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Bounds for Covering Symmetric Convex Bodies by a Lattice Congruent to a Given Lattice

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Abstract

In this paper, we focus on lattice covering of centrally symmetric convex body on $\mathbb{R}^2$. While there is no constraint on the lattice in many other results about lattice covering, in this study, we only consider lattices congruent to a given lattice to retain more information on the lattice. To obtain some upper bounds on the infimum of the density of such covering, we will say a body is a coverable body with respect to a lattice if such lattice covering is possible, and try to suggest a function of a given lattice such that any centrally symmetric convex body whose area is not less than the function is a coverable body. As an application of this result, we will suggest a theorem which enables one to apply this to a coverable body to suggesting an efficient lattice covering for general sets, which may be non-convex and may have holes.

Keywords: lattice covering, centrally symmetric convex body, density of covering, minkowski sum

1. Introduction

The covering problem of centrally symmetric convex bodies, especially related to the density of covering, is a famous problem in discrete geometry. In this paper, we will deal with lattice coverings, which is fundamental when we deal with centrally symmetric convex bodies. For a body A and a lattice Λ, $C = \{A + \lambda | \lambda \in \Lambda\}$ is called a lattice arrangement. If the members of C cover the whole plane, C is called a lattice covering. The density of a lattice covering can be expressed as $\frac{S(A)}{\det \Lambda}$ (Pach & Agarwal, 2011), where $S(A)$ is the area of A and $\det \Lambda$ is the area of the smallest lattice parallelogram of Λ. There are many studies about the upper bounds on the infimum of the density of lattice covering when A is a given body and Λ is any lattice. Because we may choose an appropriate Λ for minimizing the density of covering, the upper bounds are near 1(Fary, 1950). Especially when A is a centrally symmetric convex body, it is well known that it is $\frac{2\pi}{\sqrt{27}} \simeq 1.2092$ (Pach & Agarwal, 2011). In this study, we will consider the same problem when A is a given centrally symmetric convex body and Λ is any lattice congruent with a given lattice Λ_0. Since the condition of Λ is stronger, this upper bound is a lot bigger than 1.2092. This cannot be a constant, since it can be arbitrarily big depending on the given lattice. Thus, we aim to suggest a function of Λ_0 and $S(A)$ which is always less than or equal to

$$\inf_{A+\Lambda=\mathbb{R}^2, \Lambda \equiv \Lambda_0} \frac{S(A)}{\det \Lambda}$$

This is equivalent to suggesting a function f of lattice Λ such that for every centrally symmetric convex body A whose area is not less than $f(\Lambda)$, there exists $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$. We will call A a coverable body with respect to Λ if A is a centrally symmetric convex body and there exists a lattice $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$.

To suggest the function f, we will first prove some properties of centrally symmetric convex bodies. Then, we will define several new functions related to Λ and prove some properties of them. Using these, we will prove the main result of this paper, which gives the function f.

The condition that the lattice is congruent to a given lattice can be used to suggesting an efficient lattice covering of general sets which need not be convex and may have holes. This will be discussed in the application chapter of this paper.

2. Results

2.1 Geometric Properties of Centrally Symmetric Convex Bodies

In this section, some properties of centrally symmetric convex bodies, which are important lemmas for the main results, are suggested.

The next lemma states a method to determine whether a given set A satisfies $A + \Lambda = \mathbb{R}^2$.

Lemma 1. *Given a lattice $\Lambda \subset \mathbb{R}^2$, the followings hold:*

(i) Given a closed connected set A, if $A + \Lambda = \mathbb{R}^2$, there exists $\Lambda' \equiv \Lambda$ and a lattice triangle XYZ of Λ' such that $X, Y, Z \in A$.

(ii) Given a centrally symmetric convex set A, if there exists a lattice triangle $XYZ \subset A$, $A + \Lambda = \mathbb{R}^2$.

Proof. (i) For any set S, denote its boundary by ∂S. For any two distinct points P, Q, $\overleftrightarrow{PQ}$ denotes the line containing both of them, and $\overline{PQ}$ may denote either the segment connecting P, Q or the length of such segment.

Since A is closed, there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $(A + \lambda_1) \cap (A + \lambda_2) \neq \emptyset$. Let L be $\overleftrightarrow{\lambda_1\lambda_2} \cap \Lambda$. Let A_1 be a connected component of $A + L$ which includes $A + \{\lambda_1, \lambda_2\}$. Since $A + \{\lambda_1, \lambda_2\} \subset A_1$ and $\partial A_1 \subset \partial(A + L) \subset \partial A + L$, it can be shown that there exist $u, v \in L$ such that $(\partial A + u) \cap (\partial A + v) \cap \partial A_1 \neq \emptyset$. Let p be an element of the intersection. Then since $p \in \partial A_1 \subset \partial(A + L)$, any neighborhood of p contains a point p' such that $p' \notin A + L$, while $p' \in \mathbb{R}^2 = A + \Lambda$. Then $p \in A + (\Lambda \setminus L)$, there exists $w \in \Lambda \setminus L$ such that $p \in A + w$. Then $p \in A + u, A + v, A + w$, thus $-u + p, -v + p, -w + p \in A$. Also, since $w \notin L = \overleftrightarrow{uv} \cap \Lambda$, u, v, w form a triangle. Thus, $-u + p, -v + p, -w + p$ form a lattice triangle of $-\Lambda + p \equiv \Lambda$.

(ii) Since A is centrally symmetric, it can be shown that there exists a hexagon $H = XY'ZX'YZ'$ such that $H \subset A$, $XYZ \equiv X'Y'Z'$ and $\overline{XY} \parallel \overline{X'Y'}$, which shall be degenerated. Then $\mathbb{R}^2 = H + \Lambda \subset A + \Lambda$ can be shown as the following figure.

□

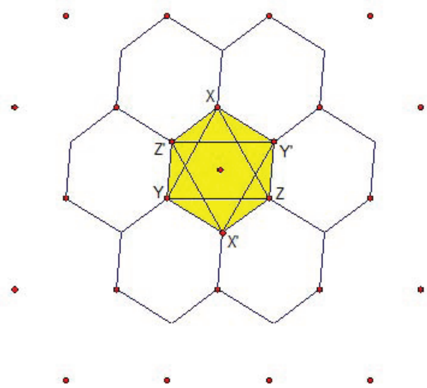


Figure 1. $H + \Lambda = \mathbb{R}^2$

The following is a corollary of Lemma 1 (ii).

Corollary 2. *If A is a centrally symmetric convex body and there exists a triangle in A which is congruent to a lattice triangle of a lattice Λ, A is a coverable body with respect to Λ.*

From now, we will denote Ω as a centrally symmetric convex body.

Lemma 3. *There exist polar coordinates such that the origin O is the center of Ω and the four rays $\theta = \frac{\pi k}{2}, k = 0, 1, 2, 3$ divide Ω into four parts of the same area.*

Proof. First consider polar coordinates whose origin is O. For $\phi \in \mathbb{R}$, let $f(\phi)$ be $S(\Omega \cap \{(r, \theta) | \theta \in (\phi, \phi + \frac{\pi}{2})\}) - S(\Omega \cap \{(r, \theta) | \theta \in (\phi - \frac{\pi}{2}, \phi)\})$. Since Ω is centrally symmetric, $f(0) = -f(\frac{\pi}{2})$. Thus there exists $t \in [0, \frac{\pi}{2}]$ such that $f(t) = 0$. Therefore, by rotating the polar coordinates through t, we obtain the polar coordinates satisfying this lemma. □

In this section, we will always use the polar coordinates suggested in Lemma 3.

Lemma 4. *If $S(\Omega) = \frac{\pi}{2}$, there exists an inscribed rhombus PQRS such that $\overline{PQ} = 1$.*

Proof. Since Ω is centrally symmetric,

$$\frac{\pi}{2} = S(\Omega) = \frac{1}{2} \int_0^{2\pi} r(\theta)^2 d\theta = \int_0^{\frac{\pi}{2}} r(\theta)^2 + r(\theta + \frac{\pi}{2})^2 d\theta,$$

thus there exists ϕ such that $r(\phi)^2 + r(\phi + \frac{\pi}{2})^2 = 1$. Let W, X, Y, Z the intersections of the boundary of Ω and the rays $\theta = \phi + \frac{\pi k}{2}, k = 0, 1, 2, 3$. Then $\overline{WX} = \overline{XY} = \overline{YZ} = \overline{ZW} = 1$, since

$$\sqrt{r(\phi)^2 + r(\phi + \frac{\pi}{2})^2} = \sqrt{r(\phi + \frac{\pi}{2})^2 + r(\phi + \pi)^2} = 1$$

□

The following lemma is a key theorem in showing the existence of a certain inscribed parallelogram.

Lemma 5. *For any function $f : [0, \frac{\pi}{8}] \to (0, \pi)$ such that its derivative f' exists and is continuous on $[0, \frac{\pi}{8}]$, and $f(0) = f(\frac{\pi}{8}) = \frac{\pi}{2}$, the following holds:*

$$\int_0^{\frac{\pi}{8}} \sqrt{64\sin^2 f(x) + f'(x)^2}dx \geq \pi$$

Proof. Since

$$\int_0^{\frac{\pi}{8}} \sqrt{64\sin^2 f(x) + f'(x)^2}dx = \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 f(x) + f'(x)^2}dx + \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 f\left(\frac{\pi}{8} - x\right) + f'\left(\frac{\pi}{8} - x\right)^2}dx,$$

it is sufficient to prove $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y + y'^2}dx \geq \frac{\pi}{2}$ for all function $y : [0, \frac{\pi}{16}] \to (0, \pi)$ such that its derivative y' exists and is continuous on $[0, \frac{\pi}{16}]$ and $y(0) = \frac{\pi}{2}$.

Let $y_0(x)$ be $\frac{\pi}{2} - \int_0^x |y'(t)|dt$. If $\int_0^{\frac{\pi}{16}} |y'(t)|dt \geq \frac{\pi}{2}$, $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y + y'^2}dx \geq \frac{\pi}{2}$ also holds, thus we will suppose $\int_0^{\frac{\pi}{16}} |y'(t)|dt < \frac{\pi}{2}$. Then for all $x \in [0, \frac{\pi}{16}]$,

$$\left|\frac{\pi}{2} - y_0(x)\right| = \int_0^x |y'(t)|dt \geq \left|\int_0^x y'(t)dt\right| = \left|\frac{\pi}{2} - y(x)\right|,$$

thus $\sin y(x) \geq \sin y_0(x)$. For all x, since $|y'(x)| = |y_0'(x)|$, $\sqrt{64\sin^2 y + y'^2} \geq \sqrt{64\sin^2 y_0 + y_0'^2}$. Therefore, it is sufficient to prove $\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_0 + y_0'^2}dx \geq \frac{\pi}{2}$. Suppose that this y_0 doesn't satisfy this inequality.

For any t, let y_t be $y_t(x) = y_0(x) - tx$. Since

$$\lim_{t\to 0} \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_t + y_t'^2}dx = \int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_0 + y_0'^2}dx < \frac{\pi}{2}$$

there exists $a > 0$ such that

$$\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_a + y_a'^2}dx < \frac{\pi}{2}$$

Since y_0 is a decreasing function, y_a is a strictly decreasing function.

Let z be $\frac{\pi}{2} - y_a$ and let h be $z(\frac{\pi}{16})$. Since z is a strictly increasing function and $z(0) = \frac{\pi}{2} - y_a(0) = \frac{\pi}{2} - y_0(0) = 0$,

$$\int_0^{\frac{\pi}{16}} \sqrt{64\sin^2 y_a + \left(\frac{dy_a}{dx}\right)^2}dx = \int_0^{\frac{\pi}{16}} \sqrt{64\cos^2 z + \left(\frac{dz}{dx}\right)^2}dx = \int_0^h \sqrt{64\left(\frac{dx}{dz}\right)^2\cos^2 z + 1}dz$$

Define a function v of z as $\sqrt{64\left(\frac{dx}{dz}\right)^2\cos^2 z + 1}$. Since $\frac{dz}{dx} = -\left(\frac{dy_0}{dx} - a\right)$ is bounded and continuous, v is bounded, continuous and $\inf_{z\in[0,h]} v \geq 1$.

Since $\int_0^h \sec z\sqrt{v^2 - 1}dz = 8\int_0^h \left(\frac{dx}{dz}\right)dz = \frac{\pi}{2}$, it is sufficient to prove the following statement for bounded continuous function v whose infimum is at least 1.

$$\int_0^h \sec z\sqrt{v^2 - 1}dz = \frac{\pi}{2} \Rightarrow \int_0^h vdz \geq \frac{\pi}{2}$$

Then, it is sufficient to prove the following:

$$\int_0^h vdz < \frac{\pi}{2} \Rightarrow \int_0^h \sec z\sqrt{v^2 - 1}dz < \frac{\pi}{2}$$

Since $\int_0^h vdz < \frac{\pi}{2}$ and $v \geq 1$, $h < \frac{\pi}{2}$. Thus there exists $\tau \in \left(\max\left\{h, \int_0^h vdz\right\}, \frac{\pi}{2}\right)$. For all $n \in \mathbb{N}$, let

$$D_n := \left\{(a_1, ..., a_n) | \frac{h}{n} \sum_{k=1}^{n} a_k \leq \tau, a_1, ..., a_n \geq 1\right\}$$

Since D_n is a compact set, there exists a pair $(b_1, ..., b_n) \in D_n$ such that for all $(a_1, ..., a_n) \in D_n$,

$$\sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{b_k^2 - 1} \geq \sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{a_k^2 - 1}$$

It can be easily shown that $\frac{h}{n} \sum_{k=1}^{n} b_k = \tau$. If there exist $i, j \in \{1, ..., n\}$ such that

$$\frac{d \sec \frac{hi}{n} \sqrt{b_i^2 - 1}}{db_i} > \frac{d \sec \frac{hj}{n} \sqrt{b_j^2 - 1}}{db_j}$$

then for sufficiently small $\epsilon > 0$, it can be shown that

$$\sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{b_k^2 - 1} < \sum_{k<n, k \neq i,j} \sec \frac{hk}{n} \sqrt{b_k^2 - 1} + \sec \frac{hi}{n} \sqrt{(b_i + \epsilon)^2 - 1} + \sec \frac{hj}{n} \sqrt{(b_j - \epsilon)^2 - 1}$$

Therefore, the values of $\frac{d \sec \frac{hk}{n} \sqrt{b_k^2 - 1}}{db_k}$, $1 \leq k \leq n$ should be a constant c_n(possibly infinite). If $c_n = \infty$, then $b_1 = ... = b_n = 1$, $h = \tau$. Thus c_n is a finite constant.

Solving the equation we obtain $b_k = \frac{c_n}{\sqrt{c_n^2 - \sec^2 \frac{hk}{n}}}$, where c_n is the solution of $\frac{h}{n} \sum_{k=1}^{n} \frac{c_n}{\sqrt{c_n^2 - \sec^2 \frac{hk}{n}}} = \tau$.

As n goes to infinity, c_n converges to the solution c of $\int_0^h \frac{c}{\sqrt{c^2 - \sec^2 z}} dz = \tau$. Here since $\tau < \frac{\pi}{2}$, $c > \sec h$. Thus

$$\lim_{n \to \infty} \frac{h}{n} \sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{b_k^2 - 1} = \int_0^h \sec z \sqrt{\left(\frac{c}{\sqrt{c^2 - \sec^2 z}}\right)^2 - 1} dz < \frac{\pi}{2}$$

can be shown.

Meanwhile, since $\int_0^h vdz < \tau$, $\frac{h}{n} \sum_{k=1}^{n} v(\frac{hk}{n}) \leq \tau$ holds for sufficiently big n. Therefore

$$\int_0^h \sec z \sqrt{v^2 - 1} dz = \lim_{n \to \infty} \frac{h}{n} \sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{v\left(\frac{hk}{n}\right)^2 - 1} \leq \lim_{n \to \infty} \frac{h}{n} \sum_{k=1}^{n} \sec \frac{hk}{n} \sqrt{b_k^2 - 1} < \frac{\pi}{2}$$

□

Theorem 6. *If $S(\Omega) = \frac{\pi}{2}$, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) \geq 1$ and $\overline{PR}, \overline{QS}$ divide Ω into four parts of the same area.*

Proof. Let $f(x)$ be $\frac{1}{2} \int_0^x r^2(\theta) d\theta$, let g be its inverse, let $\psi(x)$ be the parallelogram whose vertices are the intersections of the lines whose directions are $g(x), g(x + \frac{\pi}{8})$ and the boundary of Ω, and let $s(x)$ be $S(\psi(x))$.

Suppose that $s(x) < 1$ holds for all x. Define functions p, q as $p(x) = g(x) + g(x + \frac{\pi}{8})$, $q(x) = g(x + \frac{\pi}{8}) - g(x)$. Then since

$$s(x) = 4 \cdot \frac{1}{2} \sin\left(g(x + \frac{\pi}{8}) - g(x)\right) \cdot \frac{1}{\sqrt{\frac{1}{2} g'(x + \frac{\pi}{8})}} \cdot \frac{1}{\sqrt{\frac{1}{2} g'(x)}} = \frac{8 \sin q(x)}{\sqrt{p'(x)^2 - q'(x)^2}},$$

$\sqrt{64 \sin^2 q(x) + q'(x)^2} < p'(x)$ always holds. Therefore,

$$\int_0^{\frac{\pi}{2}} \sqrt{64 \sin^2 q(x) + q'(x)^2} dx < \int_0^{\frac{\pi}{2}} p'(x) dx = p(\frac{\pi}{2}) - p(0) = 4\pi$$

Since $q(\frac{n\pi}{8}) = \frac{\pi}{2}$ holds by Lemma 3 for all $n \in \mathbb{Z}$, by Lemma 5, the following inequality holds:

$$\int_0^{\frac{\pi}{2}} \sqrt{64 \sin^2 q(x) + q'(x)^2} dx = \sum_{n=0}^{3} \int_{\frac{n\pi}{8}}^{\frac{(n+1)\pi}{8}} \sqrt{64 \sin^2 q(x) + q'(x)^2} dx \geq 4\pi$$

This is a contradiction, thus there exists x such that $s(x) \geq 1$. It can be easily shown that $\psi(x)$ satisfies the theorem. □

As it is proved above that there exists an inscribed parallelogram ψ such that $S(\psi) \geq 1$ and the two diagonals of ψ divide a given centrally symmetric convex body Ω into four parts of equal areas, we will try to constrict ψ to satisfy $S(\psi) = 1$. However, not all Ω satisfy such property, thus we will call Ω admissible if there exists an inscribed parallelogram ψ such that $S(\psi) = 1$ and the two diagonals of ψ divide Ω into four parts of equal areas. From now, we will focus on the properties of the bodies which are not admissible.

Let $S(XY)$ denote the area of the arc XY for any chord XY of Ω and let X^* denote the reflection of X with respect to O for any point X. By Theorem 4 and Theorem 6, there exists an inscribed rhombus $P_1Q_1P_1{}^*Q_1{}^*$ such that $\overline{P_1Q_1} = 1$ and an inscribed parallelogram $P_2Q_2P_2{}^*Q_2{}^*$ such that $S(P_2Q_2P_2{}^*Q_2{}^*) \geq 1$, $S(P_2Q_2) = S(Q_2P_2{}^*)$.

In the following lemmas, for all t, the intersection of the boundary of Ω and the ray $\theta = t$ is denoted by $X(t)$.

Lemma 7. *If $S(\Omega) = \frac{\pi}{2}$ and Ω is not admissible, for all chord PQ such that $S(PQ) \geq \frac{\pi}{8} - \frac{1}{4}$, $S(P^*Q) < \frac{\pi}{8} - \frac{1}{4}$.*

Proof. Suppose that $S(P^*Q) \geq \frac{\pi}{8} - \frac{1}{4}$. Let T be a point on $\widehat{PQ} \cup \widehat{QP^*}$ such that $S(PT) = S(P^*T)$. Since $S(PT) \geq S(PQ)$ or $S(P^*T) \geq S(P^*Q)$, $S(PT) = S(P^*T) \geq \frac{\pi}{8} - \frac{1}{4}$, thus $S(PTP^*T^*) \leq 1$. For all t, let $Y(t)$ be the point on the boundary of Ω such that $S(X(t)Y(t)) = S(X^*(t)Y(t))$, then let $\psi(t)$ be $X(t)Y(t)X^*(t)Y^*(t)$. Let α, β be the angles such that $\psi(\alpha) = PTP^*T^*, \psi(\beta) = P_2Q_2P_2{}^*Q_2{}^*$. Since $S(\psi(\alpha)) \leq 1 \leq S(\psi(\beta))$, there exists γ such that $S(\psi(\gamma)) = 1$, thus Ω is admissible. □

Lemma 8. *If $S(\Omega) = \frac{\pi}{2}$ and Ω is not admissible, there exists an inscribed parallelogram PQP^*Q^* such that $\overline{PQ} = 1$ and $S(PQP^*Q^*) \geq 1$.*

Proof. Without loss of generality, suppose $\angle P_1Q_1P_1{}^* \geq \frac{\pi}{2}$. Since $\overline{P_1Q_1} = \overline{P_1{}^*Q_1} = 1$, $S(P_1Q_1P_1{}^*Q_1{}^*) \leq 1$, thus without loss of generality we may assume $S(P_1Q_1) \geq \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(P_1{}^*Q_1) < \frac{\pi}{8} - \frac{1}{4}$. Let α, β be the angles such that $X(\alpha) = P_1$, $X(\beta) = Q_1$. For θ between α and β, since $\angle P_1Q_1P_1{}^* \geq \frac{\pi}{2}$, there exists $Y(\theta) \in \widehat{P_1{}^*Q_1}$ such that $\overline{X(\theta)Y(\theta)} = 1$. Let $\psi(\theta)$ be $X(\theta)Y(\theta)X^*(\theta)Y^*(\theta)$. Since $S(X(\alpha)Y(\alpha)) \geq \frac{\pi}{8} - \frac{1}{4} \geq S(X(\beta)Y(\beta))$, there exists ϕ such that $S(X(\phi)Y(\phi)) = \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(X(\phi)Y^*(\phi)) \leq \frac{\pi}{8} - \frac{1}{4}$, thus $S(\psi(\phi)) \geq 1$. Therefore, $\psi(\phi)$ satisfies this lemma. □

Lemma 9. *If $S(\Omega) = \frac{\pi}{2}$, there exists an inscribed parallelogram PQP^*Q^* such that $\overline{P^*Q} \geq 1$, $S(PQP^*Q^*) \geq 1$, $\frac{S(PQ)}{S(PQP^*Q^*)} = \frac{\pi}{8} - \frac{1}{4}$.*

Proof. If Ω is admissible, there exists ABA^*B^* such that $S(AB) = S(BA^*) = \frac{\pi}{8} - \frac{1}{4}$. Since $\overline{AB} \cdot \overline{A^*B} \geq S(ABA^*B^*) = 1$, without loss of generality assume that $\overline{A^*B} \geq 1$. Then ABA^*B^* satisfies this lemma. Therefore, we will suppose that Ω is not admissible.

Without loss of generality, suppose $S(P_1Q_1) \geq S(P_1{}^*Q_1)$. Since $S(P_1Q_1P_1{}^*Q_1{}^*) \leq \overline{P_1Q_1} \cdot \overline{P_1{}^*Q_1} = 1$, $S(P_1Q_1) \geq \frac{\pi}{8} - \frac{1}{4}$. By Lemma 7, $S(P_1{}^*Q_1) < \frac{\pi}{8} - \frac{1}{4}$. Let α be the angle such that $X(\alpha) = P_1$. For all t, let $Y(t)$ be the point on the boundary of Ω such that $\overline{X^*(t)Y(t)} \parallel \overline{P_1{}^*Q_1}$ and let $\psi(t)$ be the parallelogram $X(t)Y(t)X^*(t)Y^*(t)$. Since $S(X^*(\alpha)Y(\alpha)) < \frac{\pi}{8} - \frac{1}{4}$, there exists β such that $S(X^*(\beta)Y(\beta)) = \frac{\pi}{8} - \frac{1}{4}$, $\widehat{X^*(\alpha)Y(\alpha)} \subset \widehat{X^*(\beta)Y(\beta)}$. Then by Lemma 7, $S(X(\beta)Y(\beta)) < \frac{\pi}{8} - \frac{1}{4}$, $S(\psi(\beta)) \geq 1$. Since $\frac{S(X(\alpha)Y(\alpha))}{S(\psi(\alpha))} \geq \frac{\pi}{8} - \frac{1}{4} \geq \frac{S(X(\beta)Y(\beta))}{S(\psi(\beta))}$, there exists γ between α, β such that $\frac{S(X(\gamma)Y(\gamma))}{S(\psi(\gamma))} = \frac{\pi}{8} - \frac{1}{4}$. Since $S(X^*(\beta)Y(\beta)) = \frac{\pi}{8} - \frac{1}{4}$ and $\widehat{X^*(\gamma)Y(\gamma)} \subset \widehat{X^*(\beta)Y(\beta)}$, $S(X^*(\gamma)Y(\gamma)) \leq \frac{\pi}{8} - \frac{1}{4}$. Then $2S(X(\gamma)Y(\gamma)) + S(\psi(\gamma)) \geq \frac{\pi}{4} + \frac{1}{2}$, $S(\psi(\gamma)) \geq 1$. Since $\widehat{X^*(\alpha)Y(\alpha)} \subset \widehat{X^*(\gamma)Y(\gamma)}$ and $\overline{X^*(\alpha)Y(\alpha)} \parallel \overline{X^*(\gamma)Y(\gamma)}$, $\overline{X^*(\gamma)Y(\gamma)} \geq \overline{X^*(\alpha)Y(\alpha)} = 1$. Thus $\psi(\gamma)$ satisfies all conditions of this lemma. □

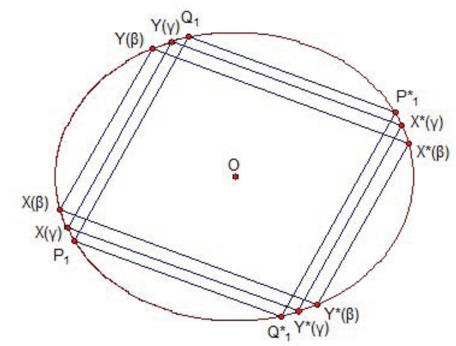
2.2 Upper Bounds on the Area of Non-coverable Set

In this section, we will suggest a function f such that for any given lattice Λ, any centrally symmetric convex body Ω is a coverable body with respect to Λ if $S(\Omega) \geq f(\Lambda)$. Also, for more efficient covering, we will suggest a certain lattice Λ^* such that $\det \Lambda^* = 1$ and any centrally symmetric convex body Ω is a coverable body with respect to Λ^* if $S(\Omega) \geq \frac{\pi}{2}$.

The followings are definitions related to the lattice, which are required to construct the function f.

Definition 10. *An elementary segment is a segment connecting two lattice points X, Y such that no lattice point exists on $\overline{XY} \setminus \{X, Y\}$. An elementary triangle is a triangle whose vertices are lattice points X, Y, Z such that no lattice point exists on $XYZ \setminus \{X, Y, Z\}$.*

For any lattice Λ, define elementary segments $d_1, d_2, \ldots$ as follows:



For all $i \in \mathbb{N}$, d_i is a shortest segment among all the elementary segments which are not parallel with $d_1, ..., d_{i-1}$.

Definition 11. *For any lattice Λ, $D(\Lambda)$ is the set of the lengths of $d_2, d_3, d_4, d_5....$*

For any set S of positive real numbers, if $S = \{s_1, s_2, ...\}$ and $s_1 < s_2 < ...$, $\mu(S) := \sup \frac{s_{i+1}}{s_i}$.

The length of d_1 is excluded from $D(\Lambda)$ to make $\mu(D(\Lambda))$ be bounded. The next theorem shows an upper bound of $\mu(D(\Lambda))$.

Theorem 12. *For all lattice Λ, $\mu(D(\Lambda)) \leq \sqrt{3}$.*

Proof. Let X, Y be the points such that $\overline{OX} = d_1, \overline{OY} = d_2, \overline{OX} \parallel d_1, \overline{OY} \parallel d_2$, $0 < \angle XOY \leq \frac{\pi}{2}$. For all k, denote Y_k as $Y + kX$. Since $\overline{OY} \leq \overline{OY_{-1}} \leq \overline{OY_1} \leq \overline{OY_{-2}} \leq \overline{OY_2} \leq ...$ and all these segments are in D, it is sufficient to show $\frac{\overline{OY_k}}{\overline{OY_{-k}}}, \frac{\overline{OY_{-k}}}{\overline{OY_{k-1}}} \leq \sqrt{3}$ for every $k \in \mathbb{N}$. Let Y' be the midpoint of $\overline{YY_{-1}}$. Let W, W' be the points such that $WW'Y'Y$ is a rectangle and $W \in \overline{OX}$. Since $\overline{W'Y'}^2 = \overline{WY}^2 = \overline{OY}^2 \sin^2 \angle XOY \geq \frac{3}{4}\overline{OX}^2$, the followings can be shown:

$$\frac{\overline{OY_k}}{\overline{OY_{-k}}} \leq \frac{\overline{W'Y_k}}{\overline{W'Y_{-k}}} = \frac{\sqrt{(k+\frac{1}{2})^2\overline{OX}^2 + \overline{W'Y'}^2}}{\sqrt{(k-\frac{1}{2})^2\overline{OX}^2 + \overline{W'Y'}^2}} \leq \frac{\sqrt{(k+\frac{1}{2})^2 + \frac{3}{4}}}{\sqrt{(k-\frac{1}{2})^2 + \frac{3}{4}}} \leq \sqrt{3}$$

$$\frac{\overline{OY_{-k}}}{\overline{OY_{k-1}}} \leq \frac{\overline{WY_{-k}}}{\overline{WY_{k-1}}} = \frac{\sqrt{k^2\overline{OX}^2 + \overline{WY}^2}}{\sqrt{(k-1)^2\overline{OX}^2 + \overline{WY}^2}} \leq \frac{\sqrt{k^2 + \frac{3}{4}}}{\sqrt{(k-1)^2 + \frac{3}{4}}} \leq \sqrt{3}$$

□

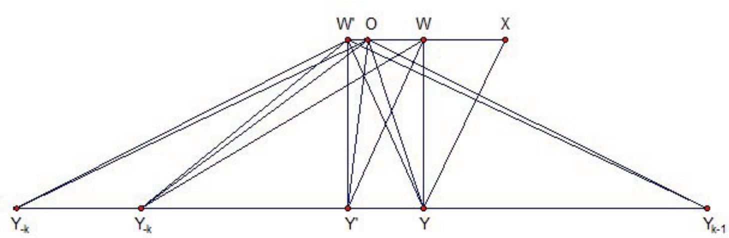


Figure 2. Proof of Theorem 12

Lemma 13. *Let O be a point and let l be a line such that $O \notin l$. Let H be the foot of the perpendicular from O to l. Let $A, B, C, D \in l$ be the points in order A, B, H, C, D, such that $\overline{AB} = \overline{CD}$, $\overline{AH} \leq \overline{DH}$. If $\angle BOD \geq \frac{\pi}{2}$, $\frac{\overline{OD}}{\overline{OA}} \leq \frac{\overline{OC}}{\overline{OB}}$.*

Proof. Let a, b, c, d, h be $\overline{AH}, \overline{BH}, \overline{CH}, \overline{DH}, \overline{OH}$, respectively. Since $\angle BOD \geq \frac{\pi}{2}$, $h^2 \leq bd$. Also, $a - b = d - c$ and $b \leq c \leq a \leq d$ hold by the given conditions. Thus $(h^2 + b^2)(h^2 + d^2) \leq (h^2 + a^2)(h^2 + c^2)$ can be shown, and this is equivalent to $\frac{\overline{OD}}{\overline{OA}} \leq \frac{\overline{OC}}{\overline{OB}}$. □

The following theorem shows how to find $\mu(D(\Lambda))$ in finite steps.

Theorem 14. *Let center O be a lattice point and let $\overline{OX}, \overline{OY}$ be the shortest two elementary segments such that $\overline{OX} \le \overline{OY}$ and $\angle XOY \le \frac{\pi}{2}$. Let $D'(\Lambda) = D(\Lambda) \cap \{\overline{OP} | \overline{OP} < 12d_2\}$. Then $\mu(D(\Lambda)) = \mu(D'(\Lambda))$.*

Proof. For $k \in \mathbb{N}$, let Y_{2k-1} be $Y - kX$ and Y_{2k} be $Y + kX$. Let Z be $Y + Y_1$. Let n be the integer such that $\overline{OY_n} \le \overline{OZ} < \overline{OY_{n+1}}$. Suppose there exists $k \ge \max\{4, n\}$ such that $\angle Y_{k-2}OY_{k+1} < \frac{\pi}{2}$. Since $\overline{OY_{k+1}} > \overline{OZ} \ge 2\overline{OH}$, $\angle OY_{k+1}H < \frac{\pi}{6}$. Then $\angle OY_{k-2}H > \frac{\pi}{3}$, thus $\overline{Y_{k+1}H} > 3\overline{Y_{k-2}H}$. This contradicts $\overline{Y_{k+1}H} \le \frac{k+2}{2}\overline{YY_1}$ and $\overline{Y_{k-2}H} \ge \frac{k-2}{2}\overline{YY_1}$. Thus $\angle Y_{k-2}OY_{k+1} \ge \frac{\pi}{2}$, and by Lemma 13, $\frac{\overline{OY_{k+1}}}{\overline{OY_k}} \le \frac{\overline{OY_{k-1}}}{\overline{OY_{k-2}}}$ holds for all $k \ge \max\{4, n\}$.

Meanwhile, it can be shown that d_1, d_2, d_3, d_4, d_5 are $\overline{OX}, \overline{OY}, \overline{OY_1}, \overline{OY_2}, \overline{OY_3}$, respectively. Thus we only need to consider the following cases.

(i) $n \ge 4$: Since $\overline{OY_1}, ..., \overline{OY_n}$ are the smallest elements of $D(\Lambda)$ and $\frac{\overline{OY_n}}{\overline{OY_{n-1}}} \ge \frac{\overline{OY_{n+2}}}{\overline{OY_{n+1}}} \ge ...$ and $\frac{\overline{OY_{n-1}}}{\overline{OY_{n-2}}} \ge \frac{\overline{OY_{n+1}}}{\overline{OY_n}} \ge ...$ hold, $\mu(D(\Lambda)) = \mu(\{\overline{OY_1}, ..., \overline{OY_n}\})$. Since $\overline{OY_n} \le \overline{OZ} \le 2d_2 + d_1 < 12d_2$, $\mu(D'(\Lambda)) = \mu(D(\Lambda))$.

(ii) $n = 3$: $\frac{\overline{OY_5}}{\overline{OY_4}} \le \frac{\overline{OY_3}}{\overline{OY_2}} \le \mu(D(\Lambda))$. Also, it can be easily shown that $d_5 = \overline{OY_3}$, $d_6 = \overline{OZ}$, $d_7 = \overline{OY_4}$. Thus $\frac{\overline{OY_5}}{\overline{OY_3}} = \frac{\overline{OY_5}}{\overline{OY_4}}\frac{\overline{OY_4}}{\overline{OZ}}\frac{\overline{OZ}}{\overline{OY_3}} \le \mu(D(\Lambda))^3$. Since $\overline{HY_5}^2 = \overline{OY_5}^2 - \overline{OH}^2 > \overline{OZ}^2 - \overline{OH}^2 \ge 3\overline{OH}^2$ and $\overline{HY_5} \ge \frac{3}{2}\overline{HY_3}$,

$$\mu(D(\Lambda)) \ge \sqrt[3]{\frac{\overline{OY_5}}{\overline{OY_3}}} = \sqrt[6]{\frac{\overline{HY_5}^2 + \overline{OH}^2}{\overline{HY_3}^2 + \overline{OH}^2}} \ge \sqrt[6]{\frac{\overline{HY_5}^2 + \frac{1}{3}\overline{HY_5}^2}{\overline{HY_3}^2 + \frac{1}{3}\overline{HY_5}^2}} \ge \sqrt[6]{\frac{12}{7}} > \frac{12}{11}$$

Let S be $\{\overline{OY_{2k}} | k \ge 11\}$. Then since $S \subset D(\Lambda)$, $S \cap D'(\Lambda) \ne \emptyset$ and $\mu(S) \le \frac{12}{11} < \mu(D(\Lambda))$, $\mu(D'(\Lambda)) = \mu(D(\Lambda))$.

□

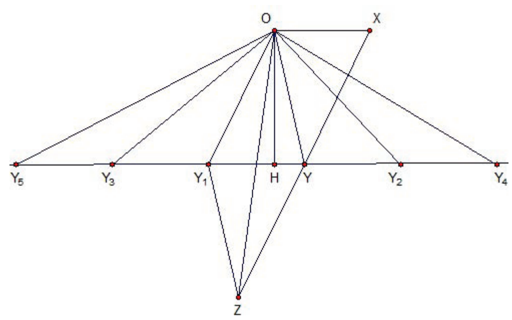


Figure 3. Arrangement of O, X, Y_1, Y_2, Y_3...

Example 15. *Let Λ_3 be $\{m[1, 0] + n[\frac{1}{2}, \frac{\sqrt{3}}{2}] | m, n \in \mathbb{Z}\}$ and let Λ_4 be $\mathbb{Z}^2$. Then $\mu(D(\Lambda_3)) = \sqrt{3}$, $\mu(D(\Lambda_4)) = \frac{\sqrt{5}}{\sqrt{2}}$ can be shown using Theorem 14.*

The next lemma shows two inequalities related to the chords of Ω. For any two sets $X, Y \subset \mathbb{R}^2$ we will denote $d(X, Y)$ as the distance between X, Y.

Lemma 16. *Suppose $S(\Omega) = \frac{\pi}{2}$. Let $PQRS$ be an inscribed parallelogram such that $S(PQRS) \ge 1$, $\frac{S(PQ)}{S(PQRS)} = \frac{\pi}{8} - \frac{1}{4}$. Given $\alpha \in [1, \frac{\pi}{4} + \frac{1}{2}]$ and $\beta \in [\frac{1}{2}, 1]$, let $\overline{U_1V_1}$ be a chord between $\overleftrightarrow{PQ}$ and $\overleftrightarrow{MN}$ such that $\overline{U_1V_1} \parallel \overline{PQ}$, $\overline{U_1V_1} = \alpha\overline{PQ}$ and let $\overline{XY}$ be a chord such that $\overline{XY} \parallel \overline{PQ}$, $\overline{XY} = \beta\overline{PQ}$, which is nearer to $\overleftrightarrow{RS}$ than $\overleftrightarrow{PQ}$. Then the followings hold:*

$$d(\overleftrightarrow{U_1V_1}, \overleftrightarrow{PQ}) \le \frac{\alpha - 1}{\pi - 2}S(PQRS) \cdot \frac{1}{\overline{PQ}}, \ d(\overleftrightarrow{XY}, \overleftrightarrow{RS}) \ge (1 - \beta)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)S(PQRS)\right) \cdot \frac{1}{\overline{PQ}}$$

Proof. Let l_0 be the line such that $l_0 \parallel \overline{PQ}$ and $O \in l_0$. Let $\overline{UV}$ be a chord between $\overleftrightarrow{PQ}$ and l_0 such that $\overline{UV} \parallel \overline{PQ}$, $d(\overleftrightarrow{UV}, \overleftrightarrow{PQ}) = \frac{\alpha-1}{\pi-2}d(\overleftrightarrow{PQ}, \overleftrightarrow{RS})$. Let L, M, N be $\overleftrightarrow{RX} \cap \overleftrightarrow{SY}, l_0 \cap \overleftrightarrow{PU}, l_0 \cap \overleftrightarrow{QV}$, respectively.

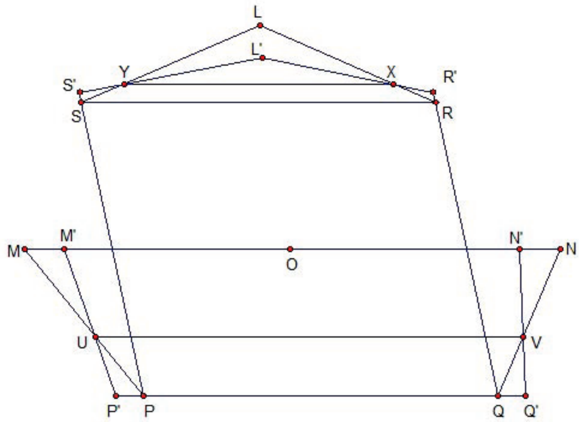


Figure 4. Proof of Lemma 16

Let u, v, x, y be the tangent lines of Ω at U, V, X, Y, respectively. Let $M', N', P', Q', R', S', L'$ be $u \cap \overleftrightarrow{MN}, v \cap \overleftrightarrow{MN}, u \cap \overleftrightarrow{PQ}, v \cap \overleftrightarrow{PQ}, x \cap \overleftrightarrow{QR}, y \cap \overleftrightarrow{PS}, x \cap y$, respectively. Since $d(\overleftrightarrow{PQ}, \overleftrightarrow{UV}) \leq d(\overleftrightarrow{UV}, \overleftrightarrow{MN})$,

$$\frac{1}{2}(\overline{MN}+\overline{PQ})d(\overleftrightarrow{PQ}, \overleftrightarrow{MN}) = S(MNQP) \geq S(M'N'Q'P') \geq \frac{S(\Omega)}{2} - S(PQ) = \left(\frac{\pi}{8}+\frac{1}{4}\right)S(PQRS) = \left(\frac{\pi}{4}+\frac{1}{2}\right)\overline{PQ}\cdot d(\overleftrightarrow{PQ}, \overleftrightarrow{MN})$$

Thus $\overline{MN} \geq \frac{\pi}{2}\overline{PQ}$. Then since

$$\overline{UV} = \frac{2(\alpha-1)}{\pi-2}\overline{MN} + \left(1 - \frac{2(\alpha-1)}{\pi-2}\right)\overline{PQ} \geq \alpha\overline{PQ} = \overline{U_1V_1},$$

$d(\overleftrightarrow{U_1V_1}, \overleftrightarrow{PQ}) \leq d(\overleftrightarrow{UV}, \overleftrightarrow{PQ}) = \frac{\alpha-1}{\pi-2}d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) = \frac{\alpha-1}{\pi-2}S(PQRS)\cdot\frac{1}{\overline{PQ}}$.

Meanwhile, since $2\overline{XY} \geq \overline{RS}$,

$$\overline{RS}\cdot d(L, \overleftrightarrow{RS}) = 2S(SLR) \geq 2S(SS'L'R'R) \geq 2S(RS) = \frac{\pi}{2} - S(PQRS)\left(\frac{\pi}{4}+\frac{1}{2}\right),$$

$$d(\overleftrightarrow{RS}, \overleftrightarrow{XY}) = (1-\beta)d(L, \overleftrightarrow{RS}) \geq (1-\beta)\left(\frac{\pi}{2} - S(PQRS)\left(\frac{\pi}{4}+\frac{1}{2}\right)\right)\cdot\frac{1}{\overline{RS}}$$

□

Definition 17. *Given an elementary segment $\overline{XY}$ of a lattice Λ, let l be a line such that $l \parallel \overline{XY}$ and $d(l, \overline{XY}) = \frac{1}{\overline{XY}}\det\Lambda$. Let T be the union of $l \cap \Lambda$ and its reflection with respect to the orthogonal bisector of $\overline{XY}$. Let k be the maximum distance between two adjacent points in T. Then the lattice rate of $\overline{XY}$ is $\frac{k}{\overline{XY}}$.*

Remark 18. *Let Z be a point on $l \cap \Lambda$ such that $\angle ZXY, \angle ZYX \leq \frac{\pi}{2}$ and let H be the point on $\overline{XY}$ such that $\overline{ZH} \perp \overline{XY}$. Let H' be the reflection of H with respect to the midpoint of $\overline{XY}$. Then since the projection of T onto $\overleftrightarrow{XY}$ is $\{H + i(Y-X)|i \in \mathbb{Z}\} \cup \{H' + i(Y-X)|i \in \mathbb{Z}\}$, the lattice rate of $\overline{XY}$ is*

$$\frac{\max\{\overline{HH'}, \overline{XY} - \overline{HH'}\}}{\overline{XY}}$$

Theorem 19. *For any lattice Λ, Ω is a coverable body if $S(\Omega)$ is not less than $f(\Lambda) = \frac{\pi}{2}\max\left\{\left(\frac{\det\Lambda}{d_1}\right)^2, {d_1}^2, \left(\frac{d_2}{\tau\mu}\right)^2, \frac{\det\Lambda}{\tau^2\mu}\right\}$, where $\mu := \mu(D(\Lambda))$, $\tau := \frac{\pi}{\pi-2+2\mu}$.*

Proof. Consider a scaling which transforms the area of Ω to $\frac{\pi}{2}$. It is sufficient to prove that Ω is a coverable body with respect to Λ if $S(\Omega) = \frac{\pi}{2}$ and $\max\left\{\left(\frac{\det\Lambda}{d_1}\right)^2, {d_1}^2, \left(\frac{d_2}{\tau\mu}\right)^2, \frac{\det\Lambda}{\tau^2\mu}\right\} \leq 1$.

Suppose that Ω is not admissible. Then by Lemma 8, there exists a parallelogram $P_0Q_0R_0S_0 \subset \Omega$ such that $\overline{P_0Q_0} = 1$ and $S(P_0Q_0R_0S_0) \geq 1$. Since $d_1 \leq 1$ and $d_1 \geq \det\Lambda$, it can be shown that there exists a parallelogram $WXYZ \subset P_0Q_0R_0S_0$

such that $S(WXYZ) = \det\Lambda$ and $\overline{WX} = d_1$. Since the lattice rate of d_1 is at most 1 and $d(\overleftrightarrow{WX}, \overleftrightarrow{YZ}) = \frac{1}{d_1}\det\Lambda$, there exists a point $T \in \overline{YZ}$ such that WXT is congruent to a lattice triangle. Then by Corollary 2, Ω is a coverable body, thus we will now suppose Ω is admissible.

Since Ω is admissible, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) = 1$ and $S(PQ) = S(QR)$. Without loss of generality, suppose $\overline{PQ} \geq \overline{QR}$. Since $\overline{PQ} \cdot \overline{QR} \geq S(PQRS) = 1$, $\overline{PQ} \geq 1$. Since $\frac{d_2}{\tau\mu} \leq 1 \leq \overline{PQ}$, there exists $u \in D(\Lambda)$ such that $\alpha := \frac{u}{\overline{PQ}} \in [\tau, \tau\mu]$. We will consider two cases : when $\alpha \geq 1$ and when $\alpha < 1$.

(i) When $\alpha \geq 1$: Since $1 < \mu < \sqrt{3}$, $\alpha \leq \tau\mu < \frac{1}{2} + \frac{\pi}{4}$. Thus by Lemma 16, there exists a chord X_1Y_1 such that $\overline{X_1Y_1} \parallel \overline{PQ}$, $\overline{X_1Y_1} = u$, $d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{PQ}) \leq \frac{\alpha-1}{\pi-2} \cdot \frac{1}{\overline{PQ}}$. Then $S(X_1Y_1X_1{}^*Y_1{}^*) = \overline{X_1Y_1}d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{X_1{}^*Y_1{}^*}) = \alpha\overline{PQ}(d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) - 2d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{PQ})) \geq \alpha\left(1 - 2 \cdot \frac{\alpha-1}{\pi-2}\right) = \frac{\alpha(\pi-2\alpha)}{\pi-2} \geq \frac{\tau\mu(\pi-2\tau\mu)}{\pi-2} = \tau^2\mu \geq \det\Lambda$.

(ii) When $\alpha < 1$: Since $1 < \mu < \sqrt{3}$, $\frac{1}{2} < \tau \leq \alpha$. Thus by Lemma 16, there exists a chord X_2Y_2 such that $\overline{X_2Y_2} \parallel \overline{PQ}$, $\overline{X_2Y_2} = u$, $d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{RS}) \geq (1-\alpha)\left(\frac{\pi}{4} - \frac{1}{2}\right) \cdot \frac{1}{\overline{PQ}}$. Then $S(X_2Y_2X_2{}^*Y_2{}^*) = \overline{X_2Y_2}d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{X_2{}^*Y_2{}^*}) = \alpha\overline{PQ}(d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) + 2d(\overleftrightarrow{X_2Y_2}, \overleftrightarrow{RS})) = \alpha\left(1 + (1-\alpha)\left(\frac{\pi}{2} - 1\right)\right) \geq \tau\left(1 + (1-\tau)\left(\frac{\pi}{2} - 1\right)\right) = \tau^2\mu \geq \det\Lambda$.

Therefore, there exists a parallelogram $XYX'Y' \subset \Omega$ such that $\overline{XY} = u$, $S(XYX'Y') = \det\Lambda$. Since $\overline{XY} \in D(\Lambda)$, $d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'}) = \frac{1}{\overline{XY}}\det\Lambda$ and the lattice rate of $\overline{XY}$ is at most 1, there exists a point $W \in \overline{X'Y'}$ such that WXY is congruent to a lattice triangle. Therefore, by Corollary 2, Ω is a coverable body. □

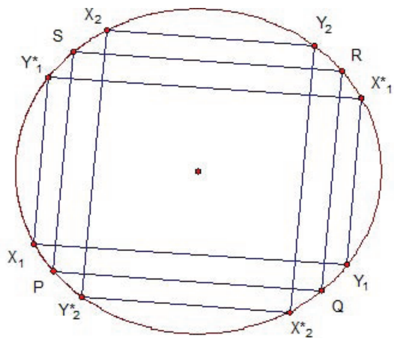


Figure 5. Proof of Theorem 19

The following example shows how we apply this theorem and the theorem's accuracy.

Example 20. *If* $S(\Omega) \geq \frac{(\pi-2+2\sqrt{3})^2}{4\pi} \simeq 1.69$, *by Theorem 19 and Example 15,* Ω *is a coverable body with respect to* Λ_3. *Similarly, if* $S(\Omega) \geq \frac{(\pi-2+\sqrt{10})^2}{\sqrt{10}\pi} \simeq 1.86$, *by Theorem 19 and Example 15,* Ω *is a coverable body with respect to* Λ_4.

Let Ω_3 *be* $\left\{(x,y)|x^2+y^2 < \frac{3}{4}, y^2 < \frac{3}{16}\right\}$ *and let* Ω_4 *be* $\left\{(x,y)|x^2+y^2 < \frac{1}{2}\right\}$. *Then it can be shown that no lattice triangle can be inscribed in each of these, thus* Ω_3, Ω_4 *are not coverable bodies. Since* $S(\Omega_3) = \frac{\pi}{4} + \frac{3\sqrt{3}}{8} > 1.43$ *and* $S(\Omega_4) = \frac{\pi}{2} > 1.57$, $S(\Omega)$ *should be at least* 1.43, 1.57 *to certify that* Ω *is always a coverable body with respect to* Λ_3, Λ_4, *respectively, while the constants we obtained from Theorem 19 were* 1.69 *and* 1.86.

To find out an efficient covering, we may apply Theorem 19 to an appropriate lattice. However, there exists a certain lattice which enables us get a more efficient covering. The followings are the processes of suggesting such lattice, denoted by Λ^*, and showing that Ω whose area is $\frac{\pi}{2}$ is always a coverable body with respect to Λ^*.

Definition 21. Λ^* *is a lattice such that* $\det\Lambda^* = 1$, $d_2 = \sqrt{2}\ d_1$ *and* $\|\boldsymbol{d}_1 + \boldsymbol{d}_2\| = \sqrt[4]{2}\ \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|$, *where* $\boldsymbol{d}_1, \boldsymbol{d}_2$ *are the vectors satisfying* $\boldsymbol{d}_1 \parallel d_1$, $\|\boldsymbol{d}_1\| = d_1$, $\boldsymbol{d}_2 \parallel d_2$, $\|\boldsymbol{d}_2\| = d_2$, $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 > 0$.

Theorem 22. *A centrally symmetric convex body* Ω *is a coverable body with respect to* Λ^* *if* $S(\Omega) = \frac{\pi}{2}$.

Proof. Let Φ be $A \cup B$, where A, B are the following sets :

$$A := \{p\mathbf{d}_1 + q\mathbf{d}_2 | 0 \leq p \leq 6, q = \pm 1, p, q \in \mathbb{Z}\} \cup \{4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1 | 1 \leq p \leq 3, p \in \mathbb{Z}\},$$

$$B := \{4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1 | p \geq 3, p \in \mathbb{Z}\}$$

For all $t \geq 6$, since $\|4\mathbf{d}_2 + (2t+1)\mathbf{d}_1\| - \|4\mathbf{d}_2 + (2t-1)\mathbf{d}_1\| \leq 2\|\mathbf{d}_1\| < \frac{1}{5}\|4\mathbf{d}_2 + 11\mathbf{d}_1\|$, $\|4\mathbf{d}_2 + (2t+1)\mathbf{d}_1\| < \frac{6}{5}\|4\mathbf{d}_2 + (2t-1)\mathbf{d}_1\|$. Thus $\mu(B) < \frac{6}{5}$. Also, $\mu(A) < \frac{6}{5}$ can be shown by checking all elements. Therefore, $\mu(\Phi) < \frac{6}{5}$.

For any $p \geq 3$, let X, Y, Z be the lattice points such that $\overleftrightarrow{XY} = 4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1$, $\overleftrightarrow{XZ} = \mathbf{d}_2 + p\mathbf{d}_1$ and let H be the point on $\overline{XY}$ such that $\overline{ZH} \perp \overline{XY}$. Let H' be the reflection of H with respect to the midpoint of $\overline{XY}$. Since $S(XYZ) = \frac{1}{2}$, $d(Z, \overleftrightarrow{XY}) = \frac{1}{\overline{XY}}$. Since

$$\left|\frac{1}{4} - \frac{\overline{XH}}{\overline{XY}}\right| = \left|\frac{1}{4} - \frac{(4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1)\cdot(\mathbf{d}_2 + p\mathbf{d}_1)}{\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|^2}\right| = \frac{(4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1)\cdot \mathbf{d}_1}{4\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|^2} \leq \frac{\|\mathbf{d}_1\|}{4\|4\mathbf{d}_2 + (4p \pm 1)\mathbf{d}_1\|} \leq \frac{1}{16},$$

$\max\{\overline{HH'}, \overline{XY} - \overline{HH'}\} \leq \frac{3}{4}\overline{XY}$. Also, since $\overleftrightarrow{XZ} \cdot \overleftrightarrow{ZY} \geq (\mathbf{d}_2 + p\mathbf{d}_1)\cdot(3\mathbf{d}_2 + (3p \pm 1)\mathbf{d}_1) \geq 0$, $\angle XYZ \geq \frac{\pi}{2}$, $\angle ZXY, \angle ZYX \leq \frac{\pi}{2}$. Thus the lattice rate of $\overline{XY}$ is at most $\frac{3}{4}$. Also, it can be shown that the lattice rate of any element of A is at most $\frac{3}{4}$ by checking all elements. Therefore, every element of Φ has lattice rate not bigger than $\frac{3}{4}$.

By Lemma 9, there exists an inscribed parallelogram $PQRS$ such that $S(PQRS) \geq 1$, $\frac{S(PS)}{S(PQRS)} = \frac{\pi}{8} - \frac{1}{4}$ and $\overline{PQ} \geq 1$. Let s be $S(PQRS)$. Since $\frac{5}{6}d_2 < 1 \leq \overline{PQ}$ and $d_2 \in \Phi$ and $\mu(\Phi) < \frac{6}{5}$, there exists $d_i \in \Phi$ such that $\overline{PQ} \leq d_i < \frac{6}{5}\overline{PQ}$.

Let XY be a chord between $\overleftrightarrow{PQ}$ and O such that $\overline{XY} \parallel \overline{PQ}$ and $\overline{XY} = d_i$. Let $X'Y'$ be a chord such that $\overline{X'Y'} \parallel \overline{PQ}$ and $\overline{X'Y'} = \frac{3}{4}d_i$. Let t be $\frac{d_i}{\overline{PQ}}$. Then by Lemma 16,

$$d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'}) = d(\overleftrightarrow{X'Y'}, \overleftrightarrow{RS}) + d(\overleftrightarrow{PQ}, \overleftrightarrow{RS}) - d(\overleftrightarrow{XY}, \overleftrightarrow{PQ}) \geq \left(1 - \frac{3}{4}t\right)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)s\right)\frac{1}{\overline{PQ}} + \frac{s}{\overline{PQ}} - \frac{t-1}{\pi - 2}\cdot\frac{s}{\overline{PQ}}$$

Thus, $d(\overleftrightarrow{XY}, \overleftrightarrow{X'Y'})d_i \geq t\left((1 - \frac{3}{4}t)\left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\right)s\right) + s - \frac{t-1}{\pi-2}s\right)$ and this is always bigger than 1, since $t \in [1, \frac{6}{5})$ and $s \geq 1$.

Since Ω is convex, there exists $\overline{X_1Y_1} \subset \Omega$ such that $d(\overleftrightarrow{X_1Y_1}, \overleftrightarrow{XY})d_i = 1$, $\overline{X_1Y_1} \parallel \overline{XY}$ and $\overline{X_1Y_1} = \overline{X'Y'} = \frac{3}{4}d_i$. Since the lattice rate of d_i is at most $\frac{3}{4}$ and $\overline{X_1Y_1} = \frac{3}{4}\overline{XY}$, it can be shown that there exists a point $Z \in \overline{X_1Y_1}$ such that XYZ is congruent to a lattice triangle. Since $XYZ \subset \Omega$, by Corollary 2, Ω is a coverable body. □

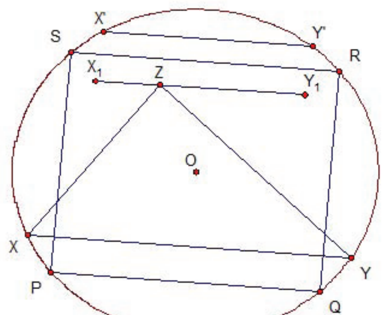


Figure 6. Proof of Theorem 22

2.3 Application

An interesting property of the coverable body is that we can suggest a reasonable upper bound on the infimum of the density of lattice covering with the minkowski sum of a coverable body and an uniformly coverable set with respect to the same lattice. Here, the uniformly coverable set is a new definition, which indicates any bounded closed set $A \subset \mathbb{R}^2$ such that for all $\Lambda' \equiv \Lambda$, $A + \Lambda' = \mathbb{R}^2$.

Theorem 23. *Let A be a coverable body and let B be an uniformly coverable set with respect to the same given lattice Λ. Then there exists a lattice covering of $A + B$ whose density is $\frac{S(A+B)}{3\det\Lambda}$.*

Proof. Since A is a coverable body with respect to Λ, there exists $\Lambda' \equiv \Lambda$ such that $A + \Lambda' = \mathbb{R}^2$. By Lemma 1, there exists $\Lambda_1 \equiv \Lambda$ such that A includes a lattice triangle of Λ_1. Since A is convex, there also exists an elementary triangle $LMN \subset A$. Let T be the lattice $\{pL + qM + rN | p + q + r = 0, p \equiv q \equiv r \mod 3\}$. Then since $\Lambda_1 = \{pL + qM + rN | p + q + r = 1\}$, $\Lambda_1 = T + \{L, M, N\}$, thus $\Lambda_1 \subset T + A$. Therefore, $\mathbb{R}^2 = B + \Lambda_1 \subset B + T + A = (A + B) + T$, $\{A + B + t | t \in T\}$ is a covering whose density is $\frac{S(A+B)}{\det T} = \frac{S(A+B)}{3\det\Lambda_1}$. □

This theorem is beneficial to general sets, since the uniformly coverable set needs not be connected and may have holes. The following is an example of this.

Example 24. *Let A be $\Gamma \setminus \Gamma'$, where $\Gamma := \left\{P|\overline{OP} \leq \frac{2}{\sqrt{3}}\right\}$, $\Gamma' := \left\{P|\overline{OP} < \frac{\sqrt{3}}{2}\right\}$. We will show that A is an uniformly coverable body with respect to Λ_3. Let X be any point on the plane. For $i, j \in \{0, 1\}$, let $\Lambda(i, j)$ be the lattice $\{(2m + i)[1, 0] + (2n + j)[\frac{1}{2}, \frac{\sqrt{3}}{2}] \,|\, m, n \in \mathbb{Z}\}$. Since a right triangle congruent to a lattice triangle of $\Lambda(i, j)$ can be inscribed in Γ, by Corollary 2, there exists $\lambda \in \Lambda(i, j)$ such that $X \in \Gamma + \lambda$. Meanwhile, since the diameter of Γ' is $\sqrt{3}$, it can be shown that there are at most three elements of $\{\lambda|X \in \Gamma' + \lambda, \lambda \in \Lambda_3\}$. Therefore, there exists a lattice point λ such that $X \in (\Gamma \setminus \Gamma') + \lambda = A + \lambda$. Thus $A + \Lambda_3 = \mathbb{R}^2$. Since A is the region between two concentric circles, $A + \Lambda' = \mathbb{R}^2$ holds for all $\Lambda' \equiv \Lambda_3$, thus A is an uniformly coverable set with respect to Λ_3.*

Let B be any centrally symmetric convex body whose area is bigger than $\frac{(\pi-2+2\sqrt{3})^2}{4\pi}$. B is a coverable body with respect to Λ_3, as it was shown in Example 20. Thus by Theorem 23, there exists a lattice covering of $A + B$ whose density is $\frac{S(A+B)}{3 \det \Lambda_3} = \frac{2}{3\sqrt{3}} S(A + B)$.

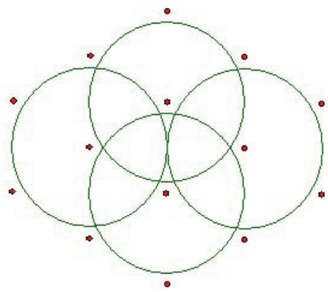


Figure 7. Covering by Γ'

3. Conclusion

In this paper, we suggested a function f such that any centrally symmetric convex body Ω is a coverable body with respect to a lattice Λ if $S(\Omega) \geq f(\Lambda)$. Also, we discovered a lattice Λ^* such that any centrally symmetric convex body Ω is a coverable body with respect to Λ^* if $S(\Omega) \geq \frac{\pi}{2}$. To apply the coverable body to more general problems, we also suggested a method to prove the existence of an efficient lattice covering using a coverable body.

References

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