Journal of Mathematics Research; Vol. 9, No. 2; April 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

# On Commutativity of Semiprime Rings with Multiplicative (generalized)-derivations

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Received: December 5, 2016 Accepted: January 18, 2017 Online Published: March 5, 2017

doi:10.5539/jmr.v9n2p9 URL: https://doi.org/10.5539/jmr.v9n2p9

**Abstract**

The aim of this paper is to explore the commutativity of semiprime rings admitting multiplicative (generalized)-derivations and satisfy certain hypotheses on appropriate subsets.

**Keywords:** semiprime ring, ideals, derivation, multiplicative (generalized)-derivation.

## 1. Introduction

Throughout this paper $R$ denotes an associative ring with center $Z(R)$. Recall, a ring $R$ is said to be prime ring if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime ring if $aRa = (0)$ implies $a = 0$. For any $x, y \in R$, we shall denote the commutator and anti-commutator by the symbols $[x, y] = xy - yx$ and $(x \circ y) = xy + yx$ respectively. We shall frequently use the basic commutator and anti-commutator identities : $[xy, z] = x[y, z]+[x, z]y$, $[x, yz] = y[x, z]+[x, y]z$ and $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x.y]z$, $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$. An additive map $f : R \to R$ is called a derivation of $R$ if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in R$. Let $F : R \to R$ be a map together with another map $f : R \to R$ so that $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$. If $F$ is additive and $f$ a derivation of $R$, then $F$ is called generalized derivation of $R$ and if $f = 0$, then $F$ is called left multiplier of $R$. The notion of generalized derivation was introduced by Brešar (Brešar, 1991) . In (Havala, 1998), author gave an algebraic study of these mappings in prime rings. Obviously, every derivation is a generalized derivation. In this way generalized derivation covers both concepts of derivation and left multiplier of $R$. Let $K$ be a nonempty subset of $R$, a map $f : K \to R$ is said to be centralizing on $K$, if $[f(x), x] \in Z(R)$ for all $x \in K$. In particular, if $[f(x), x] = 0$ for all $x \in K$, then $f$ is called commuting on $K$.

In the literature, a number of authors have discussed the commutativity of prime rings and semiprime rings admitting derivations and generalized derivations satisfying certain algebraic identities, see (Ali, Kumar & Miyan, 2011), (Ali, Dhara & Fošner, 2011), (Andima & Pajoohesh, 2010), (Ashraf et al, 2007, 2001), (Daif & Bell, 1992), (Dhara & Pattanayak, 2011), (Hongan, 1997), where further references can be found.

Let us swing to the foundation examination of multiplicative (generalized)-derivations of associative rings. Inspired by the work of Martindale III (Martindale, 1969), Daif (Daif, 1991) introduced the concept of multiplicative derivations. Accordingly, a map $f : R \to R$ is called multiplicative derivation of R if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in R$. Of course, these maps are not necessarily additive. Goldmann and Sěmrl (Goldmann & Sěmrl, 1996) presented complete description of these maps. Further, Daif and Tammam-El-Sayiad (Daif & Tammam-El-Sayiad, 1997) extended the notion of multiplicative derivation to multiplicative generalized derivation as follows: A map $F : R \to R$ is called multiplicative generalized derivation of R if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where $f$ is a derivation of R. Recently, Dhara and Ali (Dhara & Ali, 2013) made a slight generalization in above definition of multiplicative generalized derivation by relaxing the conditions on $f$. A map $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xf(y)$ holds for all $x, y \in R$, where $f$ can be any map (not necessarily additive nor a derivation). For convenience we denote it by a pair $(F, f)$. In the previous couple of years many outcomes has been gotten in prime and semi-prime rings involving multiplicative (generalized)-derivations, see (Ali et al, 2015), (Ali et al, 2014), (Dhara & Ali, 2013), (Dhara et al, 2014) and (Khan, 2016). As multiplicative (generalized)-derivation is an extended notion of generalized derivation, so it is noteworthy to demonstrate the consequences of generalized derivations for multiplicative (generalized)-derivations.

The main objective of this paper is to take care of the issue raised by author in (Khan, 2016) and investigate the commutativity of $R$. Precisely, we concentrate on the following central-valued conditions: $f(x)F(y) \pm yx \in Z(R)$, $f(x)F(y) \pm xy \in Z(R)$, $f(x)F(y) \pm (x \circ y) \in Z(R)$, $f(x)F(y) \pm [x, y] \in Z(R)$, $F(xy) \pm F(x)F(y) \in Z(R)$, $F[x, y] \pm (x \circ y) \in Z(R)$, $F(x \circ y) \pm [x, y] \in Z(R)$, $F[x, y] \pm xy \in Z(R)$, $F(x \circ y) \pm xy \in Z(R)$, $F[x, y] \pm f(x) \circ y \in Z(R)$, $F(x \circ y) \pm [f(x), y] \in Z(R)$ where $x$ and $y$ are from an appropriate subset of $R$.

## 2. Main Results

**Theorem 1.** *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm yx \in Z(R)$for all $x, y \in I$, then $f$ is commuting on I and I is commutative.*

*Proof.* We consider

$$f(x)F(y) \pm yx \in Z(R) \text{ for all } x, y \in I. \tag{1}$$

Replace $y$ by $yz$ in (1) to get $(f(x)F(y) \pm yx)z + f(x)yf(z) \pm y[z, x] \in Z(R)$ for all $x, y, z \in I$. On commuting with $z$ we obtain

$$[f(x)yf(z), z] \pm [y[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{2}$$

In particular, putting $x = z$ to obtain

$$[f(z)yf(z), z] = 0 \text{ for all } y, z \in I. \tag{3}$$

Which implies that

$$f(z)yf(z)z = zf(z)yf(z) \text{ for all } x, y, z \in I \tag{4}$$

Substituting $yf(z)w$ for y in (4), we have

$$f(z)yf(z)wf(z)z = zf(z)yf(z)wf(z) \text{ for all } x, y, z, w \in I. \tag{5}$$

Using (4) in (5), we obtain $f(z)yzf(z)wf(z) = f(z)yf(z)zwf(z)$ for all $x, y, z, w \in I$. That is $xf(z)y[f(z), z]wf(z) = 0$ for all $x, y, z, w \in I$. It implies that $x[f(z), z]y[f(z), z]w[f(z), z] = 0$ for all $x, y, z, w \in I$. Therefore, $(I[f(z), z])^3 = (0)$ for all $z \in I$. But R has no nonzero nilpotent ideal, we conclude that $I[f(z), z] = (0)$ for all $z \in I$. Thus, $[f(z), z] = 0$ for all $z \in I$(See, (Herstein, 1976)).

Now, Replace $y$ by $yz$ in (2) and we get

$$[f(x)yzf(z), z] \pm [yz[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{6}$$

Right multiply (2) by $z$ and subtract (6) from it, we obtain $[f(x)y[f(z), z], z] \pm [y[[z, x], z], z] = 0$ for all $x, y, z \in I$. Using the fact that $I[f(z), z] = (0)$ for all $z \in I$, we get

$$[y[[z, x], z], z] = 0 \text{ for all } x, y, z \in I. \tag{7}$$

Replace $y$ by $xy$ in (7), we obtain

$$x[y[[z, x], z], z] + [x, z]y[[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{8}$$

Using (7), it reduces to

$$[x, z]y[[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{9}$$

Replace $y$ by $zy$ in (9), we get

$$[x, z]zy[[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{10}$$

Left multiply (9) by $z$ and subtract from (10), we get $[[x, z], z]y[[x, z], z] = 0$ for all $x, y, z \in I$. That is $[[x, z], z]I[[x, z], z] = (0)$ for all $x, z \in I$. Semiprimeness of I yields that

$$[[x, z], z] = 0 \text{ for all } x, z \in I. \tag{11}$$

Linearizing (11) with respect to z and using (11), we have

$$[[x, z], t] + [[x, t], z] = 0 \text{ for all } x, t, z \in I. \tag{12}$$

Replace $z$ by $zt$ in (12), we get $z[[x, t], t] + [z, t][x, t] + ([[x, z], t] + [[x, t], z])t + z[[x, t], t] = 0$ for all $x, t, z \in I$. Using (11) and (12), we obtain

$$[z, t][x, t] = 0 \text{ for all } x, t, z \in I. \tag{13}$$

Replace $x$ by $xy$ in (13) to get $[z, t]x[y, t] + [z, t][x, t]y = 0$ for all $x, y, t, z \in I$. Using (13), we obtain $[z, t]x[y, t] = 0$ for all $x, y, t, z \in I$. In particular, $[y, t]I[y, t] = (0)$ for all $y, t \in I$. It implies that $[y, t] = 0$ for all $y, t \in I$. Hence, $[I, I] = (0)$ as desired.

**Theorem 2.** *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of R. If $f(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then $f$ is commuting on I.*

*Proof.* We consider

$$f(x)F(y) \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{14}$$

Replace $y$ by $yz$ in (14), we get

$$(f(x)F(y) \pm xy)z + f(x)yf(z) \in Z(R) \text{ for all } x, y, z \in I. \tag{15}$$

On commuting with $z$ in (15), we obtain $[f(x)yf(z), z] = 0$ for all $x, y, z \in I$. In particular, put $x = z$, we get $[f(z)yf(z), z]$ for all $y, z \in I$. It coincides with (3), hence Theorem 1. insures the conclusion.

**Theorem 3.** *Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $f(x)F(y) \pm (x \circ y) \in Z(R)$for all $x, y \in I$, then $f$ is commuting on $I$ and $I$ is commutative.*

*Proof.* We consider

$$f(x)F(y) \pm (x \circ y) \in Z(R) \text{ for all } x, y \in I \tag{16}$$

Replace $y$ by $yz$ in (16) to obtain $(f(x)F(y) \pm (x \circ y))z + f(x)yf(z) \mp y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting both sides by $z$, we get $[f(x)yf(z), z] \mp [y[z, x], z] = 0$ for all $x, y, z \in I$. It coincides with (2), hence Theorem 1. insure the conclusions.

**Theorem 4.** *Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $f(x)F(y) \pm [x, y] \in Z(R)$for all $x, y \in I$, then $f$ is commuting on $I$ and $I$ is commutative.*

*Proof.* We consider

$$f(x)F(y) \pm [x, y] \in Z(R) \text{ for all } x, y \in I \tag{17}$$

Replace $y$ by $yz$ in (17) to obtain $(f(x)F(y) \pm [x, y])z + f(x)yf(z) \pm y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting both sides by $z$, we have

$$[f(x)yf(z), z] \pm [y[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{18}$$

Substituting $x = z$ and we get $[f(z)yf(z), z] = 0$ this is same as (3) so by theorem 1, we obtain $[f(z), z] = 0$ for all $z \in I$. Replace $y$ by $yz$ in (18), we get

$$[f(x)yzf(z), z] \pm [yz[x, z], z] = 0 \text{ for all } x, y, z \in I \tag{19}$$

Right multiply (18) by $z$ and subtract (19) from it and we get $[f(x)y[f(z), z], z] \pm [y[[x, z], z], z] = 0$ for all $x, y, z \in I$. Using the fact that $I[f(z), z] = 0$ for all $z \in I$, we obtain $[y[[x, z], z], z] = 0$ for all $x, y, z \in I$. It coincides with (7), hence Theorem 1. insures the conclusion.

**Corollary 5.** *Let $R$ be a semiprime ring. If $(F, f)$ is a multiplicative (generalized) -derivation of $R$ such that any one of the following*

*i.* $f(x)F(y) \pm [x, y] \in Z(R)$

*ii.* $f(x)F(y) \pm (x \circ y) \in Z(R)$

*iii.* $f(x)F(y) \pm yx \in Z(R)$

*holds for all $x, y \in R$, then $R$ is commutative.*

**Theorem 6.** *Let $R$ be a semiprime ring and $I$ a nonzero left ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $F(xy) \pm F(x)F(y) \in Z(R)$ holds for all $x, y \in I$, then $I[f(z), z] = (0)$ for all $z \in I$.*

*Proof.* We consider

$$F(xy) \pm F(x)F(y) \in Z(R) \text{ for all } x, y, z \in I. \tag{20}$$

Replace $y$ by $yz$ in (20), we get $(F(xy) \pm F(x)F(y))z + xyf(z) \pm F(x)yf(z) \in Z(R)$ for all $x, y, z \in I$. On commuting with $z$ and using (20), we obtain

$$[xyf(z), z] \pm [F(x)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{21}$$

Replace $x$ by $xz$ in (21) to get

$$[xzyf(z), z] \pm [F(x)zyf(z), z] \pm [xf(z)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{22}$$

Replace $y$ by $zy$ in (21) and subtract it from (22), we have

$$[xf(z)yf(z), z] = 0 \text{ for all } x, y, z \in I. \tag{23}$$

Substitute $f(z)x$ for $x$ in (23), we get $f(z)[xf(z)yf(z), z] + [f(z), z]xf(z)yf(z) = 0$ for all $x, y, z \in I$. Relation (23) reduce it to

$$[f(z), z]xf(z)yf(z) = 0 \text{ for all } x, y, z \in I. \tag{24}$$

Replace $x$ by $xz$ in (24) and we get

$$[f(z), z]xzf(z)yf(z) = 0 \text{ for all } x, y, z \in I. \tag{25}$$

Replace $y$ by $yz$ in (24), we have

$$[f(z), z]xf(z)zyf(z) = 0 \text{ for all } x, y, z \in I \tag{26}$$

Subtract (25) from (26)to obtain $[f(z), z]x[f(z), z]yf(z) = 0$ for all $x, y, z \in I$. It implies that $(I[f(z), z])^3 = (0)$ for all $z \in I$. Hence, we conclude that $I[f(z), z] = (0)$ for all $z \in I$.

**Corollary 7.** Let $R$ be a semiprime ring and $(F, f)$ a multiplicative (generalized)-derivation of $R$. If $F(xy) \pm F(x)F(y) \in Z(R)$ holds for all $x, y \in R$, then $f$ is a commuting map.

**Theorem 8.** *Let $R$ be a semiprime ring and $I$ a nonzero left ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $F[x, y] \pm (x \circ y) \in Z(R)$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

*Proof.* We consider

$$F[x, y] \pm (x \circ y) \in Z(R) \text{ for all } x, y \in I. \tag{27}$$

If $Z(R) = (0)$ then

$$F[x, y] \pm (x \circ y) = 0 \text{ for all } x, y \in I. \tag{28}$$

Replace $y$ by $yx$ in (28) and we get $(F[x, y] \pm (x \circ y))x + [x, y]f(x) = 0$ for all $x, y \in I$. It reduces to

$$[x, y]f(x) = 0 \text{ for all } x, y \in I \tag{29}$$

Replace $y$ by $f(x)y$ in (29), we have $f(x)[x, y]f(x) + [x, f(x)]yf(x) = 0$ for all $x, y \in I$. Using (29), we obtain

$$[x, f(x)]yf(x) = 0 \text{ for all } x, y \in I. \tag{30}$$

Replace $y$ by $yx$ in (30) and we get

$$[x, f(x)]yxf(x) = 0 \text{ for all } x, y \in I. \tag{31}$$

Right multiply (30) by $x$ and subtract from (31), to obtain $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. Since I is a left ideal of R, so we have $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. Hence, we conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (27), we get $(F[x, y] \pm (x \circ y))t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (27), we get $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we have

$$[[x, y]f(t), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{32}$$

Replace $x$ by $yx$ in (32), we get $[y[x, y]f(t), r] = y[[x, y]f(t), r] + [y, r][x, y]f(t) = 0$ for all $x, y \in I$ and $r \in R$. Using (32), we obtain

$$[y, r][x, y]f(t) = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{33}$$

Replace $r$ by $pr$ in (33) where $p \in R$, we get $p[y, r][x, y]f(t) + [y, p]r[x, y]f(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using (33), we get $[y, p]r[x, y]f(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Substitute $f(t)r$ for $r$ and in particular, we get $[x, y]f(t)R[x, y]f(t) = (0)$ for all $x, y \in I$. Semiprimeness of R implies that

$$[x, y]f(t) = 0 \text{ for all } x, y \in I. \tag{34}$$

Replace $y$ by $f(t)y$ in (34), we get $f(t)[x, y]f(t)+[x, f(t)]yf(t) = 0$ for all $x, y \in I$. Equation (34) forces that $[x, f(t)]yf(t) = 0$ for all $x, y \in I$. It implies $[x, f(t)]y[x, f(t)] = 0$ for all $x, y \in I$. Since I is a left ideal of R so we have $y[x, f(t)]Ry[x, f(t)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(t)] = 0$ for all $x, y \in I$ and $t \in Z(R)$. Hence, we conclude that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

**Theorem 9.** *Let $R$ be a semiprime ring and $I$ a nonzero left ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $F(x \circ y) \pm [x, y] \in Z(R)$ for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

*Proof.* We consider

$$F(x \circ y) \pm [x, y] \in Z(R) \text{ for all } x, y \in I. \tag{35}$$

If $Z(R) = (0)$ then

$$F(x \circ y) \pm [x, y] = 0 \text{ for all } x, y \in I. \tag{36}$$

Replace $y$ by $yx$ in (36), we get $(F(x \circ y) \pm [x, y])x + (x \circ y)f(x) = 0$ for all $x, y \in I$. Using (36) to obtain

$$(x \circ y)f(x) = 0 \text{ for all } x, y \in I \tag{37}$$

Replace $y$ by $f(x)y$ in (37) and we get $f(x)(x \circ y)f(x) + [x, f(x)]yf(x) = 0$ for all $x, y \in I$. Relation (37) implies that

$$[x, f(x)]yf(x) = 0 \text{ for all } x, y \in I. \tag{38}$$

Replace $y$ by $yx$ in (38), we obtain

$$[x, f(x)]yxf(x) = 0 \text{ for all } x, y \in I. \tag{39}$$

Right multiply (38) by $x$ and subtract from (39), we get $[x, f(x)]y[x, f(x)] = 0$ for all $x, y \in I$. Since I is a left ideal of R, so we have $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. Hence, we conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (27) to get $(F[x, y] \pm (x \circ y))t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (27), we left with $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we obtain

$$[(x \circ y)f(t), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{40}$$

Replace $y$ by $xy$ in (40), we get $x[(x \circ y)f(t), r] + [x, r](x \circ y)f(t) = 0$ for all $x, y \in I$ and $r \in R$. Equation (40) reduce it to

$$[x, r](x \circ y)f(t) = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{41}$$

Replace $y$ by $py$ in (41) where $p \in R$, we have $[x, r]p(x \circ y)f(t) + [x, r][x, p]yf(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using the fact that $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$, we get $[x, r](x \circ y)f(t)p + [x, r][x, p]yf(t) = 0$ for all $x, y \in I$ and $r, p \in R$. Using (41) to obtain

$$[x, r][x, p]yf(t) = 0 \text{ for all } x, y \in I \text{ and } r, p \in R. \tag{42}$$

Replacing $r$ by $sr$ where $s \in R$ in (42) and we have $s[x, r][x, p]yf(t) + [x, s]r[x, p]yf(t) = 0$ for all $x, y \in I$ and $p, r, s \in R$. Using (42) to obtain

$$[x, s]r[x, p]yf(t) = 0 \text{ for all } x, y \in I \text{ and } p, r, s \in R. \tag{43}$$

Replace $y$ by $yx$ in (43), we get

$$[x, s]r[x, p]yxf(t) = 0 \text{ for all } x, y \in I \text{ and } p, r, s \in R. \tag{44}$$

Right multiply (43) by $x$ and subtract from (44) to get $[x, s]r[x, p]y[x, f(t)] = 0$ for all $x, y \in I$ and $p, r, s \in I$. Replace $r$ by $ry$ and $y$ by $ry$, we obtain $[x, s]ry[x, p]ry[x, f(t)] = 0$ for all $x, y \in I$ and $p, r, s \in I$. In particular, $[x, f(t)]ry[x, f(t)]ry[x, f(t)] = 0$ for all $x, y \in I$, $r \in I$ and $t \in Z(R)$. It implies $(Ry[x, f(Z(R))])^3 = (0)$ for all $x, y \in I$. But R has no nonzero nilpotent ideal, so we have $Ry[x, f(Z(R))] = (0)$ for all $x, y \in I$. Hence, we conclude that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

**Theorem 10.** *Let $R$ be a semiprime ring and $I$ a nonzero left ideal of $R$. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If $F[x, y] \pm xy \in Z(R)$ holds for all $x, y \in I$, then $I[x, f(x)] = (0)$ or $I[x, f(Z(R))] = (0)$ for all $x \in I$.*

*Proof.* We consider

$$F[x, y] \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{45}$$

If $Z(R) = (0)$ then it is easy to prove that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (45) to obtain $(F[x, y] \pm xy)t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (45), we get $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we have $[[x, y]f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (32), hence Theorem 9. insure the conclusions.

**Theorem 11.** *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* $(F, f)$ *is a multiplicative (generalized)-derivation of R. If* $F(x \circ y) \pm xy \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))] = (0)$ *for all* $x \in I$*.*

*Proof.* We consider

$$F(x \circ y) \pm xy \in Z(R) \text{ for all } x, y \in I. \tag{46}$$

If $Z(R) = (0)$ then it is easy to prove that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (46) and we get $(F[x, y] \pm xy)t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (46), we get $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. On commuting with $r \in R$, we obtain $[(x \circ y)f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (40), hence Theorem 10. insure the conclusions.

**Theorem 12.** *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* $(F, f)$ *is a multiplicative (generalized)-derivation of R. If* $F[x, y] \pm f(x) \circ y \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))]$ *for all* $x \in I$*.*

*Proof.* We consider

$$F[x, y] \pm f(x) \circ y \in Z(R) \text{ for all } x, y \in I. \tag{47}$$

If $Z(R) = (0)$ then we have

$$F[x, y] \pm f(x) \circ y = 0 \text{ for all } x, y \in I. \tag{48}$$

Substitute $yx$ for $y$ in (48) to get $(F[x, y] \pm f(x) \circ y)x + [x, y]f(x) \mp y[f(x), x] = 0$ for all $x, y \in I$. By (48), it reduces to

$$[x, y]f(x) \mp y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{49}$$

Replace $y$ by $f(x)y$ in (49), we get

$$f(x)[x, y]f(x) + [x, f(x)]yf(x) \mp f(x)y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{50}$$

Left multiply (49) by $f(x)$ and subtract from (50), we obtain $[x, f(x)]yf(x) = 0$ for all $x, y \in I$.. Since I is a left ideal in R, it implies that $y[x, f(x)]Ry[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = 0$ for all $x, y \in I$. We conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If $Z(R) \neq (0)$ then there exist some $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (47), we get $(F[x, y] + f(x) \circ y)t + [x, y]f(t) \in Z(R)$ for all $x, y \in I$. Using (47) to obtain $[x, y]f(t) \in Z(R)$ for all $x, y \in I$. That is $[[x, y]f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (32), hence Theorem 9. yields that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

**Theorem 13.** *Let R be a semiprime ring and I a nonzero left ideal of R. Suppose that* $(F, f)$ *is a multiplicative (generalized)-derivation of R. If* $F(x \circ y) \pm [f(x), y] \in Z(R)$ *holds for all* $x, y \in I$*, then* $I[x, f(x)] = (0)$ *or* $I[x, f(Z(R))] = (0)$*for all* $x \in I$*.*

*Proof.* We consider

$$F(x \circ y) \pm [f(x), y] \in Z(R) \text{ for all } x, y \in I. \tag{51}$$

If $Z(R) = (0)$ then we have

$$F(x \circ y) \pm [f(x), y] = 0 \text{ for all } x, y \in I. \tag{52}$$

Replace $y$ by $yx$ in (52) and we obtain $F(x \circ y)x + (x \circ y)f(x) \pm [f(x), y]x \pm y[f(x), x] = 0$ for all $x, y \in I$. Using (52), we left with

$$(x \circ y)f(x) \pm y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{53}$$

Replace $y$ by $f(x)y$ in (53) and we get

$$f(x)(x \circ y)f(x) + [x, f(x)]yf(x) \pm f(x)y[f(x), x] = 0 \text{ for all } x, y \in I. \tag{54}$$

Left multiply (53) by $f(x)$ and subtract it from (54), we obtain $[x, f(x)]yf(x) = 0$ for all $x, y \in I$. It implies that $[x, f(x)]y[x, f(x)] = (0)$ for all $x, y \in I$. Semiprimeness of R yields that $y[x, f(x)] = (0)$ for all $x, y \in I$. We conclude that $I[x, f(x)] = (0)$ for all $x \in I$.

If If $Z(R) \neq (0)$ then there exist some $0 \neq t \in Z(R)$. Replace $y$ by $yt$ in (51), we get $(F(x \circ y) + [f(x), y])t + (x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. Using (51), we obtain $(x \circ y)f(t) \in Z(R)$ for all $x, y \in I$. That is $[(x \circ y)f(t), r] = 0$ for all $x, y \in I$ and $r \in R$. It coincides with (40), hence Theorem 10. yields that $I[x, f(Z(R))] = (0)$ for all $x \in I$.

**Corollary 14.** Let $R$ be a semi-prime ring. Suppose that $(F, f)$ is a multiplicative (generalized)-derivation of $R$. If any one of the following

i. $F[x, y] \pm (x \circ y) \in Z(R)$

ii. $F(x \circ y) \pm [x, y] \in Z(R)$

iii. $F[x, y] \pm xy \in Z(R)$

iv. $F(x \circ y) \pm xy \in Z(R)$

v. $F[x, y] \pm (f(x) \circ y) \in Z(R)$

vi. $F(x \circ y) \pm [f(x), y] \in Z(R)$

holds for all $x, y \in R$, then either $f$ is commuting map or $f(Z(R)) \subseteq Z(R)$.

## 3. Examples

In this section, we build a few examples to show that the condition of semiprimeness in our results is not superfluous. **Example 1.** Consider

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in S \right\},$$

where $S$ is any arbitrary ring.

We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & bc \\ 0 & 0 & 0 \end{pmatrix}, f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it is verified that $F$ is a multiplicative (generalized)-derivations associated with the maps $f$ and it is easy to see that the identities $f(x)F(y) \pm [x, y] \in Z(R)$,$f(x)F(y) \pm (x \circ y) \in Z(R)$ and $f(x)F(y) \pm yx \in Z(R)$ are satisfied for all $x, y \in R$. Here R is not a semiprime ring because

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0).$$

Note that $R$ is not commutative. Hence, the condition of semi-primeness in Corollary 5. can not be omitted.

**Example 2.** Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$ be a ring over integers modulo 2 and let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$, be a left ideal in R. We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & nb \\ 0 & 0 \end{pmatrix}, f\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & (n-1)b \\ 0 & 0 \end{pmatrix},$$

where n is any positive integer. Then it is verified that $F$ is a multiplicative (generalized)-derivations associated with the maps $f$ and it is easy to see that the identities $F(xy) \pm F(x)F(y) \in Z(R)$ are satisfied for all $x, y \in I$. Here R is not a semiprime ring, but observe that $I[f(x), x] \neq (0)$ for all $x \in I$. Hence, the condition of semiprimeness in Theorem 6. is essential.

**Example 3.** Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ stands for the ring of integers. We define maps $F, f : R \to R$ by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then it is verified that $F$ is a multiplicative (generalized)-derivations associated with the maps $f$ and it is easy to see that the identities $F[x,y]\pm(x\circ y) \in Z(R)$,$F(x\circ y)\pm[x,y] \in Z(R)$,$F[x,y]\pm xy \in Z(R)$, $F(x\circ y)\pm xy \in Z(R)$, $F[x,y]\pm(f(x)\circ y) \in Z(R)$ and $F(x \circ y) \pm [f(x), y] \in Z(R)$ are satisfied for all $x, y \in R$. Clearly, R is not a semiprime ring. Note that $f$ is neither commuting on R nor maps $Z(R)$ into $Z(R)$. Hence, the condition of semiprimeness in Corollary 14. can not be removed.

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