Journal of Mathematics Research; Vol. 9, No. 2; April 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Nilpotency of the Ordinary Lie-algebra of an n-Lie Algebra

Côme Chancel Likouka[1]

[1] Université Marien Ngouabi, BP: 69, Brazzaville, Congo

Correspondence: Côme Chancel Likouka, Université Marien Ngouabi, BP: 69, Brazzaville, Congo. E-mail: clikouka@gmail.com

Received: November 11, 2016 Accepted: February 3, 2017 Online Published: March 16, 2017

doi:10.5539/jmr.v9n2p79 URL: https://doi.org/10.5539/jmr.v9n2p79

The author was supported by the International Mathematical Union under the project IMU-Congo.

Abstract

In this paper, we generalize to n-Lie algebras a corollary of the well-known Engel's theorem which offers some justification for the terminology "nilpotent" and we construct a nilpotent ordinary Lie algebra from a nilpotent n-Lie algebra.

Keywords: Lie algebra, n-Lie algebra, nilpotency

1. Introduction

(Filipov, 1985) Introduced a generalization of a Lie algebra, which he called an n-Lie algebra. The Lie product is taken between n elements of the algebra instead of two. This new bracket is n-linear, anti-symmetric and satisfies a generalization of the Jacobi identity.

(Bossoto,Okassa, & Omporo, 2013) Associate to an n-Lie algebra, a Lie algebra called the ordinary Lie algebra.

In this paper, we generalize to n-Lie algebras a corollary of the well-known Engel's theorem and we construct a nilpotent ordinary Lie algebra from a nilpotent n-Lie algebra.

1.1 n-Lie Algebra Structure

In the following, K will denote a commutative field with characteristic zero.

An n-Lie algebra $\mathcal{G}$ over K is a vector space together with a multilinear fully skewsymmetric map

$$\{,...,\} : \mathcal{G}^n = \mathcal{G} \times \mathcal{G} \times ... \times \mathcal{G} \longrightarrow \mathcal{G}, (x_1, x_2, ..., x_n) \longmapsto \{x_1, x_2, ..., x_n\},$$

such that

$$\{x_1, x_2, ..., x_{n-1}, \{y_1, y_2, ..., y_n\}\} = \sum_{i=1}^{n} \{y_1, y_2, ..., y_{i-1}, \{x_1, x_2, ..., x_{n-1}, y_i\}, y_{i+1}, ..., y_n\}$$

for all $x_1, x_2, ..., x_{n-1}, y_1, y_2, ..., y_n$ elements of $\mathcal{G}$.

The above equation is called the generalized Jacobi Identity.

A subspace $\mathcal{G}_0$ of $\mathcal{G}$ is called an n-Lie subalgebra if for any $y_1, y_2, ..., y_n \in \mathcal{G}_0, \{y_1, y_2, ..., y_n\} \in \mathcal{G}_0$.

Let $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$ be subalgebras of n-Lie algebra $\mathcal{G}$ and let $\{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n\}$ denote the subspace of $\mathcal{G}$ generated by all vectors $\{x_1, x_2, ..., x_n\}$, where $x_i \in \mathcal{G}_i$ for $i = 1, 2, ..., n$. The subalgebra $\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}\}$ is called the derived algebra of $\mathcal{G}$, and is denoted by $\mathcal{G}^1$. If $\mathcal{G}^1 = 0$, then $\mathcal{G}$ is called an abelian n-Lie algebra.

Using the derivation $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \longrightarrow \mathcal{G}, y \longmapsto \{x_1, x_2, ..., x_{n-1}, y\}$, we can rephrase this definition as follows:

A vector subspace $\mathcal{G}_0$ of $\mathcal{G}$ is an n-Lie subalgebra of $\mathcal{G}$ if $ad(x_1, x_2, ..., x_{n-1})(\mathcal{G}_0) \subset \mathcal{G}_0$ for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}_0$.That is, $ad(\mathcal{G}_0, \mathcal{G}_0, ..., \mathcal{G}_0)(\mathcal{G}_0) \subset \mathcal{G}_0$.

A subspace $\mathcal{I}$ of $\mathcal{G}$ is called an ideal if $\{x, y_1, y_2, ..., y_{n-1}\} \in \mathcal{I}$ for any $x \in \mathcal{I}$, and for any $y_1, y_2, ..., y_{n-1} \in \mathcal{G}$. That is equivalent to say that $ad(\mathcal{G}, ..., \mathcal{G})(\mathcal{I}) \subset \mathcal{I}$.

1.2 The Ordinary Lie Algebra of an n-Lie Algebra

Let $\mathcal{G}$ be an n-Lie algebra over a field K. (Bossoto et al., 2013) associate to $\mathcal{G}$ a Lie algebra called the ordinary Lie algebra. This construction goes as presented below:

Consider the map

$$\mathcal{G}^{n-1} \longrightarrow Der_K(\mathcal{G}), (x_1, x_2, ..., x_{n-1}) \longmapsto ad(x_1, x_2, ..., x_{n-1}),$$

where $Der_K(\mathcal{G})$ denote the set of $K-$derivations of $\mathcal{G}$.

Denote by $\Lambda_K^{n-1}(\mathcal{G})$, the $(n-1)$-exterior power of the K-vector space $\mathcal{G}$, there exists a unique K-linear map

$$ad_{\mathcal{G}} : \Lambda_K^{n-1}(\mathcal{G}) \longrightarrow Der_K(\mathcal{G})$$

such that

$$ad_{\mathcal{G}}(x_1 \Lambda x_2 \Lambda ... \Lambda x_{n-1}) = ad(x_1, x_2, ..., x_{n-1})$$

for all $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

When $f : W \longrightarrow W$ is an endomorphism of a K-vector space W and when $\Lambda_K(W)$ is the K-exterior algebra of W, then there exists a unique derivation of degree *zero*

$$D_f : \Lambda_K(W) \longrightarrow \Lambda_K(W)$$

such that, for $p \in \mathbb{N}$,

$$D_f(w_1 \Lambda w_2 \Lambda ... \Lambda w_p) = \sum_{i=1}^{p} w_1 \Lambda w_2 \Lambda ... \Lambda w_{i-1} \Lambda f(w_i) \Lambda w_{i+1} \Lambda ... \Lambda w_p$$

for all $w_1, w_2, ..., w_p$ elements of W.

Proposition 1 For all s_1 and s_2 elements of $\Lambda_K^{n-1}(\mathcal{G})$, then we have simultaneously

$$[ad_{\mathcal{G}}(s_1), ad_{\mathcal{G}}(s_2)] = ad_{\mathcal{G}} \left(D_{ad_{\mathcal{G}}(s_1)}(s_2)\right)$$

and

$$[ad_{\mathcal{G}}(s_1), ad_{\mathcal{G}}(s_2)] = ad_{\mathcal{G}} \left(-D_{ad_{\mathcal{G}}(s_2)}(s_1)\right)$$

where [,] denotes the usual bracket of endomorphisms.

We denote by $\mathcal{V}_K(\mathcal{G})$ the K-subspace of $\Lambda_K^{n-1}(\mathcal{G})$ generated by the elements of the form $D_{ad_{\mathcal{G}}(s_1)}(s_2) + D_{ad_{\mathcal{G}}(s_2)}(s_1)$ where s_1 and s_2 describe $\Lambda_K^{n-1}(\mathcal{G})$.

Let

$$\Lambda_K^{n-1}(\mathcal{G}) \longrightarrow \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), s \longmapsto \overline{s},$$

be the canonical surjection. Given the foregoing, we conclude that

$ad_{\mathcal{G}} [\mathcal{V}_K(\mathcal{G})] = 0.$

We denote by

$$\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}(\mathcal{G}) \longrightarrow Der_K(\mathcal{G})$$

the unique linear map such that

$$\widetilde{ad_{\mathcal{G}}}(\overline{s}) = ad_{\mathcal{G}}(s)$$

for all $s \in \Lambda_K^{n-1}(\mathcal{G})$.

Theorem 2 When $(\mathcal{G}, \{, ..., \})$ is a n-Lie algebra, then the map

$$[,] : \left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]^2 \longrightarrow \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), (\overline{s_1}, \overline{s_2}) \longmapsto \overline{D_{ad_{\mathcal{G}}(s_1)}(s_2)},$$

depends only on $\overline{s_1}$ and $\overline{s_2}$, and defines an ordinary Lie algebra structure on $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$.

Proposition 3 If a subspace $\mathcal{G}_0$ of an n-Lie algebra $\mathcal{G}$ is stable for the representation

$$\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s),$$

then $\mathcal{G}_0$ is an ideal of the n-Lie algebra $\mathcal{G}$.

2. Nilpotency of the Ordinary Lie Algebra

An n-Lie algebra $\mathcal{G}$ is nilpotent if $\mathcal{G}$ satisfies $\mathcal{G}^r = 0$ for some $r \geq 0$, where $\mathcal{G}^0 = \mathcal{G}$ and $\mathcal{G}^r$ is defined by induction, $\mathcal{G}^{r+1} = [\mathcal{G}^r, \mathcal{G}, \mathcal{G}, \cdots, \mathcal{G}]$ for $r \geq 0$.

Proposition 4 Let $\mathcal{G}$ be an n-Lie algebra over a field K. If $\mathcal{G} \neq 0$ is nilpotent then $\mathcal{Z}(\mathcal{G}) \neq 0$.

Proof. Let us suppose $\mathcal{Z}(\mathcal{G}) = 0$.

Nilpotency of $\mathcal{G}$ implies that there exists an integer $k \geq 0$ such that $\mathcal{G}^{k-1} \neq 0$ and $\mathcal{G}^k = 0$.

$$\begin{aligned} 0 = \mathcal{G}^k &= \left\{\mathcal{G}^{k-1}, \mathcal{G}, \mathcal{G}, ..., \mathcal{G}\right\} \\ &= \left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-1}\right\} \\ &= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-1}) \\ &= 0 \end{aligned}$$

Then $\mathcal{G}^{k-1} \subset \mathcal{Z}(\mathcal{G})$.

Therefore $0 \neq \mathcal{G}^{k-1} \subset \mathcal{Z}(\mathcal{G}) = 0$ which is impossible.

Thus $\mathcal{Z}(\mathcal{G}) \neq 0$.

Below we give the statement of the Engel's theorem and its corollary for Lie algebras:

Theorem 5 (Engel) Let $\rho : \mathcal{G} \to End(V)$ be a linear representation of $\mathcal{G}$ on the vector space V such that $\rho(x)$ is nilpotent for each $x \in \mathcal{G}$. If $V \neq (0)$, then there

exists $v \in V, v \neq 0$ such that $\rho(x)v = 0$ for all $x \in \mathcal{G}$.

Corollary 6 $\mathcal{G}$ is nilpotent if and only if adx is nilpotent for each $x \in \mathcal{G}$.

Now we're going to give a generalization to n-Lie algebras of the above corollary:

Theorem 7 Let $\mathcal{G}$ be an n-Lie algebra over a field K. $\mathcal{G}$ is nilpotent if and only if $ad(x_1, x_2, ..., x_{n-1})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

To prove the Theorem, one needs some Lemmas:

Lemma 8 Let $\mathcal{G}$ be an n-Lie algebra, $\mathcal{Z}(\mathcal{G})$ the center of $\mathcal{G}$ and $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{Z}(\mathcal{G})$ the canonical surjection. For any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, if $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \to \mathcal{G}$ is nilpotent, then the unique linear map

$$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \to \mathcal{G}/\mathcal{Z}(\mathcal{G}), \bar{y} \mapsto \overline{\{x_1, x_2, ..., x_{n-1}, y\}}$$

such that $\pi \circ \overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} = ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}) \circ \pi$ is nilpotent.

Proof. It's clear that $ad(x_1, x_2, ..., x_{n-1})[\mathcal{Z}(\mathcal{G})] = 0$. We denote by

$$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \to \mathcal{G}/\mathcal{Z}(\mathcal{G}), \bar{y} \mapsto \overline{\{x_1, x_2, ..., x_{n-1}, y\}}$$

the unique linear map such that $\pi \circ \overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} = ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}) \circ \pi$.

$ad(x_1, x_2, ..., x_{n-1})$ nilpotent, then there exists $k \geq 0$ such that $(ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1}))^k = 0$. We have: $(\overline{ad_{\mathcal{G}}h})^k \circ \pi = \pi \circ (ad_{\mathcal{G}}h)^k = 0$. Since π is surjective $\Rightarrow (\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})})^k = 0$ ie $\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})}$ is nilpotent.

Lemma 9 If for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \to \mathcal{G}$ is nilpotent, then $\mathcal{Z}(\mathcal{G}) \neq (0)$.

Proof. Using the well-known Engel's theorem, there exists $u \in \mathcal{G}, u \neq 0$, such that
$ad(x_1, x_2, ..., x_{n-1})(u) = 0$, for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$. That implies $u \in \mathcal{Z}(\mathcal{G})$. And as $u \neq 0$, thus $\mathcal{Z}(\mathcal{G}) \neq (0)$.We are done.

The set $\{ad(x_1, x_2, ..., x_{n-1})/ad(x_1, x_2, ..., x_{n-1})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}\}$ is a Lie subalgebra of $End_{\mathbb{k}}(\mathcal{G})$.

Proof. " $\Rightarrow$ ":

$\mathcal{G}$ nilpotent implies that there exists $k \geq 0$ such that $\mathcal{G}^{k-1} \neq 0$ and $\mathcal{G}^k = 0$.

$$
\begin{aligned}
0 &= \mathcal{G}^k = \left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-1}\right\} \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-1}) \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})\left\{\mathcal{G}, \mathcal{G}, ..., \mathcal{G}, \mathcal{G}^{k-2}\right\} \\
&= ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})\left[ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})(\mathcal{G}^{k-2})\right] \\
&= \underbrace{[ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G}) \circ ... \circ ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})]}_{k-times}(\mathcal{G}) \\
&= [ad(\mathcal{G}, \mathcal{G}, ..., \mathcal{G})]^k (\mathcal{G})
\end{aligned}
$$

i.e $[ad(x_1, x_2, ..., x_{n-1})]^k = 0$ for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.

Thus $ad(x_1, x_2, ..., x_{n-1})$ is nilpotent.

"$\Longleftarrow$ " we prove by induction on the dimension of $\mathcal{G}$.

• $\dim \mathcal{G} = 1$, $ad(x_1, x_2, ..., x_{n-1}) : \mathcal{G} \to \mathcal{G}$ is nilpotent $\Rightarrow ad(x_1, x_2, ..., x_{n-1})(y) = 0$ for any $x_1, x_2, ..., x_{n-1}, y \in \mathcal{G}$, that is $\mathcal{G}$ is commutative. Thus $ad(\mathcal{G}^{n-1})\ (\mathcal{G}) = 0$ ie $\mathcal{G}^1 = 0$.Therefore $\mathcal{G}$ is nilpotent.

• Suppose the assumption true for $\dim \mathcal{G} = n$.Let's verify the assumption for $\dim \mathcal{G} = n + 1$.

$ad(x_1, x_2, ..., x_{n-1})$ nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$, then from *Lemma* 8,$\overline{ad_{\mathcal{G}}(x_1, x_2, ..., x_{n-1})} : \mathcal{G}/\mathcal{Z}(\mathcal{G}) \to \mathcal{G}/\mathcal{Z}(\mathcal{G})$ is nilpotent for any $x_1, x_2, ..., x_{n-1} \in \mathcal{G}$.$\Rightarrow \mathcal{G}/\mathcal{Z}(\mathcal{G})$ is nilpotent and $\mathcal{Z}(\mathcal{G}) \neq 0$ from *Lemma* 9. $\mathcal{G}/\mathcal{Z}(\mathcal{G})$ nilpotent, there exists $k \geq 0$ such that $[\mathcal{G}/\mathcal{Z}(\mathcal{G})]^k = 0$. As $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{Z}(\mathcal{G})$, then $[\mathcal{G}/\mathcal{Z}(\mathcal{G})]^k = \pi(\mathcal{G}^k) = 0$ since π is surjective. Thus $\mathcal{G}^k \subset \mathcal{Z}(\mathcal{G})$. $\mathcal{G}^{k+1} = ad(\mathcal{G}^{n-1})(\mathcal{G}^k) \subset ad(\mathcal{G}^{n-1})(\mathcal{Z}(\mathcal{G})) = 0$.Therefore $\mathcal{G}$ is nilpotent.That ends the proof.

Below we give the statement of the main theorem we obtained:

Theorem 10 If $\mathcal{G}$ is a nilpotent n-Lie algebra over a field $\mathbb{k}$ and if $\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s)$,is the canonical representation of $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$ in $\mathcal{G}$, then $\left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]/Ker\widetilde{ad_{\mathcal{G}}}$ is a nilpotent Lie algebra.

Proof. Let $\mathcal{G}$ be an n-Lie algebra. Then the mapping

$$\mathcal{G}^{n-1} \longrightarrow Der_K(\mathcal{G}), (x_1, x_2, ..., x_{n-1}) \longmapsto ad(x_1, x_2, ..., x_{n-1}),$$

induces a representation $\widetilde{ad_{\mathcal{G}}} : \Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \longmapsto ad_{\mathcal{G}}(s)$ of $\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$ in $\mathcal{G}$. When $\mathcal{G}$ is a nilpotent n-Lie algebra then $\widetilde{ad_{\mathcal{G}}}(\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}))$ is a Lie subalgebra of $Der_K(\mathcal{G})$ whose all elements are nilpotent.Thus $\widetilde{ad_{\mathcal{G}}}(\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}))$ is a nilpotent Lie algebra. Therefore $\left[\Lambda_K^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]/Ker\widetilde{ad_{\mathcal{G}}}$ is a nilpotent Lie algebra.

Acknowledgements

We are very grateful to Professor Eugène Okassa for his advice and for his help.

References

Bossoto, B. G. R., Okassa, E., & Omporo, M. (2013). *Lie algebra of an n-Lie algebra.*

Bourbaki, N. (1974). *Elements of Mathematics: Algebra 1, chapters 1-3* . Hermann.

Filipov, V. T. (1985). n-Lie algebras. *Sibirsk. Mat. Zh., 39*(6), 126-140. (English translation: (1985). *Siberian Math. J., 26*(6), 879-891).

Filipov, V. T. (1998). On n-Lie algebra of jacobians. *Sibirsk. Mat. Zh., 26*(6), 126-140. (English translation: (1985). *Siberian Math. J.*, *26*(6), 879-891.)

Kasymov, S. M. (1987). Theory of n-Lie algebras. *Algebra i Logika*, *26*(3), 277-297. (English translation: (1988). Algebra and Logic *26*, 155-166.) https://doi.org/10.1007/BF02009328

Ling, W. X. (1993). *On the structure of n-Lie algebras* . Siegen, PhD thesis.

Pozhidaev, A. P. (1993). *n*-Lie algebras . *Algebra and Logic*, *38*(3), 181-192. https://doi.org/10.1007/BF02671742

Rotkiewicz, M. L. (2005). Cohomology Ring of n-Lie Algebras. *Extracta mathematicae*, *20*(3), 219-232.

Williams, M. P. (2009, June). Nilpotent n-Lie Algebras . *Communications in Algebra*, *37*(6), 1843-1849. https://doi.org/10.1080/00927870802108007