its derivation algebra and $f ∈ Aut(A)$, then $f(e) = e$ and $f(\text{lin}(e_1, e_2, e_3)) ⊆ \text{lin}(e_1, e_2, e_3)$. Moreover $[f, ψ_a] = 0$.

Proof. We notice that $f(e) ∈ I(A)$ and $f(e_i)$ for all $i ∈ \{1, 2, 3\}$, satisfy to (a) of corollary 1. Then $f(e) = e$ and $f(e_i) ∈ \text{lin}(e_1, e_2, e_3)$. It’s easy to show that $[f, ψ_a] = 0$.

Theorem 3 Let A be a 4-dimensional real division algebra with $su(2)$ as its derivation algebra, then the following propositions are equivalent:

1. $Aut(A) ≅ SO(3)$;
2. $Der(A) ≅ su(2)$;
3. $A_{ψ_0, ω}$ is isomorphic to $\mathbb{H}$ with $μ = \frac{1}{2 - \sqrt{3}}$ + $\frac{1}{2}$.

Proof. (1) ⇒ (2) $Der(A) = Lie(Aut(A)) = Lie(SO(3)) ≅ so(3) ≅ su(2)$.

(2) ⇒ (3) See the Theorem 2.

(3) ⇒ (1) All automorphisms of $A$ commute with $ψ$ and $ψ = ω$ according to Proposition 1 and also all automorphisms of $A_{ψ_0, ψ_0}$ commute with $ψ_0$ and $ψ = ω$ according to theorem 2, then $Aut(A) = Aut(A_{ψ_0, ψ_0})$.

The Lemmas 1 and 2 give $Aut(A) = Aut(A_{ψ_0, ψ_0}) = Aut(\mathbb{H}^0) = Aut(\mathbb{H}) ≅ SO(3)$.

4. Characterization Unitary 4-dimensional Real Division Algebra with $SO(2)$ as Its Automorphisms Groups

In (Diang & all., (2016)_1), we have the following result:

**Theorem 4** Let A be an unital 4-dimensional real division algebra having a non-trivial derivation $δ$, then there exists a basis $B_1 = \{e, e_1, e_2, e_3\}$ of A for which the multiplication is given by the table (1.3):

<table>
<thead>
<tr>
<th>⊗</th>
<th>e</th>
<th>e_1</th>
<th>e_2</th>
<th>e_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>e_1</td>
<td>e_2</td>
<td>e_3</td>
</tr>
<tr>
<td>e_1</td>
<td>e_1</td>
<td>e</td>
<td>α_1 e_2 + α_2 e_3</td>
<td>−α_2 e_2 + α_1 e_3</td>
</tr>
<tr>
<td>e_2</td>
<td>−e</td>
<td>−e</td>
<td>e_2</td>
<td>e_3</td>
</tr>
<tr>
<td>e_3</td>
<td>α_3 e_2 + α_4 e_3</td>
<td>−α_2 e_2 + α_1 e_3</td>
<td>e_2 + e_3</td>
<td>e_3</td>
</tr>
</tbody>
</table>

for some real numbers $α_i, i ∈ \{1, \ldots, 7\}$.

**Corollary 3** Let A be an four-dimensional real unital division algebra A having a non-trivial derivation, then the following propositions are equivalent:

1. $α_1 = α_3 = α_5 = α_7 = 0$, $α_5 < 0$, $α_2 = −α_4 ≠ 0$ and $α_8 = −α_2 α_5 ≠ 0$;
2. A is quadratic and flexible;
3. $Der(A) = su(2)$;
4. $\text{Aut}(A) = SO(3)$; 

5. $A$ is isotope to $\mathbb{H}$.

Proof. $(1) \iff (2) \iff (3)$ results of Theorem 2 in (Diabang & all, (2016))

$(3) \iff (4) \iff (5)$ results of Theorem 3.

**Lemma 4** Let $A$ be an unital four-dimensional real division algebra having a non-trivial derivation $\mathcal{D}$ such that $A$ isn’t quadratic or isn’t flexible. If $f \in \text{Aut}(A)$, then $f(e) = e$ and $f(e_1) = e e_1$ with $e^2 = 1$.

Proof. $f$ being bijective then for all $y \in A$ there is $x \in A$ such that $f(x) = y$. We have $f(e)y = f(e)f(x) = f(ex) = f(x) = y$ and $yf(e) = f(x)f(e) = f(xe) = f(x) = y$, then $f(e)$ is an unitary element of $A$, therefore $f(e) = e$. The subalgebra of $A$ generated by $f(e_1)$, denoted $< f(e_1) >$, is isomorphic to $B_0 = \ker \mathcal{D}$. As $\dim(\text{Der}(A)) = 1$ then for all $x \in < f(e_1) >$, $\mathcal{D}(x) = 0$ consequently $f(e_1) \in B_0$. The equation $f(e_1)^2 = -e$ gives $f(e_1) = e e_1$.

**Remark 2** Let $A$ be an unital 4-dimensional real division algebra having a non-trivial derivation. Let $x = \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in A$, we have:

$$
x^2 = -e \iff \begin{cases} 
\lambda_0^2 - \lambda_1^2 + \alpha_5 (\lambda_2^2 + \lambda_3^2) = -1 & \text{(E.2)} \\
2\lambda_0 \lambda_1 + \alpha_6 (\lambda_2^2 + \lambda_3^2) = 0, & \text{(E.3)} \\
2\lambda_0 \lambda_2 + (\alpha_1 + \alpha_3) \lambda_1 \lambda_2 - (\alpha_2 + \alpha_4) \lambda_1 \lambda_3 = 0, & \text{(E.4)} \\
2\lambda_0 \lambda_3 + (\alpha_2 + \alpha_4) \lambda_1 \lambda_2 + (\alpha_1 + \alpha_3) \lambda_1 \lambda_3 = 0. & \text{(E.5)} 
\end{cases}
$$

\[ \lambda_2 \text{E.4} + \lambda_3 \text{E.5} \implies (2 \lambda_0 + (\alpha_1 + \alpha_3) \lambda_1) (\lambda_2^2 + \lambda_3^2) = 0 \quad \text{(E.6)} \]

\[ \lambda_3 \text{E.4} + \lambda_2 \text{E.5} \implies (\alpha_2 + \alpha_4) \lambda_1 (\lambda_2^2 + \lambda_3^2) = 0 \quad \text{(E.7)} \]

There are four possible cases:

**Case 1.** If $\alpha_6 (\alpha_2 + \alpha_4) \neq 0$, then $x^2 = -e \iff x = e e_1$.

**Case 2.** If $\alpha_6 = 0$ and $\alpha_2 + \alpha_4 \neq 0$, then

$$
x^2 = -e \iff \begin{cases} 
x \in \{ e e_1 \} & \text{If } \alpha_5 < 0 \\
x = e e_1 \text{ otherwise,} & \text{If } \alpha_5 > 0 
\end{cases}
$$

**Case 3.** If $\alpha_6 = \alpha_2 + \alpha_4 = 0$, then

$$
x^2 = -e \iff \begin{cases} 
x \in \{ \lambda_0 e + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 ; \lambda_1^2 = 1 + \alpha_5 \lambda_2^2 + \alpha_5 \lambda_3^2 \} & \text{If } \alpha_1 + \alpha_3 = 0 \\
x \in \{ e e_1 \} & \text{If } \alpha_1 + \alpha_3 \neq 0 \text{ and } \alpha_5 < 0 
\end{cases}
$$

**Case 4.** If $\alpha_6 \neq 0$ and $\alpha_2 + \alpha_4 = 0$, then

$$
x^2 = -e \iff \begin{cases} 
x \in \{ \lambda_0 e + \lambda_1 e_1 + k e \sqrt{k_1} e_1 + \lambda_2 e_2 + \lambda_3 e_3 ; \lambda_1^2 + \lambda_3^2 = \alpha_1 + \alpha_3 k_1 \} & \text{If } \frac{\alpha_1 + \alpha_3}{\alpha_6} > 0, k_1 > 0 \text{ with } k_1 = \frac{4 \alpha_6}{4 \alpha_6 - 4 \alpha_5 (\alpha_1 + \alpha_3) - \alpha_6 (\alpha_1 + \alpha_3)^2}, k_0 = \frac{e (\alpha_1 + \alpha_3) \sqrt{k_1}}{2} \text{ and } e \in \{-1, 1\}. 
\end{cases}
$$

**Proposition 2** Let $A$ be an unital four-dimensional real division algebra having a non-trivial derivation $\mathcal{D}$ such that $A$ isn’t quadratic or isn’t flexible. If $f \in \text{Aut}(A)$, then

$$
M(f, B_1) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos(\theta) & -\sin(\theta) \\
0 & 0 & \sin(\theta) & \cos(\theta)
\end{pmatrix}.
$$

with $\theta \in \mathbb{R}$ so $\text{Aut}(A) \cong SO(2)$.

Proof. The lemma 4, gives $f(e) = e$ and $f(e_1) = e e_1$. By the definition of the automorphism $f$ and the equations (E.2), . . . (E.7), we obtain the result.
Theorem 5 Let $A$ be an unital 4-dimensional real division algebra having a non-trivial derivation such that $A$ isn’t quadratic or isn’t flexible, then the following propositions are equivalent:

1. $\text{Aut}(A) \cong S O(2)$;
2. $\dim(\text{Der}(A)) = 1$;
3. $A$ is obtained from the unital real algebra $C$ by unit-duplication process.

Proof. (1) $\implies$ (2) $\text{Der}(A) = \text{Lie} (\text{Aut}(A)) = \text{Lie}(SO(2)) = so(2)$, so $\dim(\text{Der}(A)) = 1$.

(2) $\implies$ (3) See Corollary 1 in (Diabang et al., 2016).

(3) $\implies$ (1) $A$ admits a nonzero derivation, then $A$ satisfies the hypotheses of the Theorem 4. The proposition 2 completes the proof.

Remark 3 Let $A$ be a finite-dimensional real division algebra, whose Lie algebra of derivations is trivial, then the group $\text{Aut}(A)$ is finite.

Problem 1 Let $A$ be a four-dimensional real division algebra, whose group $\text{Aut}(A)$ is finite. Is there an upper limit to the order of the group $\text{Aut}(A)$?

References

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