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Resistance to Noise of Non-linear Observers in Canonical Form Application to a Sludge Activation Model

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Abstract

The aim of this study was to increase the resistance to noise of an observer of a non-linear MISO system transformed into canonical regulation form of order n. For this, the principle idea was to add n observers on the output equations of the main observer. By adjusting the time scale of the output observers, the resistance to noise of the final estimates is considerably increased. The proposed method is illustrated by model simulations based on a non-linear Sludge Activation Model (SAM)

Keywords: non-linear systems, state observers, continuous time

1. Introduction

State observers have been intensely exploited since (Luenberger, 1966), to model, control or identify linear and non-linear systems, including the studies of (Krener & Isidori, 1983; Zheng, Boutat, & Barbot, 2009) relating to non-linear systems transformable into a canonical form. The key idea in such approaches is to produce approximate measures of non-linearity of order 1, as in Extended *Luenberger* Observers (ELO) (Ciccarella, Mora, & Germani, 1993). Approximations of non-linearities in the canonical form (which results in ELO) have already been suggested (Bestle & Zeitz, 1983), and this approach can be extended to higher order approximations (Röbenack & Lynch, 2004). An observer using a partial non-linear observer canonical form (POCF) (Röbenack & Lynch, 2006) has weaker observability and integrability existence conditions than the well-established non-linear observer canonical form (OCF). Non-linear sliding mode observers use a quasi-Newtonian approach, applied after pseudo-derivations of the output signal (Efimov & Fridman, 2011). State observers using Extended Kalman Filters (EKF) provide another method of transforming non-linear systems (Boker & Khalil, 2013), (Rauh, Butt, & Aschemann, 2013). Finding an appropriate method for parameter synthesis remains one of the major difficulties with state observers for non-linear systems. (Tornambè, 1992), (Mobki, Sadeghia, & Rezazadehb, 2015) proposed high-gain state observers to deal with this problem. High-gain state observers reduce observation errors for a range of predetermined amplitudes or fluctuations by making the observations independent of parameters. The weak point of this method is its sensitivity to noise and uncertainty.

In network identification and encryption, observers with delays are used to synchronize chaotic oscillators, as shown in several studies (Ibrir, 2009; Martínez-Guerra, et al., 2011). Noise and uncertainty are not critical factors in such a context. This can be very different in the case of industrial processes, as shown in a recent study (Bodizs, 2011), where the performances of observers using ELO, EKF or Integrated Kalman Filters (IKF) are compared. The influence of noise and uncertainty on these observer types was emphasized, with more reliable results produced by ELO observers, which permit the exact state reconstruction of highly perturbed systems. For PI and ELO observer classes, (Söffker, et al., 2002) demonstrated a compensation effect on measurement errors ; (Khalifa & Mabrouk, 2015) addressed the problem of uncertainty of non-linear models. One way of overcoming the problem of parametric uncertainty is to use adaptive observers (Tyukina, et al., 2013; Farza, et al., 2014) in the particular case where the measurements are only available at discrete instants and have disturbances. Another approach (Mazenc & Dinh, 2014; Thabet, et al., 2014) consists of defining interval observers. Modeling observer systems by Takagi-Sugeno decomposition (Bezzaoucha, et al., 2013; Guerra, et al., 2015) is another possibility, as is the use of models using symmetries and semi-invariants (Menini & Tornambè, 2011), or the use of immersible techniques for systems transformed into a non-linear observer form (Back & Seo, 2008).

A large number of non-linear MISO systems with multiple inputs and a single output can be transformed into state equations using the form :

$$\dot{\underline{z}}(t) = \underline{s}(t) \quad (1a)$$

$$y(t) = \underline{d}^T \ \underline{z}(t) + \Phi(t) \quad (1b)$$

$$\underline{s}(t) = \begin{bmatrix} s_1 \left[\ \underline{z}(t), \underline{u}_1(t) \ \right] \\ \dots \\ s_n \left[\ \underline{z}(t), \underline{u}_1(t) \ \right] \end{bmatrix} \quad (1c)$$

$$\underline{d}^T = \left[\ d_1 \quad \dots \quad d_n \ \right] \quad (1d)$$

with the following definitions :
n : the order of the system of non-linear differential equations
m : number of independant inputs
$\underline{u}_1(t)^T$: the vector $[u_{11}(t), \dots u_{m1}(t)]$ of the m independent inputs
$y(t)$: the measurable output variable
$\underline{z}^T(t)$: the state vector $[\ z_1(t) \dots z_n(t)\ ]$
$\underline{d}^T$: the vector of the output parameters of the system
$\Phi(t)$: the non-linear function of vector $\underline{u}_1(t)$ of the inputs
$s_i \left[\ \underline{z}(t), \underline{u}_1(t) \ \right]$: one of the n non-linear functions of the state vector $\underline{s}(t)$.

Such systems are often found in nonlinear robotic systems in the form of trigonometric functions. Other systems contain non-linear polynomials (strange attractors, *Bernouilli* functions, non-linear springs), polynomial fractions, or various common simple functions ...
The n non-linear functions of vector $\underline{s}(t)$ employ a vector of m independent inputs $\underline{u}_1(t)$, as well as the state vector $\underline{z}(t)$ as input variables. Such a procedure allows amongst other possibilities the description of bi-linear systems. We limited ourselves in this study to continuous functions in all points of type C^1.
One considers that the measurable output is a linear combination of $\underline{z}(t)$, superimposed on a non-linear function $\Phi \left[\ \underline{u}_1(t) \ \right]$. For an engineer or physicist, many applications have such a form. Often, non-linear systems (1) are transformable in a regulation canonical form concieved by (Fliess, 1990), and are written :

$$\dot{\underline{x}}(t) = \underline{A} \ \underline{x}(t) + \underline{f}(t) \quad (2a)$$

$$y(t) = \underline{c}^T \ \underline{x}(t) + \Phi(t) \quad (2b)$$

$$\underline{A} = \delta_{ij} \quad j = i + 1, \quad i = 1 \dots n - 1 \quad (2c)$$

$$\underline{c}^T = \left[\ c_1 \quad \dots \quad c_n \ \right] \quad (2d)$$

$$\underline{f}(t)^T = \left[\ 0 \quad \dots \quad 0 \quad \Psi \left[\ \underline{x}(t), \ \underline{U}(t) \ \right] \ \right] \quad (2e)$$

with the following definitions :
$\underline{u}_i(t)$: the $(i-1)-th$ temporal derivative of the vector $\underline{u}_1(t)$, either $\underline{u}_i(t)^T = [\ u_{1i}(t), \dots u_{mi}(t)\ ] \quad i = 2 \dots n$
$\underline{U}(t) = \left[\ \underline{u}_1(t) \dots \underline{u}_n(t) \ \right]$: the $n \times m$ input matrix, with the group of n vectors $\underline{u}_i(t)$
$x_i(t)$: $(i-1)\ th$ temporal derivative of $x_1(t)$
$\underline{x}^T(t)$: state vector $[\ x_1(t), \dots x_n(t)\ ]$
$\underline{c}$: the output parameters vector of the transformed system
$\theta \leqslant n$: index of last coefficient $c_i \neq 0$
$\Psi [\ \underline{x}(t), \underline{U}(t)\ ]$: a scalar non-linear C^1 function
$\underline{A}$: the $n \times n$ matrix of which the last line is zero.

Conversion of the transformed version (2) to the initial representation (1) is performed using :

$$\underline{z}(t) = \underline{g}(t) \quad (3a)$$

$$\underline{g}(t)^T = \left[\ g_1 \left[\ \underline{x}(t), \underline{U}(t) \ \right] \ \dots \ g_n \left[\ \underline{x}(t), \underline{U}(t) \ \right] \ \right] \quad (3b)$$

with $\underline{g}(t)$: the vector of n non-linear inverted transformation functions $g_i [\ \underline{x}(t), \ \underline{U}(t)\ ]$ which link $\underline{x}(t)$ to $\underline{z}(t)$.

In (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) a new observer was proposed which was adapted to this transformed form, and which provided non-biased robust estimates of $\underline{x}(t)$. This is not always the case for estimates of $\underline{z}(t)$. Functions $g_i [\underline{x}(t)]$ (1b) permit linking $\underline{x}(t)$ to $\underline{z}(t)$ (2c) and are called inverted transformations. Because of the non-linearity of $\underline{g}(t)$, small perturbations of estimates of $\underline{x}(t)$ may be considerably increased and strongly disturb estimates of $\underline{z}(t)$. The main aim of this study was to solve this type of situation, by introducing the inverted observer transform functions $g_i\left[\underline{x}(t)\right]$. Doing this, the resistance to observer noise is affected (Bodizs et al., 2011), and one obtains a tool capable of limiting its impact on estimates of $\underline{z}(t)$.

Definition 1 *Let us define, for the moment, a normalised pulse $\omega_o = 2\pi/T_o$, which introduces a new time scale τ for the representation of the transformed state of the system :*

$$\dot{\underline{x}}(\tau) = \underline{A}\ \underline{x}(\tau) + \underline{f}(\tau) \tag{4a}$$

$$y(\tau) = \underline{\widetilde{c}}^T\ \underline{x}(\tau) + \Phi(\tau) \tag{4b}$$

$$\underline{\widetilde{c}}^T = \begin{bmatrix} \widetilde{c}_1 & \dots & \widetilde{c}_n \end{bmatrix} \tag{4c}$$

$$\underline{f}(\tau)^T = \begin{bmatrix} 0 & \dots & 0 & \widetilde{\Psi}\left[\underline{x}(\tau), \underline{U}(\tau)\right] \end{bmatrix} \tag{4d}$$

and for the inverse transformation system :

$$\underline{z}(\tau) = \underline{g}(\tau) \tag{5a}$$

$$\underline{g}(\tau)^T = \begin{bmatrix} g_1\left[\underline{x}(\tau), \underline{U}(\tau)\right] & \dots & g_n\left[\underline{x}(\tau), \underline{U}(\tau)\right] \end{bmatrix} \tag{5b}$$

with :

$$\tau = \omega_o\ t, \quad \dot{x}_n(t) = \dot{x}_n(\tau)\ {\omega_o}^n \tag{6a}$$

$$u_{ij}(t) = u_{ij}(\tau)\ {\omega_o}^{i-1}, \quad x_i(t) = x_i(\tau)\ {\omega_o}^{i-1} \tag{6b}$$

$$\widetilde{c}_i = c_i\ {\omega_o}^{i-1}, \quad z_i(t) = z_i(\tau), \quad i = 1 \dots n \tag{6c}$$

$\underline{f}(\tau)$ and $\underline{g}(\tau)$ are vectors with dimension n. In (4b), $\Phi(\tau) = \Phi\left[\underline{u}_1(\tau)\right] = \Phi\left[\underline{u}_1(t)\right]$. Equations (6) define time dilatation or retraction of the state representation and its new parameters, without changing the pattern of the signal $x_i(\tau)$. For the function Ψ, this is translated by the relation of changing the following scale representation :

$$\Psi\left[\underline{x}(t), \underline{U}(t)\right] = {\omega_o}^n\ \widetilde{\Psi}\left[\underline{x}(\tau), \underline{U}(\tau)\right] \tag{7}$$

The function $\widetilde{\Psi}\left[\underline{x}(\tau), \underline{U}(\tau)\right]$ is obtained by replacing every state or command variable by the corresponding one in (6) and dividing everything by ${\omega_o}^n$.

Afterwards, the procedure can be separated into several steps: in section 2, the estimation of the state of the transformed system (4) is dealt with ; in section 3 a new observation method of the inverse transformation functions which permit estimation of state variables (1) is presented ; in section 4 this new approach is applied to observe a system of management of activated sludge in a purification station ; the study is concluded in section 5.

2. Structure of the Observer in Canonical Form

To begin with, let us isolate the componant $x_1(\tau)$ of (4b) which will subsequently serve to determine the observation error. To obtain $y_1(\tau)$, the estimation of variable $x_1(\tau)$, three cases may be distinguished. For $\theta = 1$:

$$y_1(\tau) = \frac{y(\tau) - \Phi(\tau)}{\widetilde{c}_1} \tag{8}$$

For $\theta = 2$, it becomes :

$$\dot{y}_1(\tau) = -\frac{\widetilde{c}_1}{\widetilde{c}_2}\ y_1(\tau) + \frac{y(\tau) - \Phi(\tau)}{\widetilde{c}_2} \tag{9}$$

In the most general case where $\theta > 2$, $y(\tau) - \Phi(\tau)$ is filtered by :

$$\dot{\underline{w}}(\tau) = \underline{K}\ \underline{w}(\tau) + \underline{k}\ \left[\, y(\tau) - \Phi(\tau) \,\right] \tag{10a}$$

$$\underline{K} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & 0 & 0 & 1 \\ -\frac{\widetilde{c}_1}{\widetilde{c}_\theta} & \dots & \dots & -\frac{\widetilde{c}_{\theta-1}}{\widetilde{c}_\theta} \end{bmatrix} \tag{10b}$$

$$\underline{w}(\tau)^T = \begin{bmatrix} y_1(\tau) & \dots & y_{\theta-1}(\tau) \end{bmatrix}, \quad \underline{w}(0) = \underline{0} \tag{10c}$$

$$\underline{k}^T = \begin{bmatrix} 0 & \dots & 0 & 1/\widetilde{c}_\theta \end{bmatrix} \tag{10d}$$

To analyze the effect of the filter, we rewrite (4b) in scalar form, ignoring $\widetilde{c}_{\theta+1}\ \ldots\ \widetilde{c}_n$, which are all zero :

$$y(\tau) - \Phi(\tau) = \sum_{i=1}^{\theta} \widetilde{c}_i\ x_i(\tau) \tag{11}$$

If (11) is inserted in (10a), (9) or (8) as a function of θ, it becomes :

$$\sum_{i=1}^{\theta} \widetilde{c}_i\ y_i(\tau) = \sum_{i=1}^{\theta} \widetilde{c}_i\ x_i(\tau) \tag{12a}$$

$$\dot{y}_{\theta-1}(\tau) = y_\theta(\tau) \quad \theta \geqslant 2 \tag{12b}$$

The *Laplace* transformation of (12a) gives the transfer function :

$$y_1(s)/x_1(s) = 1 \tag{13}$$

To develop the rest, $y_1(\tau)$ is used to determine the observer error.

Definition 2 *To generate state estimates $\underline{v}(\tau)$ for the system (4), a PI observer structure is defined in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) with :*

$$\dot{\underline{\breve{x}}}(\tau) = \underline{A}\ \underline{\breve{x}}(\tau) + \underline{\widetilde{f}}(\tau) + \underline{\breve{h}}\ \Delta y_1(\tau) \tag{14a}$$

$$\dot{\underline{\widehat{x}}}(\tau) = \underline{A}\ \underline{\widehat{x}}(\tau) + \underline{\breve{A}}\ \underline{\breve{x}}(\tau) + \underline{\widehat{h}}\ \Delta y_1(\tau) \tag{14b}$$

$$\Delta y_1(\tau) = x_1(\tau) - \widehat{x}_1(\tau) \tag{14c}$$

$$\underline{\widetilde{f}}(\tau)^T = \begin{bmatrix} 0 & \ldots & 0 & \widetilde{f}(\tau) \end{bmatrix} \tag{14d}$$

$$\dot{I}_0(\tau) = h_0\ \Delta y_1(\tau) \tag{14e}$$

$$\widetilde{f}(\tau) = I_0(\tau) + \widetilde{\Psi}\left[\ \underline{v}(\tau), \underline{U}(\tau)\ \right] \tag{14f}$$

$$\underline{\breve{x}}(\tau)^T = \begin{bmatrix} \breve{x}_2(\tau) & \ldots & \breve{x}_n(\tau) \end{bmatrix} \tag{14g}$$

$$\underline{\widehat{x}}(\tau)^T = \begin{bmatrix} \widehat{x}_1(\tau) & \ldots & \widehat{x}_{n-1}(\tau) \end{bmatrix} \tag{14h}$$

$$\underline{v}(\tau)^T = \begin{bmatrix} \widehat{x}_1(\tau) & \underline{\breve{x}}(\tau)^T \end{bmatrix} \tag{14i}$$

$$\underline{\widehat{x}}(0) = \underline{\breve{x}}(0) = \underline{0}, \quad I_0(0) = 0 \tag{14j}$$

$$\underline{\breve{h}}^T = \begin{bmatrix} \underline{0}^T & h_1 \end{bmatrix} \tag{14k}$$

$$\underline{\widehat{h}}^T = \begin{bmatrix} h_n & \ldots & h_2 \end{bmatrix} \tag{14l}$$

$$\underline{h}^T = \begin{bmatrix} \underline{\widehat{h}}^T, & h_1 \end{bmatrix} \tag{14m}$$

$$\underline{A} = \delta_{ij}, \quad j = i+1,\ i = 1 \ldots n-1 \tag{14n}$$

$$\underline{\breve{A}} = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ldots & \vdots \\ \vdots & & 0 & \vdots \\ 0 & \ldots & \ldots & 1 \end{bmatrix} \tag{14o}$$

with $\underline{\breve{x}}(\tau)$ (14g) and $\underline{\widehat{x}}(\tau)$ (14h) as two distinct state vectors of dimension $n-1$, coupled using the matrices $\underline{A}$ (14n) and $\underline{\breve{A}}$ (14o) of dimension $(n-1)\times(n-1)$. The vectors $\underline{\breve{h}}$ and $\underline{\widehat{h}}$ are also of dimension $n-1$. The matrix $\underline{A}$ is constructed using the Kronecker operator which puts the upper diagonal at 1. The parameters $h_i,\ i = 0 \ldots n$ are the gains of the observer.

Figure 1 illustrates the functional diagram of such an observer of third order.

The augmented vector $\underline{v}(\tau)$ (14i),(14h) and (14g) is used as estimation of $\underline{x}(\tau)$ and as variable of the function $\widetilde{\Psi}\left[\ \underline{v}(\tau), \underline{U}(\tau)\ \right]$ (14f). The state $\underline{\widehat{x}}(\tau)$ (14b) is an observer exploiting the observation error $\Delta y_1(\tau)$ (14c) *via* the gains h_i (14m) serving to correct the state distances between the system and its observer.

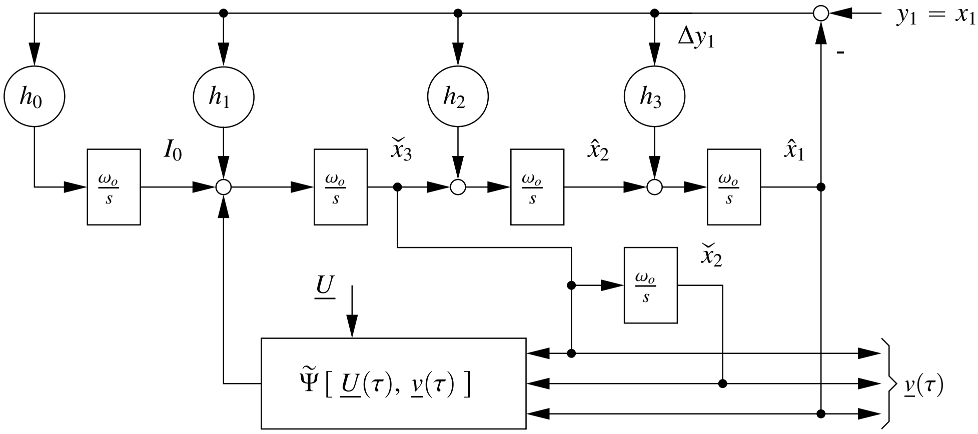


Figure 1. Third order observer

In figure 1, for example, we have :

$$
\begin{aligned}
\underline{\widehat{x}}(\tau)^T &= \left[\ \widehat{x}_1(\tau) \quad \widehat{x}_2(\tau)\ \right] \\
\underline{\breve{x}}(\tau)^T &= \left[\ \breve{x}_2(\tau) \quad \breve{x}_3(\tau)\ \right] \\
\underline{v}(\tau)^T &= \left[\ \widehat{x}_1(\tau) \quad \breve{x}_2(\tau) \quad \breve{x}_3(\tau)\ \right] \\
\underline{\widehat{h}}^T &= \left[\ h_3 \quad h_2\ \right] \\
\underline{\breve{h}}^T &= \left[\ 0 \quad h_1\ \right]
\end{aligned}
$$

The choice of using two state variables $\underline{\widehat{x}}(\tau)$ and $\underline{\breve{x}}(\tau)$ is motivated by the $n-1$ successive integrations of $\dot{\breve{x}}_n(\tau)$ in which no $\underline{\widehat{h}}\ \Delta y_1(\tau)$ re-injection error is involved. This allows an increase in the robustness of the estimations to the measurement noise, which in general affects the variable $y_1(\tau)$. One thus overcomes a common weak point of high gain observations, i.e. their sensitivity to measurement noise. The second advantage comes from the non-linear function $\widetilde{\Psi}\left[\underline{v}(\tau), \underline{U}(\tau)\right]$ which is no longer subjected to the restrictive conditions used in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2013), and covers the ensemble of the systems described by (Fliess, 1990). The vector $\underline{\widetilde{f}}(\tau)$ (14d), of dimension $n-1$, compensates the effects of $\underline{f}(\tau)$, and of possible external exogenous disturbance of (2) using the integral component $I_0(\tau)$ (14e). One notes that at the second order, for a gain $h_0 = 0$ inhibiting the integrator I_0, the observer becomes similar to that proposed by (Gauthier, Hammouri, & Othman, 1992) for a SISO system.

In (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), a full analysis was performed in order to determine the dynamics of the observation error $\Delta y_1(\tau)$ (14c) and its successive derivatives, to characterise stability conditions and also the exponential convergent nature of estimates $\underline{v}(\tau)$. A mthod to synthesize parameters $h_0 \ldots h_n$ was also proposed.

3. Observation of the Original System via the Inverted Transformation Functions

3.1 New observers definitions

In (5b), the inverted transform functions $\underline{g}(\tau)$ allow converting the system in the canonical form of regulation back to the original form (1). Using the estimates $\underline{v}(\tau)$ reconstructed by the observer (14), it is possible to define : (15)

$$\widehat{z}_i(t) = \widehat{g}_i\left[\ \underline{v}(\tau),\ \underline{U}(\tau)\ \right] \quad i = 1 \ldots n \tag{15a}$$

$$\underline{\widehat{z}}(t)^T = \left[\ \widehat{z}_1(t) \quad \ldots \quad \widehat{z}_n(t)\ \right] \tag{15b}$$

One thus obtains estimates $\underline{\widehat{z}}(t)$ of $\underline{z}(t)$ (1). If the stability conditions (Theorem 1 of (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)) are respected, $\underline{\widehat{z}}(t) \to \underline{z}(t)$ when $\Delta y_1(t) \to 0$. Similarly, $\dot{\underline{\widehat{z}}}(t) \to \dot{\underline{z}}(t)$ when $\Delta y_1(t) \to 0$. One then has :

$$\lim_{\Delta y_1(t) \to 0} \dot{\underline{\widehat{z}}}(t) = \dot{\underline{z}}(t) \tag{16a}$$

$$\dot{\underline{\widehat{z}}}(t)^T = \left[\ \widehat{s}_1(t) \quad \ldots \quad \widehat{s}_n(t)\ \right] \tag{16b}$$

$$\widehat{s}_i(t) = s_i\left[\ \underline{\widehat{z}}(t),\ \underline{u}_1(t)\ \right] \quad i = 1 \ldots n \tag{16c}$$

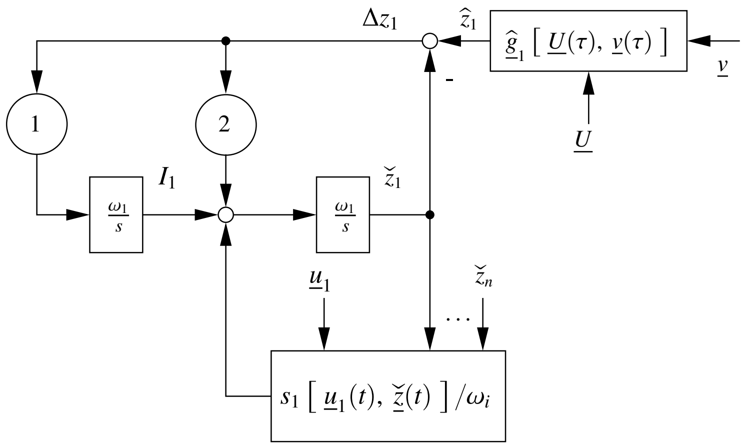


Figure 2. Observer of inverse function $\widehat{z}_1(\tau_1)$

The n estimates $\widehat{z}_i(t)$ can be used as reference inputs to observe n state variables $\breve{z}_i(t)$ which tend towards (15a). Their temporal derivatives tend towards $\dot{\underline{\widehat{z}}}(t)$, which themselves tend towards $\dot{\underline{z}}(t)$ (16). With the model (14), one defines n first order observers. Each is normalised by a pulse ω_i which leads to its dimensionless time definition (17e), possesses its own *Lipschitz* constant, and its specific stability conditions that we have to find. Synthesising the gains h_i (subsection 2.4 of (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)) gives $h_0 = 1$ and $h_1 = 2$. The n observers are written :

$$\dot{\breve{z}}_i(\tau_i) = I_i(\tau_i) + 2\ \Delta z_i(\tau_i) + \breve{s}_i(\tau_i) \tag{17a}$$

$$\dot{I}_i(\tau_i) = \Delta z_i(\tau_i) \quad i = 1 \ldots n \tag{17b}$$

$$\Delta z_i(\tau_i) = \widehat{z}_i(\tau_i) - \breve{z}_i(\tau_i) \tag{17c}$$

$$\breve{s}_i(\tau_i) = s_i \left[\underline{\breve{z}}(t),\ \underline{u}_1(t) \right] \ / \ \omega_i \tag{17d}$$

$$\tau_i = \omega_i\ t \tag{17e}$$

with $\underline{\breve{z}}(t) = \left[\breve{z}_1(t) \quad \ldots \quad \breve{z}_n(t) \right]$ the vector of the estimations of $\underline{\widehat{z}}(t)$; $\breve{s}_i(\tau_i)$ is the normalised non-linear function of $\dot{\breve{z}}_i(\tau_i)$. Figure 2 illustrates (15) and (17).

The general calculation procedure is as follows :

- estimation of $\underline{v}(\tau)$ (14i) after treatment of (14);
- estimation of $\underline{\widehat{z}}(t)$ (15) ;
- estimation of the n state distances (17c) ;
- determination of the n non-linear functions $\breve{s}_i(\tau_i)$ (17d) to access the n terms $\dot{\breve{z}}_i(\tau_i)$ (17a) and $\dot{I}_i(\tau_i)$ (17b) ;
- integration of the n equations (17a) to obtain $\underline{\breve{z}}(t)$.

The temporal derivative of (17c) and inserting (17a) in the rest obtained enables one to obtain the expression of $\Delta\dot{z}_i(\tau)$:

$$\Delta\dot{z}_i(\tau_i) = \dot{\widehat{z}}_i(\tau_i) - \dot{\breve{z}}_i(\tau_i) \quad i = 1 \ldots n \tag{18a}$$

$$= \Delta\widetilde{\Psi}_i(\tau_i) - I_i(\tau_i) - 2\ \Delta z_i(\tau_i) \tag{18b}$$

$$\Delta\widetilde{\Psi}_i(\tau_i) = \widehat{s}_i(\tau_i) - \breve{s}_i(\tau_i) \tag{18c}$$

$$\widehat{s}_i(\tau_i) = s_i \left[\underline{\widehat{z}}(t),\ \underline{u}_1(t) \right] \ / \ \omega_i \tag{18d}$$

3.2 Dynamics of the Observer Errors

We now characterise the dynamics of the observer errors by searching the n differential equations of the distances $\Delta z_i(\tau_i)$. Due to the presence of integrators $I_i(\tau_i)$, an extra temporal derivative is necessary to obtain the differential equation of the distances $\Delta z_i(\tau_i)$. To do this, it is necessary to define the following augmented vectors :

$$\underline{\Upsilon}(\tau_i) = \left[\ \underline{u}_1(\tau_i) \quad \underline{u}_2(\tau_i)\ \right] \tag{19a}$$

$$\underline{\widehat{z}}_i(\tau_i)^T = \left[\ \widehat{z}_i(\tau_i) \quad \dot{\widehat{z}}_i(\tau_i)\ \right] \quad i = 1 \dots n \tag{19b}$$

$$\underline{\breve{z}}_i(\tau_i)^T = \left[\ \breve{z}_i(\tau_i) \quad \dot{\breve{z}}_i(\tau_i)\ \right] \tag{19c}$$

$$\Delta\underline{z}_i(\tau_i)^T = \left[\ \Delta z_i(\tau_i) \quad \Delta\dot{z}_i(\tau_i)\ \right] \tag{19d}$$

$$\underline{\widehat{Z}}(\tau_i)^T = \left[\ \underline{\widehat{z}}_1(\tau_i) \quad \dots \quad \underline{\widehat{z}}_n(\tau_i)\ \right] \tag{19e}$$

$$\underline{\breve{Z}}(\tau_i)^T = \left[\ \underline{\breve{z}}_1(\tau_i) \quad \dots \quad \underline{\breve{z}}_n(\tau_i)\ \right] \tag{19f}$$

$$\Delta\underline{Z}(\tau_i)^T = \left[\ \Delta\underline{z}_1(\tau_i) \quad \dots \quad \Delta\underline{z}_n(\tau_i)\ \right] \tag{19g}$$

The temporal derivative of (18b) is written :

$$\Delta\ddot{z}_i(\tau_i) = \Delta\dot{\tilde{\Psi}}_i(\tau_i) - \Delta z_i(\tau_i) - 2\ \Delta\dot{z}_i(\tau_i) \quad i = i \dots n \tag{20a}$$

$$\Delta\dot{\tilde{\Psi}}_i(\tau_i) = \dot{s}_i \left[\ \underline{\widehat{Z}}(\tau_i), \underline{\Upsilon}(\tau_i)\ \right] - \dot{s}_i \left[\ \underline{\breve{Z}}(\tau_i), \underline{\Upsilon}(\tau_i)\ \right] \tag{20b}$$

and gives the scalar expression of the differential equations of the observation errors. Using notations (19) gives the matricial writing of (20a) in the form of state equations :

$$\Delta\dot{\underline{z}}_i(\tau_i) = \underline{A}_i\ \Delta\underline{z}_i(\tau_i) + \Delta\dot{\underline{\tilde{\Psi}}}_i(\tau_i) \quad i = 1 \dots n \tag{21a}$$

$$\underline{A}_i = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \tag{21b}$$

$$\Delta\dot{\underline{\tilde{\Psi}}}_i(\tau_i) = \begin{bmatrix} 0 \\ \Delta\dot{\tilde{\Psi}}_i(\tau_i) \end{bmatrix} \tag{21c}$$

Assuming that the non-linear functions $\dot{s}_i$ are at least locally *Lipschitz* in $\underline{Z}(\tau_i)$, and uniformly bounded in $\underline{\Upsilon}(\tau_i)$ in an invariant set, they are associated with a *Lipschitz* constant L_i :

$$\left\|\ \Delta\dot{\tilde{\Psi}}_i(\tau_i)\ \right\| \leqslant L_i\ \|\ \Delta\underline{Z}(\tau_i)\ \| \quad i = 1 \dots n \tag{22}$$

Applying the *Lipschitz* inequality to (20b) permits reduction to $\Delta\underline{Z}(\tau_i)$ the number of useful variables to characterise the perturbing difference $\Delta\dot{\tilde{\Psi}}_i(\tau_i)$. For many systems, if functions $\dot{s}_i$ are not globally of a *Lipschitz* type, they can be locally or be transformed adequately into the *Lipschitz* type.

3.3 Convergence of State Observations

Now let us try to analyse the globally asymptotic development of the observation errors and to characterise the limiting stability conditions of each observer (17).

Theorem 1 *Let us consider a MISO system decomposable as described in* (4)*, for which the observer structures* (14) *and* (17) *are used, and related to each other by the inverted transform function* (15a)*. If the system function* $\dot{s}_i \left[\ \underline{\widehat{Z}}(\tau_i), \underline{\Upsilon}(\tau_i)\ \right]$ *is locally of the Lipschitz type in* $\underline{\widehat{Z}}(\tau_i)$ *and uniformly bounded in* $\underline{\Upsilon}(\tau_i)$ *in an invariant set, with a Lipschitz constant* L_i (22)*, then the observer* (17) *will be locally stable if the Lipschitz constant* L_i *satisfies the following conditions :*

$$L_i^2 \leqslant \frac{2\ \sigma_i\ \phi_{i1} - 1}{4\ n\ \sigma_i^2\ \phi_{i1} \left(\dfrac{\lambda_i^2}{\phi_{i1}} + \dfrac{\phi_{i1}}{4} \right)} \tag{23a}$$

$$L_i^2 \leqslant \frac{2\ \sigma_i\ (\ 4\ \phi_{i2} - \phi_{i1}\ ) - 1}{4\ n\ \sigma_i^2\ \phi_{i2} \left(\phi_{i2} + \dfrac{\phi_{i1}^2}{4\ \phi_{i2}} \right)} \tag{23b}$$

$$\lambda_i > 0, \quad \sigma_i > 0, \quad \phi_{i1} > 0, \quad \phi_{i2} > 0 \quad i = 0 \dots n \tag{23c}$$

If the system function $\dot{s}_i\left[\ \underline{\widehat{Z}}(\tau_i), \underline{\Upsilon}(\tau_i)\ \right]$ *is globally of the Lipschitz type, and if the Lipschitz constant* L_i *satisfy* (23)*, then the observers* (17) *will be globally asymptotically stable.*

Proof. The proof of theorem 1 can be demonstrated by proving the stability of (21a) using an appropriate positive *Lyapunov* function, like the following quadratic function :

$$\breve{V}_n(\tau_i) = \sum_{i=1}^{n} v_i(\tau_i) \tag{24a}$$

$$v_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{P}_i\ \Delta\underline{z}_i(\tau_i) \tag{24b}$$

$$\underline{P}_i = \begin{bmatrix} \lambda_i & 0 \\ \phi_{i1} & \phi_{i2} \end{bmatrix} \tag{24c}$$

The $\underline{P}_i$ lower triangular matrix are defined as positive and satisfying the *Sylvester* criteria, with (24c). The proof of convergence is linked to the study of the sign of the derivative of the candidate for a *Lyapunov* function. This is obtained after temporal derivation of (24a), and after placing (21a) in the result obtained for terms $\Delta\dot{\underline{z}}_i(\tau_i)$:

$$\dot{\breve{V}}_n(\tau_i) = \sum_{i=1}^{n} \dot{v}_i(\tau_i) \tag{25a}$$

$$\dot{v}_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{Q}_i\ \Delta\underline{z}_i(\tau_i) + N_i(\tau_i) \quad i = 0\ldots n \tag{25b}$$

$$\begin{aligned} \underline{Q}_i &= \underline{A}_i{}^T \underline{P}_i + \underline{P}_i\ \underline{A}_i \\ &= \begin{bmatrix} -\phi_{i1} & 0 \\ 2(\lambda_i - \phi_{i1}) - \phi_{i2} & \phi_{i1} - 4\phi_{i2} \end{bmatrix} \end{aligned} \tag{25c}$$

$$N_i(\tau_i) = \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \Delta\dot{\tilde{\underline{\Psi}}}_i(\tau_i) \tag{25d}$$

$$\underline{S}_i = \underline{P}_i + \underline{P}_i^T \tag{25e}$$

An appropriate choice of ϕ_{i1}, ϕ_{i2} can provide negative diagonal coefficients for $\underline{Q}_i$. The criterion of semi-negativity of *Sylvester* is then respected, and the successive minors of $\underline{Q}_i$ will be of opposite sign, ensuring the semi-negativity of the first member on the right of (25b). Verifying the sign of the second member on the right of (25b) involves increasing $N_i(\tau_i)$ using the inequalities of *Schwartz* and *Lipschitz* (22) :

$$N_i(\tau_i) \leqslant \left\|\ \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \Delta\dot{\tilde{\underline{\Psi}}}(\tau_i)\ \right\| \tag{26a}$$

$$\leqslant \left\|\ \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \right\|\ \left\|\ \Delta\dot{\tilde{\underline{\Psi}}}(\tau_i)\ \right\| \tag{26b}$$

$$\leqslant \left\|\ \Delta\underline{z}_i(\tau_i)^T\ \underline{S}_i\ \right\|\ L\ \left\|\ \Delta\underline{z}_i(\tau_i)\ \right\| \tag{26c}$$

To determine the sign of $\dot{v}_i(\tau_i)$ function, one applies the following inequality :

$$\left\|\ \underline{a}(\tau_i)^T\underline{b}(\tau_i)\ \right\| \leqslant \frac{n\ \sigma_i}{2}\ \underline{a}(\tau_i)^T\underline{a}(\tau_i) + \frac{1}{2\ n\ \sigma_i}\ \underline{b}(\tau_i)^T\underline{b}(\tau_i) \tag{27a}$$

$$\underline{a}(\tau_i) = L\ \underline{S}_i{}^T\ \underline{z}_i(\tau_i) \tag{27b}$$

$$\underline{b}(\tau_i) = \Delta\underline{z}_i(\tau_i) \tag{27c}$$

to (26c) to obtain the desired increase of $N_i(\tau_i)$:

$$N_i(\tau_i) \leqslant \Delta\underline{z}_i(\tau_i)^T\ \underline{R}_i\ \Delta\underline{z}_i(\tau_i) \quad i = 1\ldots n \tag{28a}$$

$$\underline{R}_i = \frac{n\ \sigma_i L_i{}^2}{2}\ \underline{S}_i\ \underline{S}_i + \frac{\underline{I}}{2\ n\ \sigma_i} \tag{28b}$$

In (28a) yields a positive lower triangular matrix $\underline{R}_i$ (28b), the diagonal elements of which are written :

$$r_{jj} = \begin{cases} 2\ n\ \sigma_i\ L_i^2\left(\lambda_i{}^2 + \phi_{i1}{}^2/4\right) + \dfrac{1}{2\sigma_i} & j = 1 \\ 2\ n\ \sigma_i\ L_i^2\left(\phi_{i2}{}^2 + \phi_{i1}{}^2/4\right) + \dfrac{1}{2\sigma_i} & j = 2 \end{cases} \tag{29}$$

The inequality (28a) permits deduction of (25b) :

$$\dot{v}_i(\tau_i) \leqslant \Delta\underline{z}_i(\tau_i)^T \; \underline{M}_i \; \Delta\underline{z}_i(\tau_i) \tag{30a}$$

$$\underline{M}_i = \underline{Q}_i + \underline{R}_i \tag{30b}$$

With negative functions $\dot{v}_i(\tau_i)$, adding together the diagonal terms of (25c) and (29), and imposing $\underline{Q}_i + \underline{R}_i \leqslant 0$, one obtains the conditions (23). The sum $\underline{Q}_i + \underline{R}_i$ yields an inferior triangular matrix that satisfies *Sylvester* criteria of semi-negativity if inequalities (23a) and (23b) are satisfied. Then, if $\Delta\dot{\tilde{\Psi}}_i(\tau_i)$ (20b) is *Lipschitz* (22), $\dot{v}_i(\tau_i)$ is semi-negative and (21a) is globally and asymptotically stable ; (21a) is locally stable if (22) is locally *Lipschitz* ■
Using the (theorem 2, section 2.3, (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016)), it is easy to demonstrate that the observers (17) will be exponentially convergent.

4. Application to a Sludge Activation Model

Let us now illustrate the proposed observation method by applying it to a non-linear example with multiple inputs.

4.1 Original Model

For this we choose a simplified treatment model for activated sludge ASM1 similar to that used by (Nagy-Kiss et al., 2010), and structurally of the same types as (1) :

$$\dot{z}_i(t) = s_i(t) \quad i = 1 \ldots 3 \tag{31a}$$

$$y_1(t) = z_1(t) \quad y_2(t) = z_2(t) \tag{31b}$$

with :

$$s_1(t) = k_1 \; \ell_1(t) + k_2 \; z_3(t) - k_3 \; \ell_2(t) \; z_3(t) \tag{32a}$$

$$s_2(t) = k_4 \; \ell_3(t) - k_1 \; \ell_4(t) - k_5 \; \ell_2(t) \; z_3(t) \tag{32b}$$

$$s_3(t) = k_9 \; \ell_5(t) - k_6 \; \ell_6(t) - k_7 \; z_3(t) + k_8 \; \ell_2(t) \; z_3(t) \tag{32c}$$

and non-linear functions :

$$\ell_1(t) = u_{11}(t) \; (\; u_{31}(t) - z_1(t) \;) \tag{33a}$$

$$\ell_2(t) = \frac{z_1(t) \; z_2(t)}{(\; k_{10} + z_1(t) \;) \; (\; k_{11} + z_2(t) \;)} \tag{33b}$$

$$\ell_3(t) = u_{21}(t) \; [\; k_{12} - z_2(t) \;] \tag{33c}$$

$$\ell_4(t) = u_{11}(t) \; z_2(t) \tag{33d}$$

$$\ell_5(t) = u_{11}(t) \; u_{41}(t) \tag{33e}$$

$$\ell_6(t) = u_{11}(t) \; z_3(t) \tag{33f}$$

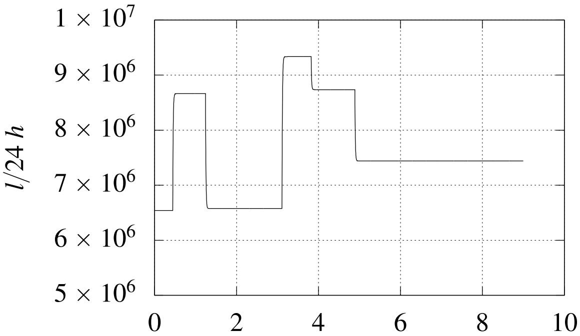
$$\underline{u}_1(t) = \begin{bmatrix} u_{11}(t) & \ldots & u_{41}(t) \end{bmatrix} \tag{33g}$$

The constants used are given by :

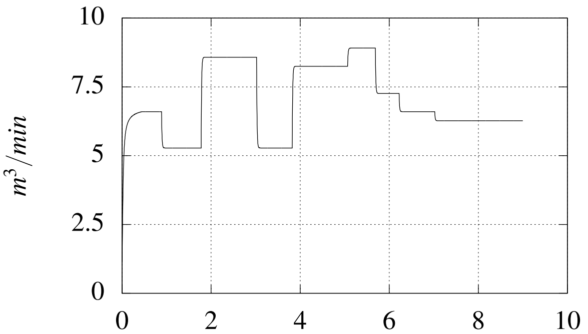
$$\begin{array}{lll} k_1 = 5 \cdot 10^{-11} & k_2 = 1,08 \cdot 10^{-5} & k_3 = 2.872 \cdot 10^{-4} \\ k_4 = 3.5 \; 10^{-4} & k_5 = 9 \cdot 10^{-5} & k_6 = 1.316 \cdot 10^{-12} \\ k_7 = 4.8 \cdot 10^{-6} & k_8 = 7.47 \cdot 10^{-5} & k_9 = 8 \cdot 10^{-11} \\ k_{10} = 20 & k_{11} = 0.2 & k_{12} = 10 \end{array} \tag{34}$$

$\underline{u}_1(t)$ (33g) represent the inputs of the system (figures 3(a),(b),(c),(d) page 10), respectively the input flow of waste water, the flow of injected air, the concentration soluble carbonated substrate recycled, the particle concentration of recycled heterotrophic biomass. All abscissas of the figures are expressed in hours.

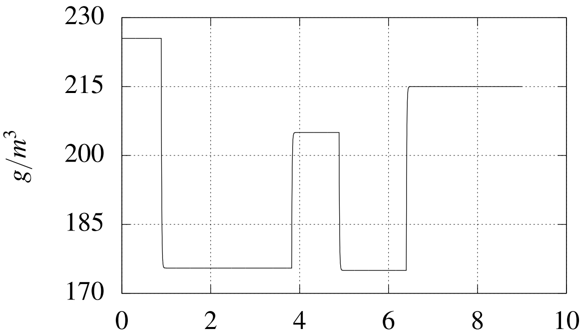
The variables $z_1(t)$, $z_2(t)$ $z_3(t)$ represent the state of the reactor (figures 3(e),(f),(g)), respectively the concentration of rapidly biodegradable substrate, the concentration of dissolved oxygen, the particle concentration of biomass, with (34) its parameters, all known, and $z_1(0) = 4.1$, $z_2(0) = 3.0$, $z_3(0) = 867$ the initial conditions. The sizes $y_1(t)$, $y_2(t)$ (31b) represent the measurable outputs.



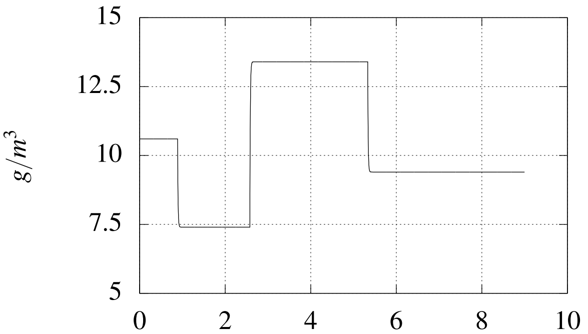
(a) $u_{11}(t)$, water flow



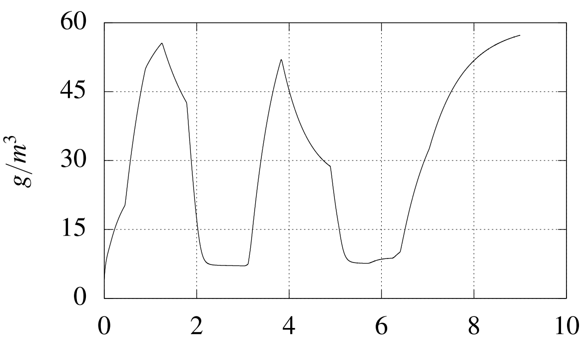
(b) $u_{21}(t)$, air flow



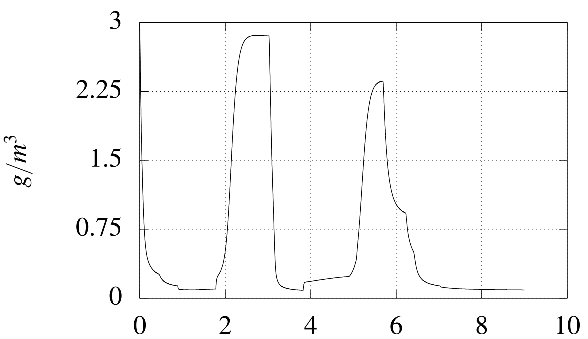
(c) $u_{31}(t)$, injection of carbonated substrate



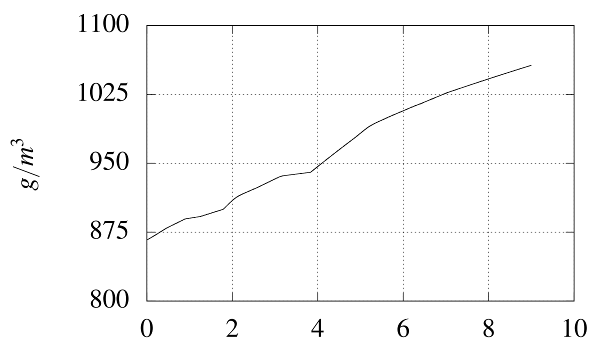
(d) $u_{41}(t)$, injection of biomass



(e) $z_1(t)$, concentration of biodegradable carbonated substrate



(f) $z_2(t)$, concentration of dissolved oxygen



(g) $z_3(t)$, concentration of biomass

Figure 3. Input variables and state variables of the bioreactor

4.2 Model Transformed in a Canonical Form of Regulation

We now transform the two first order differential equations (32b) and (32c) into a single second order differential equation. With this one we determine a function $\Psi[\ \underline{x}(t),\ \underline{U}(t)\ ]$ and a new differential equation in a canonical form of regulation. As this application of the general procedure of transformation permits passage from systems (1) to (2) (Fliess, 1990) it will permit the use of an observer similar to that proposed in (14), associated with inverted transformation (15) and with observers (17).

In our example we try to observe the measured output $y_2(t)$ and its successive derivatives, to subsequently determine an estimation of the immeasurable variable $z_3(t)$. From $s_2(t)$ (32b) we deduce $z_3(t)$:

$$z_3(t) = \frac{k_4\ \ell_3(t) - k_1\ \ell_4(t) - \dot{z}_2(t)}{k_5\ \ell_2(t)} \tag{35}$$

The temporal derivative of (35) gives an expression of $\dot{z}_3(t)$ which can be equated to $s_3(t)$ (32c). We thus deduce :

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) && \text{(36a)}\\
\dot{x}_2(t) &= \Psi[\ \underline{x}(t),\ \underline{U}(t)\ ] && \text{(36b)}\\
\Psi[\ \underline{x}(t),\ \underline{U}(t)\ ] &= k_4\ \dot{\ell}_3(t) - k_1\ \dot{\ell}_4(t) - k_5\ \ell_7(t) && \text{(36c)}\\
\ell_2(t) &= \frac{z_1(t)\ x_1(t)}{(\ k_{10} + z_1(t)\ )\ (\ k_{11} + x_1(t)\ )} && \\
&= n(t)/d(t) && \text{(36d)}\\
\ell_3(t) &= u_{21}(t)\ \ [\ k_{12} - x_1(t)\ ] && \text{(36e)}\\
\ell_4(t) &= u_{11}(t)\ x_1(t) && \text{(36f)}\\
\ell_6(t) &= u_{11}(t)\ z_3(t) && \text{(36g)}\\
\ell_7(t) &= z_3(t)\ \dot{\ell}_2(t) + s_3(t)\ \ell_2(t) && \text{(36h)}\\
z_2(t) &= x_1(t) && \text{(36i)}
\end{aligned}$$

System (36) is made up of a second order differential equation, in a canonical regulation form structurally of the same type as that described in (2). The derived functions $\dot{\ell}_3(t),\ \dot{\ell}_4(t)$ of (36c), and $\dot{\ell}_2(t),\ s_3(t)$ of (36h) are defined by :

$$\begin{aligned}
\dot{\ell}_2(t) &= \dot{n}(t)\ d(t) - \dot{d}(t)\ n(t)/d(t)^2 && \text{(37a)}\\
\dot{n}(t) &= s_1(t)\ x_1(t) + z_1(t)\ x_2(t) && \text{(37b)}\\
\dot{d}(t) &= s_1(t)\ (k_{11} + x_1(t)) + x_2(t)\ (k_{10} + z_1(t)) && \text{(37c)}\\
s_1(t) &= k_1\ \ell_1(t) + k_2\ g_3(t) - k_3\ \ell_2(t)\ g_3(t) && \text{(37d)}\\
\dot{\ell}_3(t) &= u_{22}(t)\ (\ k_{12} - x_1(t)\ ) - u_{21}(t)\ x_2(t) && \text{(37e)}\\
\dot{\ell}_4(t) &= u_{12}(t)\ x_1(t) + u_{11}(t)\ x_2(t) && \text{(37f)}\\
s_3(t) &= k_9\ \ell_5(t) - k_6\ \ell_6(t) - k_7\ g_3(t) + k_8\ \ell_2(t)\ g_3(t) && \text{(37g)}
\end{aligned}$$

Equation $\dot{z}_1(t) = s_1(t)$ defined in (31a) is conserved, and the integration of $s_1(t)$ provides $z_1(t)$, which is the measured output variable defined in (31b). The system of equation of functions of inverted transforms (3b) should permit in our example determination of $z_3(t)$. It is written :

$$\begin{aligned}
z_1(t) &= g_1(t) && \text{(38a)}\\
z_2(t) &= g_2(t) && \text{(38b)}\\
z_3(t) &= g_3(t) = \frac{k_4\ \ell_3(t) - k_1\ \ell_4(t) - x_2(t)}{k_5\ \ell_2(t)} && \text{(38c)}
\end{aligned}$$

and permits linking $\underline{x}(t)$ to $\underline{z}(t)$: $z_3(t)$ is a non-linear function of $\underline{U}(t)$ and $\underline{x}(t)$ through $\ell_2(t),\ \ell_3(t)$ and $\ell_4(t)$.

4.3 Time Scaling of $\widetilde{\Psi}\,[\underline{U}(t), \underline{x}(t)]$, Observation of $z_2(t)$ in Canonical Form of Regulation and Determination of the Inverted Transform System

Using (7), one can temporally normalise (36b), putting for $\widetilde{\Psi}(\tau)$ the definition of the following input-output variables :

$$\widetilde{\Psi}(\tau) = \widetilde{\Psi}\left[\ \underline{v}_a(t), \underline{U}(t)\ \right]/{\omega_o}^2 \tag{39a}$$

$$\underline{v}_a(t)^T = \left[\ \breve{z}_1(t) \quad \underline{v}(t)\ \right] \tag{39b}$$

$$\underline{v}(t)^T = \left[\ \widehat{x}_1(t) \quad \breve{x}_2(t)\ \right] \tag{39c}$$

$$\breve{x}_2(t) = \omega_o\ \breve{x}_2(\tau) \quad \widehat{x}_1(t) = \widehat{x}_1(\tau) \tag{39d}$$

$$\underline{U}(t) = \left[\ \underline{u}_1(t) \quad \underline{u}_2(t)\ \right] \tag{39e}$$

The scaling pulse chosen for (39) is $\omega_o = 3.927\ 10^{-2}\ rd/s$. In (39b) we define $\underline{v}_a(t)$ as the vector $\underline{v}(t)$ (14i) augmented by variable $\breve{z}_1(t)$, itself resulting from observation of the measured variable $y_1(t)$.
Note that $\underline{v}(t)$ contains two second order terms because of (36). Equation (39d) allows conversion of time scaled state variables to temporal variables. Taking (7) and definitions (39) into account, function $\widetilde{\Psi}(\tau)$ in scaled time used in (39a) is written :

$$\widetilde{\Psi}(\tau) = \Big(\ k_4\ \dot{\widehat{\ell}}_3(t) - k_1\ \dot{\widehat{\ell}}_4(t) - k_5\ \widehat{\ell}_7(t)\ \Big)/{\omega_o}^2 \tag{40a}$$

$$\widehat{\ell}_1(t) = u_{11}(t)\ (\ u_{31}(t) - \breve{z}_1(t)\ ) \tag{40b}$$

$$\widehat{\ell}_2(t) = \widehat{n}(t)/\widehat{d}(t) \tag{40c}$$

$$\widehat{\ell}_3(t) = u_{21}(t)\ \left[\ k_{12} - \widehat{x}_1(t)\ \right] \tag{40d}$$

$$\widehat{\ell}_4(t) = u_{11}(t)\ \widehat{x}_1(t) \tag{40e}$$

$$\widehat{\ell}_5(t) = u_{11}(t)\ u_{41}(t) \tag{40f}$$

$$\widehat{\ell}_6(t) = u_{11}(t)\ \widehat{z}_3(t) \tag{40g}$$

$$\widehat{\ell}_7(t) = \widehat{z}_3(t)\ \dot{\widehat{\ell}}_2(t) + \dot{\widehat{z}}_3(t)\ \widehat{\ell}_2(t) \tag{40h}$$

$$\widehat{n}(t) = \breve{z}_1(t)\ \widehat{x}_1(t) \tag{40i}$$

$$\widehat{d}(t) = \ (\ k_{10} + \breve{z}_1(t)\ )\ (\ k_{11} + \widehat{x}_1(t)\ ) \tag{40j}$$

The derived functions $\dot{\widehat{\ell}}_3(t),\ \dot{\widehat{\ell}}_4(t)$ of (40a) and $\dot{\widehat{\ell}}_2(t),\ \dot{\widehat{z}}_3(t)$ of (40h) are defined by :

$$\dot{\widehat{\ell}}_2(t) = \frac{\dot{\widehat{n}}(t)\ \widehat{d}(t) - \dot{\widehat{d}}(t)\ \widehat{n}(t)}{\widehat{d}(t)^2} \tag{41a}$$

$$\dot{\widehat{n}}(t) = \dot{\widehat{z}}_1(t)\ \widehat{x}_1(t) + \breve{z}_1(t)\ \breve{x}_2(t) \tag{41b}$$

$$\dot{\widehat{d}}(t) = \dot{\widehat{z}}_1(t)\ (\ k_{11} + \widehat{x}_1(t)\ ) + \breve{x}_2(t)\ (\ k_{10} + \breve{z}_1(t)\ ) \tag{41c}$$

$$\dot{\widehat{\ell}}_3(t) = u_{22}(t)\ (\ k_{12} - \widehat{x}_1(t)\ ) - u_{21}(t)\ \breve{x}_2(t) \tag{41d}$$

$$\dot{\widehat{\ell}}_4(t) = u_{12}(t)\ \widehat{x}_1(t) + u_{11}(t)\ \breve{x}_2(t) \tag{41e}$$

$$\dot{\widehat{z}}_1(t) = k_1\ \widehat{\ell}_1(t) + k_2\ \widehat{z}_3(t) - k_3\ \widehat{\ell}_2(t)\ \widehat{z}_3(t) \tag{41f}$$

$$\dot{\widehat{z}}_3(t) = k_9\ \widehat{\ell}_5(t) - k_6\ \widehat{\ell}_6(t) - k_7\ \widehat{z}_3(t) + k_8\ \widehat{\ell}_2(t)\ \widehat{z}_3(t) \tag{41g}$$

The observer in canonical form of regulation of the system(36) is written :

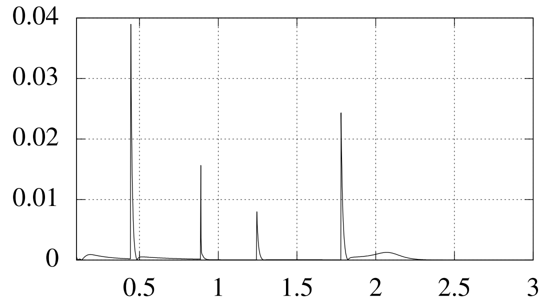
$$\dot{\widehat{x}}_1(\tau) = \breve{x}_2(\tau) + h_2\ \Delta y_1(\tau) \tag{42a}$$

$$\dot{\breve{x}}_2(\tau) = I_0(\tau) + h_1\ \Delta y_1(\tau) + \widetilde{\Psi}(\tau) \tag{42b}$$

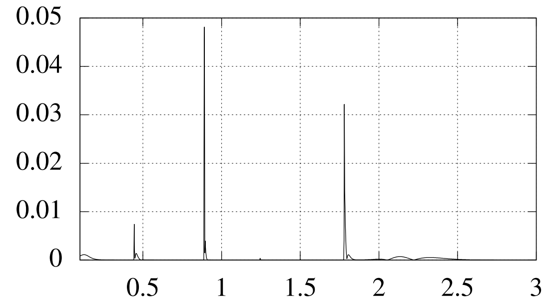
$$\dot{I}_0(\tau) = h_0\ \Delta y_1(\tau) \tag{42c}$$

$$\Delta y_1(\tau) = y_2(\tau) - \widehat{x}_1(\tau) \tag{42d}$$

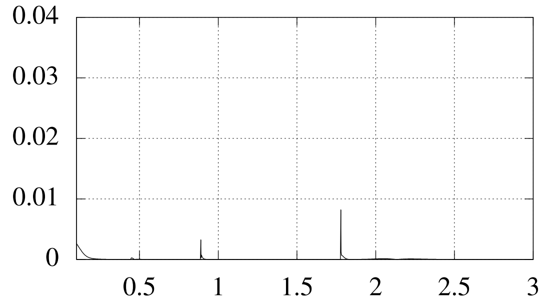
with $y_2(\tau)$ (31b) used to form the observation error $\Delta y_1(\tau)$ (14c).



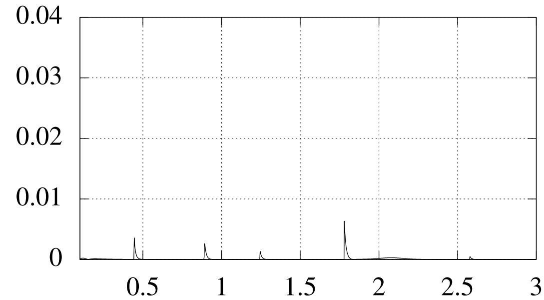
(h) Evolution of $\left\| \Delta\dot{\tilde{\Psi}}_1(\tau_1) \right\| / \| \Delta\underline{Z}(\tau_1) \|$



(i) Evolution of $\left\| \Delta\dot{\tilde{\Psi}}(\tau) \right\| / \left\| \Delta\underline{y}_a(\tau) \right\|$

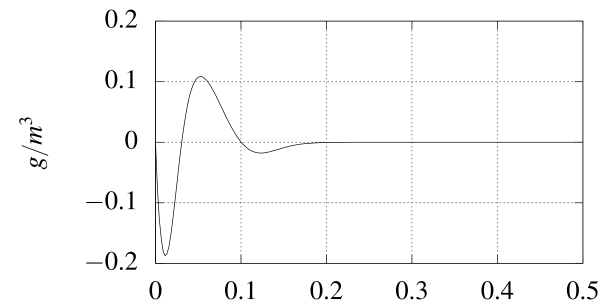


(j) Evolution of $\left\| \Delta\dot{\tilde{\Psi}}_2(\tau_2) \right\| / \| \Delta\underline{Z}(\tau_2) \|$

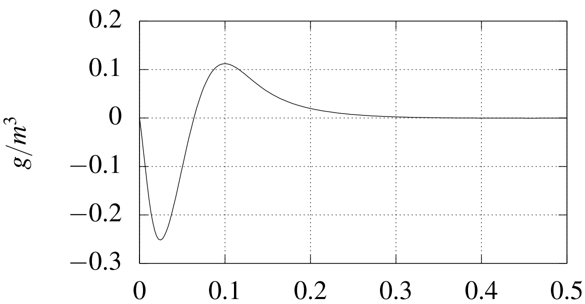


(k) Evolution of $\left\| \Delta\dot{\tilde{\Psi}}_3(\tau_3) \right\| / \| \Delta\underline{Z}(\tau_3) \|$

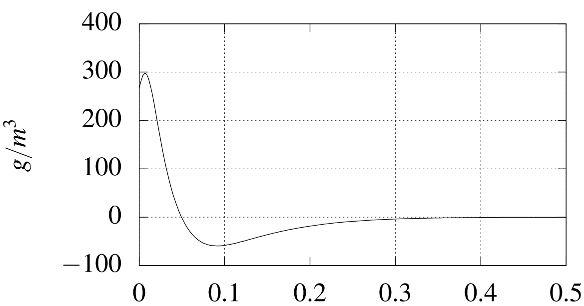
Figure 4. Search for *Lipschitz* constants



(l) Difference $z_1(t) - \breve{z}_1(t)$ in carbonated substrate

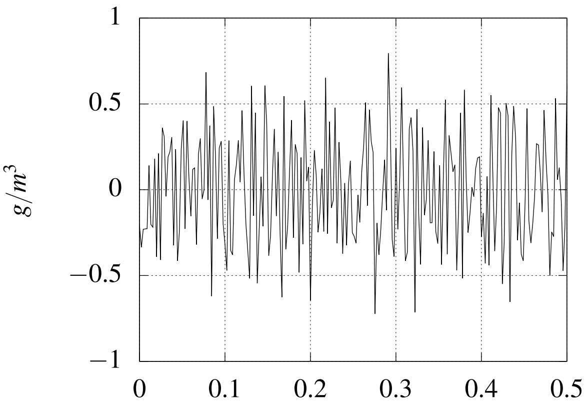


(m) Dissolved oxygen difference $z_2(t) - \breve{z}_2(t)$

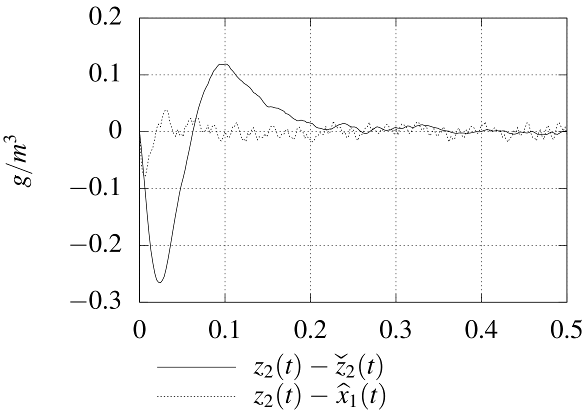


(n) Biomass difference $z_3(t) - \breve{z}_3(t)$

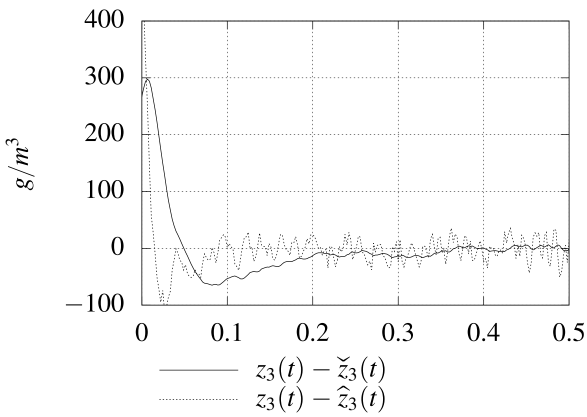
Figure 5. State distances without measurement noise



(o) Difference $z_1(t) - \breve{z}_1(t)$ in carbonated substrate



(p) Dissolved oxygen difference $z_2(t) - \breve{z}_2(t)$



(q) Biomass difference $z_3(t) - \breve{z}_3(t)$

Figure 6. State distances with measurement noise

The estimate $\widehat{z}_3(t)$ of $z_3(t)$ used in (40h), (41f) and (41g) and also the estimate $\widehat{z}_2(t)$ of $z_2(t)$ are defined by the following inverted transform functions :

$$\widehat{z}_1(t) = y_1(t) \tag{43a}$$
$$\widehat{z}_2(t) = \widehat{x}_1(t) \tag{43b}$$
$$\widehat{z}_3(t) = \frac{k_4\ \widehat{\ell}_3(t) - k_1\ \widehat{\ell}_4(t) - \breve{x}_2(t)}{k_5\ \widehat{\ell}_2(t)} \tag{43c}$$
$$\underline{\widehat{z}}(t) = \left[\ \widehat{z}_1(t) \quad \widehat{z}_2(t) \quad \widehat{z}_3(t)\ \right] \tag{43d}$$

4.4 Observation of the Inverted Transformation System

The inverted transformation system (43) serves to form the errors (14c) of three first order observers of the same type as those defined in (17), in order to estimate $\underline{\widehat{z}}(t)$.

The observed outputs and the scaling pulsation choices are given by :

$$\underline{\breve{z}}(t) = \left[\ \breve{z}_1(t) \quad \breve{z}_2(t) \quad \breve{z}_3(t)\ \right] \tag{44a}$$
$$\breve{s}_i(\tau_i) = \frac{s_i\left[\underline{u}_1(t), \underline{\breve{z}}(t)\right]}{\omega_i} \quad i = 1\ldots3 \tag{44b}$$
$$\omega_1 = \omega_o \quad \omega_2 = \omega_3 = \omega_o/5 \tag{44c}$$

the observer of $\widehat{z}_1(t)$ is written :

$$\dot{\breve{z}}_1(\tau_1) = I_1(\tau_1) + 2\ \Delta z_1(\tau_1) + \breve{s}_1(\tau_1) \tag{45a}$$

$$\breve{s}_1(\tau_1) = \frac{k_1\ \widehat{\ell}_1(t) + k_2\ \breve{z}_3(t) - k_3\ \breve{\ell}_2(t)\ \breve{z}_3(t)}{\omega_1} \tag{45b}$$

$$\breve{\ell}_2(t) = \frac{\breve{z}_1(t)\ \breve{z}_2(t)}{(\ k_{10} + \breve{z}_1(t)\ )\ (\ k_{11} + \breve{z}_2(t)\ )} \tag{45c}$$

$$\dot{I}_1(\tau_1) = \Delta z_1(\tau_1) \tag{45d}$$

$$\Delta z_1(\tau_1) = \widehat{z}_1(t) - \breve{z}_1(\tau_1) \tag{45e}$$

That of $\widehat{z}_2(t)$:

$$\dot{\breve{z}}_2(\tau_2) = I_2(\tau_2) + 2\ \Delta z_2(\tau_2) + \breve{s}_2(\tau_2) \tag{46a}$$

$$\breve{s}_2(\tau_2) = \frac{k_4\ \breve{\ell}_3(t) - k_1\ \breve{\ell}_4(t) - k_5\ \breve{\ell}_2(t)\ \breve{z}_3(t)}{\omega_2} \tag{46b}$$

$$\breve{\ell}_3(t) = u_{21}(t)\ \left[\ k_{12} - \breve{z}_2(t)\ \right] \tag{46c}$$

$$\breve{\ell}_4(t) = u_{11}(t)\ \breve{z}_2(t) \tag{46d}$$

$$\dot{I}_2(\tau_2) = \Delta z_2(\tau_2) \tag{46e}$$

$$\Delta z_2(\tau_2) = \widehat{z}_2(t) - \breve{z}_2(\tau_2) \tag{46f}$$

that of $\widehat{z}_3(t)$:

$$\dot{\breve{z}}_3(\tau_3) = I_3(\tau_3) + 2\ \Delta z_3(\tau_3) + \breve{s}_3(\tau_3) \tag{47a}$$

$$\breve{s}_3(\tau_3) = \begin{bmatrix} k_9\ \widehat{\ell}_5(t) - k_6\ u_{11}(t)\ \breve{z}_3(t) \\ -k_7\ \breve{z}_3(t) + k_8\ \breve{\ell}_2(t)\ \breve{z}_3(t) \end{bmatrix} / \omega_3 \tag{47b}$$

$$\dot{I}_3(\tau_3) = \Delta z_3(\tau_3) \tag{47c}$$

$$\Delta z_3(\tau_3) = \widehat{z}_3(t) - \breve{z}_3(\tau_3) \tag{47d}$$

The observer (45) is there to counteract the effect of measurement noise superimposed on $z_1(t)$, which has sometimes a very great impact on the estimates $\widehat{z}_3(t)$ (43c), due to the term $\widehat{\ell}_2(t)$ in the denominator.
The scaling of (46b) and (47b), parts of (45b) is defined in (44c). This has for effect to strongly reduce the noise on estimates $\breve{z}_2(t)$ and $\breve{z}_3(t)$.

We now try to determine the *Lipschitz* constants that subsequently will allow defining the stability conditions of each observer. We thus start by looking for L in (36) using the same calculation method as that explained in ((Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), section 3.1).

With (31)-(34) and (38c) we get $x_2(t)$ by using $\underline{z}(t)$, $\ell_2(t)\ \ell_3(t)$ and $\ell_4(t)$. Then it is possible with (36c) and (38b) to calculate $\underline{x}(t)$ and then $\widetilde{\Psi}\left[\underline{x}(t), \underline{U}(t)\right]/{\omega_o}^2$.

Using the initial conditions $\underline{z}(0)$ with the same method of calculation, we can determine $\widetilde{\Psi}\left[\underline{x}(0), \underline{U}(t)\right]/{\omega_o}^2$. We then calculate the state distance $\Delta\underline{\dot{y}}(\tau)$. Numerical derivation of $\widetilde{\Psi}\left[\underline{x}(t), \underline{U}(t)\right]/{\omega_o}^2 - \widetilde{\Psi}\left[\underline{x}(0), \underline{U}(t)\right]/{\omega_o}^2$ permits determination of the augmented vector $\underline{y}_a(\tau)$ of observation error and to obtain a *Lipschitz* constant adapted to the observer (42). Figure 4 (b) page 13 illustrates this procedure and allows choosing a constant $L = 0.15$. The abscissa is represented only for the first three hours of recording, the region where convergence of observers is expected. using the same stability conditions explained in ((Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016), section 2.3 and 2.4), we fix parameter $\phi_3 = 2$, $\phi_2 = 2$, $\phi_1 = 4$. We choose $\lambda = 1/8$, $\sigma = 1$ and obtain the limiting conditions to respect to synthesise the gains h_i :

$$h_0 \geqslant 0.273 \quad h_1 \geqslant 0.512 \quad h_2 \geqslant 1.3175 \tag{48}$$

Using $\nu = 1$ and $n = 2$, we get :

$$\underline{h}_a = \begin{bmatrix} 1 & 3 & 3 \end{bmatrix} \tag{49}$$

which respects the conditions (48).

We now try to determine constants $L_1,\ L_2,\ L_3$ of inverted transformation observer functions. We use a similar procedure to estimate $\Delta\underline{Z}(\tau_i),\ \Delta\dot{\tilde{\Psi}}_i(\tau_i)$ and their respective modules. Figures 4(a),(c) and (d) allows the choice of *Lipschitz* constants $L_1 = 0.1,\ L_2 = L_3 = 0.02$ (22).

If we fix $\phi_{i1} = \phi_{i2} = 2,\ \sigma_i = 1,\ n = 3,\ \lambda_i = 1/8$ for $i = 1 \ldots 3$ the stability conditions (23) are respected, and the three observers of $\underline{\widehat{g}}(\tau)$ will be stable and properly damped.

4.5 Simulations and Results Obtained

The aim of the simulation is to observe the overall stabilisation of observers to an initial difference in biomass concentration.

The initial conditions of (39) are fixed at

$$I_0(0) = 0,\ \breve{x}_2(0) = -0.164,\ \widehat{x}_1(0) = z_2(0) = 3$$

Those of (45) at $\breve{z}_1(0) = z_1(0) = 4.1$, those of (46) at $\breve{z}_2(0) = 3$, those of (47) at $\breve{z}_3(0) = 600$.

In figures 5(a)- 5(c) the exponential convergent reduction of the state distances $z_i(t) - \breve{z}_i(t)$ are visualised for zero measured noise on the outputs $y_1(t)$ and $y_2(t)$. The same test is performed by adding two bandwidth limited white noise to outputs $y_1(t)$ and $y_2(t)$. These uncorrelated noises have an amplitude of 1% on each of the variables. Figure 4(h) permits verification that the dynamics of convergence of $\breve{z}_3(t)$ is conserved. The normative pulse ω_o chosen for (39) and (45) allow reduction of residual noise by about 10% compared with the measured variables and to contain that still present in $\widehat{z}_3(t)$. This setting permits however to have a rate of convergence of $\breve{z}_1(t)$ and $\widehat{x}_1(t)$ of the same order as the abrupt variations that are seen in $z_1t)$ and $z_2(t)$.
In figures 6(a)-(c) page 14 the smoothing effect on the estimates of observer (46) and (47) is illustrated : division by 5 on the noise on $y_2(t)$ for $\breve{z}_2(t)$ and fluctuations of 2% on superimposed noise compared with the full scale for $\breve{z}_3(t)$.

5. Conclusions and Perspectives

Observation in canonical form of regulation that is proposed in (Schwaller, Ensminger, Dresp-Langley, & Ragot, 2016) did not take into account the effect of measurement noise on the inverted transformation, which allowed passing from the observation of transformed systems to the non-transformed state space. Certain non-linear functions, because of their nature, can greatly amplify the effect of extraneous perturbations on the final estimations. Observers of inverted transformation functions limit this type of effect. The time scale of each observer affects the stability conditions of each observer, *via* the value of the *Lipschitz* constant. This also greatly influences the existing noise on the estimated variables. By reducing the pulse ω_i of the observers (17), the *Lipschitz* constant L_i is reduced, and similarly the magnitude of remaining noise on estimates $\breve{z}_i(t)$ and one increases the convergence time. Setting the rate of convergence of each observer can be done independently.
Observer stability and synthesising observer gains employ demonstrations published in the previous study.
The proposed technique can be applied to other observers (Gauthier, Hammouri, & Othman, 1992) or to different high gain observers. Observation of inverted transformation functions opens the route to identification on line of parameters of n equation of the state of vector $\underline{s}(t)$ (1c). In fact, it is possible to consider using the n functional distances between $\dot{\breve{z}}_i(t)$ and $s_i\left[\underline{u}_1(t), \underline{\breve{z}}(t)\right]$ to identify parameters of the n functions $s_i\left[\ \underline{u}_1(t), \underline{z}(t)\ \right]$ (1c). This could provide a means of dealing with parametric uncertainty in state equations of the system (1), as well as external perturbations, which are already compensated by the integral component of the observer (14e).
Finally, by slightly modifying the filter (8)-(10), it can be envisaged to extend the proposed method to multivariable MIMO systems with multiple outputs.

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