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Convergence Analysis for Mixed Finite Element Method of Positive Semi-definite Problems

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Abstract

A mixed element-characteristic finite element method is put forward to approximate three-dimensional incompressible miscible positive semi-definite displacement problems in porous media. The mathematical model is formulated by a nonlinear partial differential system. The flow equation is approximated by a mixed element scheme, and the pressure and Darcy velocity are computed at the same time. The concentration equation is treated by the method of characteristic finite element, where the convection term is discretized along the characteristics and the diffusion term is computed by the scheme of finite element. The method of characteristics can confirm strong computation stability at the sharp fronts and avoid numerical dispersion and nonphysical oscillation. Furthermore, a large step is adopted while small time truncation error and high order accuracy are obtained. It is an important feature in numerical simulation of seepage mechanics that the mixed volume element can compute the pressure and Darcy velocity simultaneously and the accuracy of Darcy velocity is improved one order. Using the form of variation, energy method, L^2 projection and the technique of priori estimates of differential equations, we show convergence analysis for positive semi-definite problems. Then a powerful tool is given to solve international famous problems.

Keywords: three-dimensional incompressible miscible displacement, positive semi-definite problem, mixed element with characteristic finite element, error estimate in L^2-norm

1. Introduction

The incompressible miscible positive semi-definite displacement problem in porous media consists of two partial different equations: an elliptic equation for the pressure, a convection-diffusion equation for the concentration, where the concentration equation has strong hyperbolic feature (Douglas, 1983; Dougals, Ewing & Wheeler, 1983; Ewing, Russell & Wheeler, 1984; Russell, 1985),

$$-\nabla\cdot\left(\frac{\kappa(X)}{\mu(X)}(\nabla p-\gamma(c)\nabla d(X))\right)\equiv\nabla\cdot\mathbf{u}=q,\ X\in\Omega,t\in J=(0,T],\tag{1a}$$

$$\mathbf{u}=-\frac{\kappa(X)}{\mu(X)}(\nabla p-\gamma(c)\nabla d(X)),X\in\Omega,t\in J.\tag{1b}$$

$$\phi\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(\mathbf{D}(X,\mathbf{u})\nabla c)=(\tilde{c}-c)\tilde{q},\ X\in\Omega,t\in J,\tag{2}$$

$$\mathbf{u}\cdot\nu=(\mathbf{D}(X,\mathbf{u})\nabla c)\cdot\nu=0,\ X\in\partial\Omega,t\in J,\tag{3}$$

$$c(X,0)=c_0(X),\ X\in\Omega.\tag{4}$$

Ω denotes a bounded domain of R^3, and ν is the normal outer vector to the boundary surface, denoted by $\partial\Omega$. The pressure, $p(X,t)$, Darcy velocity, $\mathbf{u}=(u_1,u_2,u_3)^T$ and the concentration of water, $c(X,t)$, are objective functions. $q(X,t)$, the quantity, is greater than zero at injection wells and is less than zero at production wells, and $\tilde{q}=\max\{q,0\}$. $\phi(X)$ is the porosity of rock, and $\kappa(X)$ is absolute permeability. $\mu(c)$, the viscosity of mixture, depends on c. $\tilde{c}$, the concentration of injected fluid, is equal to c at production wells, $\tilde{c}=c$. $\gamma(c)$ and $d(X)=(0,0,z)^T$ denote the gravitational coefficient and vertical coordinates, respectively. The diffusion matrix, $\mathbf{D}(X,\mathbf{u})$, is generally defined by (Yuan & Han, 2008; Yuan, Wang & Han, 2010),

$$\mathbf{D}(X,\mathbf{u}) = D_m(X)\mathbf{I} + \alpha_l|\mathbf{u}|^\beta \begin{pmatrix} \hat{u}_x^2 & \hat{u}_x\hat{u}_y & \hat{u}_x\hat{u}_z \\ \hat{u}_x\hat{u}_y & \hat{u}_y^2 & \hat{u}_y\hat{u}_z \\ \hat{u}_x\hat{u}_z & \hat{u}_y\hat{u}_z & \hat{u}_z^2 \end{pmatrix} + \alpha_t|\mathbf{u}|^\beta \begin{pmatrix} \hat{u}_y^2+\hat{u}_z^2 & -\hat{u}_x\hat{u}_y & -\hat{u}_x\hat{u}_z \\ -\hat{u}_x\hat{u}_y & \hat{u}_x^2+\hat{u}_z^2 & -\hat{u}_y\hat{u}_z \\ -\hat{u}_x\hat{u}_z & -\hat{u}_y\hat{u}_z & \hat{u}_x^2+\hat{u}_y^2 \end{pmatrix}. \quad (5)$$

D_m is the molecular diffusivity. $\mathbf{I}$ denotes a 3×3 unit matrix. α_l and α_t are the longitudinal and the transverse dispersivities, respectively. $\hat{u}_x, \hat{u}_y, \hat{u}_z$ denote three direction cosines of $\mathbf{u}$. Generally speaking, the symbol, $\beta \geq 2$, is a positive constant. The mathematical model is usually used to describe numerical simulation of oil reservoir and contaminant-transport problem, and the diffusion matrix is supposed to be positive definite. While in actual numerical simulation applications such as oil-gas resources basin assessment (Yuan & Han, 2008; Yuan, Wang & Han, 2010) and numerical computation of enhanced (chemical) oil recovery (Yuan, Cheng, Yang & Li, 2014,2015), the diffusion matrix is only positive semi-definite (Dawson, Russell & Wheeler, 1989; Ewing, 1983; Yuan, 2013),

$$\mathbf{D}(X,\mathbf{u}) \geq 0. \quad (6)$$

The present paper mainly considers a positive semi-definite problem, and the discussion gives more theoretical reference in terms of mathematics and mechanics (Ewing, 1983; Yuan, 2013).

Oil-water two phase seepage displacement is a primary topic in numerical simulation of oil reservoir. For two-dimensional positive definite problems, Douglas and Russell presented well-known numerical methods such as characteristic finite difference and characteristic finite element (Russell, 1985; Douglas, 1983). Douglas, Ewing and Wheeler put forward the method of mixed element (Douglas, Ewing & Wheeler, 1983), and Ewing, Russell, Wheeler discussed the characteristics-mixed element (Ewing, Russell & Wheeler, 1984). The above arguments were based on the positive definite assumption, but the diffusion matrix was only positive semi-definite in some actual applications (Dawson, Russell & Wheeler, 1989; Ewing, 1983; Yuan & Han, 2008; Yuan, Wang & Han, 2010; Yuan, 2013; Yuan, Cheng, Yang & Li, 2014,2015). Therefore, the framework of theoretical analysis is not feasible. It is hard and difficult to show convergence analysis of semi-definite problem. The characteristic finite element method was presented by Dawson (Dawson, Russell & Wheeler, 1989). For three-dimensional positive semi-definite problems, Yuan discussed characteristic finite element and characteristic finite difference (Yuan, 1997,1999). Based on the above discussions, we present a method of mixed element-characteristic finite element to simulate three-dimensional incompressible miscible positive semi-definite displacement problem of (1)-(4). The flow equation is treated by a conservative mixed element method, and the pressure and Darcy velocity are obtained at the same time. The characteristic finite element is used to solve the concentration equation, where the convection term is discretized along the characteristics and the diffusion term is approximated by the finite element method. The characteristics can confirm strong stability at the fronts and can avoid numerical dispersion. Moreover, it can adopt large spatial step while computational accuracy is not decreased. More important in numerical simulation of seepage mechanics, the pressure and Darcy velocity are obtained simultaneously by using the scheme of mixed element and the computational accuracy of Darcy velocity is developed one order. Using variation form, energy method, L^2 projection and theoretical framework of priori estimate, we show convergence analysis in L^2 norm. Then the well-known difficult problem is solved numerically, and a basic theoretical reference is given for actual numerical simulations.

common symbols and notations of Sobolev space are adopted. Suppose that the problem of (1)-(4) is regular,

$$\text{(R)} \qquad \begin{cases} c \in L^\infty(H^{l+1}) \bigcap H^1(H^{l+1}) \bigcap L^\infty(W_\infty^1) \bigcap H^2(L^2), \\ p \in L\infty(H^{k+1}), \\ \mathbf{u} \in L^\infty(H^{k+1}(\mathrm{div})) \bigcap L^\infty(W_\infty^1) \bigcap W_\infty^1(L^\infty) \bigcap H^2(L^2), \end{cases} \quad (7)$$

where $l \geq 3, k \geq 1$.

And suppose that the problem is positive semi-definite

$$\text{(C)} \qquad 0 < a_* \leq \tfrac{\kappa(X)}{\mu(c)} \leq a^*,\ 0 < \phi_* \leq \phi(X) \leq \phi^*,\ \mathbf{D}(X,\mathbf{u}) \geq 0, \quad (8)$$

where a_*, a^*, ϕ_* and ϕ^* are positive constants.

In this paper, the symbols K and ε denote a generic positive constant and a generic small positive number, respectively. They have different definitions at different places.

2. The Computation Program

2.1 The Mixed Element for the Pressure

The form of variation is discussed. Let $H(\mathrm{div};\Omega)$ denote a space consisting of vector functions, $\mathbf{v} \in L^2(\Omega)^3$, satisfying $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, then define

$$V = H(\mathrm{div};\Omega) \bigcap \{\mathbf{v} \cdot \nu = 0 \text{ on the boundary } \partial\Omega\}. \tag{9a}$$

The pressure $p(X,t)$ is determined only except an additive constant. For simplicity, consider a factor space

$$W = L^2(\Omega)/\{\varphi \equiv \text{const. on the boundary } \partial\Omega\}. \tag{9b}$$

For $\alpha, \beta \in V$, $\varphi \in W$ and $\theta \in L^\infty(\Omega)$, define the following bilinear function

$$\mathcal{A}(\theta,\alpha,\beta) = (\frac{\mu(\theta)}{k}\alpha,\beta), \tag{10a}$$

$$\mathcal{B}(\alpha,\varphi) = -(\nabla \cdot \alpha, \varphi). \tag{10b}$$

Then the pressure equation is equivalent to a family of saddle point problems:

$$\mathcal{A}(c,\mathbf{u},\mathbf{v}) + \mathcal{B}(\mathbf{v},p) = (\gamma(c)\nabla d, \mathbf{v}),\ \forall v \in V, \tag{11a}$$

$$\mathcal{B}(\mathbf{u},w) = -(q,w),\ \forall w \in W. \tag{11b}$$

The problem of (11) is considered. Let $h_p > 0$ be the spatial step for the pressure, and let J_{h_p} be a quasi-regular partition of Ω, consisting of tetrahedrons or cubes with the greatest diameter at most h_p. Let $V_h \times W_h \subset V \times W$ be a Raviar-Thomas space on the partition (Raviart & Thomase, 1977; Thomase, 1977), with the index k and the approximation $O(h_p^{k+1})$, whose approximations satisfy

$$(A_p) \qquad \begin{cases} \inf\limits_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_{L^2(\Omega)^3} \le K\|\mathbf{v}\|_{H^{k+1}(\Omega)^3} h_p^{k+1}, \\ \inf\limits_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_V \le K\{\|\mathbf{v}\|_{H^{k+1}(\Omega)^3} + \|\nabla \cdot \mathbf{v}\|_{H^{k+1}(\Omega)}\} h_p^{k+1}, \\ \inf\limits_{w_h \in W_h} \|w - w_h\| \le K\|w\|_{H^{k+1}(\Omega)} h_p^{k+1}, \end{cases} \tag{12a}$$

$$(I_p) \qquad \|w\|_{L^\infty(\Omega)} \le K h_p^{-3/2}\|w\|_{L^2(\Omega)}, \|w\|_{H^1(\Omega)} \le K h_p^{-1}\|w\|_{L^2(\Omega)}. \tag{12b}$$

Introduce elliptic projection of $(\mathbf{u}, p)$ to find $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$: $[0,T] \to V_h \times W_h$ such that

$$\mathcal{A}(c,\tilde{\mathbf{u}}_h,\mathbf{v}) + \mathcal{B}(\mathbf{v},\tilde{p}_h) = (\gamma(c)\nabla d, \mathbf{v}),\ \forall \mathbf{v} \in V, \tag{13a}$$

$$\mathcal{B}(\tilde{\mathbf{u}}_h,w) = -(q,w),\ \forall w \in W, \tag{13b}$$

where c is the exact concentration.

It is seen that in the references (Douglas, Ewing & Wheeler, 1983; Ewing, Russell & Wheeler, 1984) the solution $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ exists solely and is estimated as follows

$$\|\tilde{\mathbf{u}}_h - \mathbf{u}\|_{L^\infty(H(\mathrm{div}))} + \|\tilde{p}_h - p\|_{L^\infty(L^2)} \le K h_p^{k+1}. \tag{14}$$

Then it follows from (14) and (I_p), for $k \ge 1$,

$$\|\tilde{\mathbf{u}}\|_{L^\infty(L^\infty)} \le K. \tag{15}$$

The mixed element scheme is constructed. When the approximate concentration c_h at $t \in J$ is known, then $(\mathbf{u}_h, p_h) \in V_h \times W_h$ is aimed to find

$$\mathcal{A}(c_h,\mathbf{u}_h,\mathbf{v}) + \mathcal{B}(\mathbf{v},p_h) = (\gamma(c_h)\nabla d, \mathbf{v}),\ \forall \mathbf{v} \in V_h, \tag{16a}$$

$$\mathcal{B}(\mathbf{u}_h,w) = -(q,w),\ \forall w \in W_h. \tag{16b}$$

It has been proved that numerical solutions of (16) exist solely (Brezzi, 1974). Using (14) and (15),

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{H(\mathrm{div})} + \|p_h - \tilde{p}_h\| \le K(1 + \|\tilde{\mathbf{u}}_h\|_{L^\infty})\|c - c_h\|_{L^2}. \tag{17}$$

The concentration equation (2) is discretized later.

2.2 The Finite Element Approximation for the Concentration

For convenience to interpret the approximation of concentration, we suppose that Darcy velocity $\mathbf{u} = (u_1, u_2, u_3)^T$ is given. The procedures are constructed by combining the discretization of characteristics and the approximation of finite element.

Let $M_h \subset H^1(\Omega)$ be a normal finite element space with index l, and let $h_c > 0$ be a spatial step of quasi-regular partition J_{h_c}. The largest diameter of tetrahedron elements or cube elements is not exceeding h_c. Approximation order is $O(h_c^{l+1})$ (Ciarlet, 1978),

$$(A_c) \qquad \inf_{z_h \in M_h} \left\{ \|z - v_h\|_{L^2(\Omega)} + h_c \|z - z_h\|_{H^1(\Omega)} \right\} \leq K h_c^{l+1}, \tag{18a}$$

$$(I_c) \qquad \|w\|_{L^\infty(\Omega)} \leq K h_c^{-3/2} \|w\|_{L^2(\Omega)}, \|w\|_{H^1(\Omega)} \leq K h_c^{-1} \|w\|_{L^2(\Omega)}. \tag{18b}$$

Let $\tau(X, t)$ denote a unit vector of the characteristics, and let $\psi = [\phi^2 + |\mathbf{u}|^2]^{1/2} = (\phi^2 + \sum_{i=1}^{3} u_i^2)^{1/2}$. Then the characteristic derivative is formulated by

$$\psi \frac{\partial c}{\partial \tau} = \phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c. \tag{19}$$

Let $\Delta t_c = T/N$ denote the time step for the concentration, and $t^n = n\Delta t_c$. N is a positive integer. For $X \in \Omega$, define

$$\check{X}^{n-1} = X - \phi^{-1} \mathbf{u}^n \Delta t_c, \ \check{c}^{n-1}(X) = c^{n-1}(\check{X}^{n-1}). \tag{20}$$

The characteristic derivative, $\frac{\partial c^n}{\partial \tau}(X) = \frac{\partial c}{\partial \tau}(X, t^n)$, is approximated by a backward difference quotient

$$\frac{\partial c^n}{\partial \tau}(X) \approx \frac{c^n(X) - \check{c}^{n-1}(X)}{\Delta t_c \psi^n}, \tag{21}$$

where $\psi^n = [\phi^2 + |\mathbf{u}^n|^2]^{1/2}$.

The variation of (2) is defined as follows. A function, $c : J \to H^1(\Omega)$, is determined by

$$(\phi \frac{\partial c}{\partial t}, z) + (\mathbf{u} \cdot \nabla c, z) + (\mathbf{D}(\mathbf{u}) \nabla c, \nabla z) = ((\tilde{c} - c)\tilde{q}, z), \ \forall z \in H^1(\Omega), t \in J. \tag{22}$$

By using (19), (22) is restated by

$$(\psi \frac{\partial c}{\partial \tau}, z) + (\mathbf{D}(\mathbf{u}) \nabla c, \nabla z) = ((\tilde{c} - c)\tilde{q}, z), \ \forall z \in H^1(\Omega), t \in J, \tag{23a}$$

$$c(X, 0) = c_0(X), \ X \in \Omega. \tag{23b}$$

Since the diffusion matrix $\mathbf{D}(\mathbf{u})$ is positive semi-definite, so L^2 projection is introduced, replacing elliptic projection, to show convergence analysis. For $t \in J$, $\bar{c}_h \in M_h$ is defined by

$$(\phi \bar{c}_h, \chi) = (\phi c, \chi), \forall \chi \in M_h. \tag{24}$$

Then the estimate holds (Ciarlet,1978)

$$\|c - \bar{c}_h\|_{L^2(L^2)} + h_c \|c - \bar{c}_h\|_{L^2(H^1)} \leq K h_c^{l+1} \|c\|_{L^2(H^{l+1})}. \tag{25}$$

The characteristic finite element scheme of (23) is constructed. $\{c_h^n \in M_h\}$ is computed by

$$(\phi \frac{c_h^n - \hat{c}_h^{n-1}}{\Delta t_c}, z_h) + (\mathbf{D}(\mathbf{u^n}) \nabla c_h^n, \nabla z_h) + (\tilde{q} c_h^n, z_h) = ((\tilde{q} \tilde{c}(t^n), z_h), \ \forall z_h \in M_h, \tag{26a}$$

$$c_h^0 = \bar{c}_h^0, \tag{26b}$$

where $\bar{c}_h^0$ is an L^2 projection of initial solution $c_0(X)$.

2.3 The Composite Scheme

Combining (16) and (26), we state a composite scheme to solve (1)-(4). In actual computation Darcy velocity changes more slowly than the saturation with respect to time t, so spatial large step is adopted for computing (16). Time interval J is partitioned $0 = t_0 < t_1 < \cdots < t_L = T$, with $\Delta t_p^m = t_m - t_{m-1}$. All the steps except for the first step Δt_p^1 are supposed to be uniform $\Delta t_p^m = \Delta t_p, m \geq 2$. Each pressure node t_m is also a saturation node t^n where m, n are positive integers, and let

$j = \Delta t_p/\Delta t_c$, $j_1 = \Delta t_p^1/\Delta t_c$. For a function $\varphi_m(X) = \varphi(X, t_m)$ related with saturation step t^n for $t_{m-1} < t^n \le t_m$, we require a velocity approximation $\mathbf{u}_h$ in (26). If $m \ge 2$, define a linear extrapolation of $\mathbf{u}_{h,m-1}$ and $\mathbf{u}_{h,m-2}$ as follows

$$E\mathbf{u}_h^n = (1 + \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}})\mathbf{u}_{h,m-1} - \frac{t^n - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{u}_{h,m-2}. \tag{27}$$

If $m = 1$, set $E\mathbf{u}_h^n = \mathbf{u}_{h,0}$.

Combining (16) with (26), replacing exact solution by numerical approximations, then we can obtain full discrete coupled scheme of (1)-(4) to find $\{c_h^n\} : (t^0, t^1, \cdots, t^L) \to M_h$ and $\{\mathbf{u}_{h,m}, p_{h,m}\} : (t_0, t_1, \cdots, t_{L/j}) \to V_h \times W_h$,

$$(\phi\frac{c_h^n - \hat{c}_h^{n-1}}{\Delta t_c}, z_h) + (\mathbf{D}(E\mathbf{u}_h^n)\nabla c_h^n, \nabla z_h) + (\tilde{q}^n c_h^n, z_h) = (\tilde{q}c_h^n, z_h),\ \forall z_h \in M_h, \tag{28a}$$

$$c_h^0 = \bar{c}_h^0,\ \ \forall X \in \Omega, \tag{28b}$$

$$\mathcal{A}(c_{h,m}, \mathbf{u}_{h,m}, \mathbf{v}) + \mathcal{B}(\mathbf{v}, p_{h,m}) = (\gamma(c_{h,m})\nabla d, \mathbf{v}),\ \forall \mathbf{v} \in V_h, \tag{29a}$$

$$\mathcal{B}(\mathbf{u}_{h,m}, w) = -(q_m, w),\ \forall w \in W_h, \tag{29b}$$

where $\hat{c}_h^{n-1}(X) = c_h^{n-1}(X - \phi^{-1}E\mathbf{u}_h^n\Delta t_c)$.

The procedures of (28) and (29) run as follows.
Step 1. Given initial approximation c_h^0, then by (29a) and (29b) the numerical values of $(\mathbf{u}_{h,0}, p_{h,0})$ are obtained.
Step 2. Applying (28a) and (28b) to find $c_h^1, c_h^2, \cdots, c_h^{j_1}$.
Step 3. By the fact of $c_h^{j_1} = c_{h,1}$, and by (29a), (29b), we have $(\mathbf{u}_{h,1}, p_{h,1})$.
Step 4. Similarly, we get the values of $c_h^{j_1+1}, c_h^{j_1+2}, \cdots, c_h^{j_1+j}$, $(\mathbf{u}_{h,2}, p_{h,2})$.
Step 5. The program runs repeatedly as above, then all the numerical solutions are obtained.

3. Convergence Analysis

Based on the discussions of Darcy velocity $\mathbf{u}_h$ and the pressure p_h, (14) and (17), convergence analysis is shown as follows. Let $\zeta = c_h - \bar{c}_h$ and $\xi = c - \bar{c}_h$. From (28a), (23a) ($t = t^n$), (21) and (24), taking $z_h = \zeta^n$, we have

$$\begin{aligned}
&(\phi\frac{\zeta^n - \check{\zeta}^{n-1}}{\Delta t_c}, \zeta^n) + (\mathbf{D}(E\mathbf{u}_h^n)\nabla\zeta^n, \nabla\zeta^n) + (\tilde{q}^n\zeta^n, \zeta^n)\\
&= (\sigma^n, \zeta^n) - (\phi\frac{\check{\xi}^{n-1} - \xi^{n-1}}{\Delta t_c}, \zeta^n) + (\tilde{q}^n\xi^n, \zeta^n) + (\mathbf{D}(E\mathbf{u}_h^n)\nabla\xi^n, \nabla\zeta^n)\\
&+ (\phi\frac{\check{F}_h^{n-1} - \hat{F}_h^{n-1}}{\Delta t_c}, \zeta^n) - (\phi\frac{\check{\zeta}^{n-1} - \hat{\zeta}^{n-1}}{\Delta t_c}, \zeta^n) + ([\mathbf{D}(\mathbf{u}^n) - \mathbf{D}(E\mathbf{u}_h^n)]\nabla c^n, \nabla\zeta),
\end{aligned} \tag{30}$$

where $\sigma^n = [\phi\frac{\partial c^n}{\partial t} + E\mathbf{u}^n \cdot \nabla c^n] - \phi\frac{c^n - \check{c}^{n-1}}{\Delta t_c}$, $F_h = \bar{c}_h$, $\check{\zeta}^{n-1} = \zeta^{n-1}(\check{X}^{n-1}), \cdots$.

The first term on the left-hand side of (30) is estimated. Noting that

$$\begin{aligned}
(\phi\frac{\zeta^n - \check{\zeta}^{n-1}}{\Delta t_c}, \zeta^n) &\ge \frac{1}{2\Delta t_c}\{(\phi\zeta^n, \zeta^n) - (\phi\check{\zeta}^{n-1}, \check{\zeta}^{n-1})\}\\
&= \frac{1}{2\Delta t_c}\{(\phi\zeta^n, \zeta^n) - (\phi\zeta^{n-1}, \zeta^{n-1})\} + \frac{1}{2\Delta t_c}\{(\phi\zeta^{n-1}, \zeta^{n-1}) - (\phi\check{\zeta}^{n-1}, \check{\zeta}^{n-1})\}.
\end{aligned} \tag{31}$$

The second term of (31), $\frac{1}{2\Delta t_c}\{(\phi\zeta^{n-1}, \zeta^{n-1}) - (\phi\check{\zeta}^{n-1}, \check{\zeta}^{n-1})\}$, is discussed. Let $Y = \check{X} = X - \mathbf{u}(X, t^n)\phi^{-1}\Delta t_c \triangleq R(X)$, and let

$\check{\Omega} = R(\Omega)$ be the range of the mapping R. Since $\det DR(X) = 1 - \nabla \cdot (\frac{\mathbf{u}}{\phi})\Delta t_c + O((\Delta t_c)^2)$, then

$$\begin{aligned}
&\frac{1}{2\Delta t_c}\{(\phi\check{\zeta}, \check{\zeta}) - (\phi\zeta, \zeta)\} \\
&= \frac{1}{2\Delta t_c}\{\int_{\check{\Omega}} \phi(x)\zeta(y)\zeta(y)(\det DR(X))^{-1}dy - \int_{\Omega} \phi(X)\zeta(x)\zeta(x)dx\} \\
&= \frac{1}{2\Delta t_c}\{\int_{\check{\Omega}} \phi(x)\zeta(y)\zeta(y)[1 + \nabla \cdot (\frac{\mathbf{u}}{\phi})\Delta t_c + O((\Delta t_c)^2)]dy - \int_{\Omega} \phi(x)\zeta(x)\zeta(x)dx\} \\
&= \frac{1}{2\Delta t_c}\{\int_{\check{\Omega}} \phi(y)\zeta(y)\zeta(y)dy - \int_{\Omega} \phi(X)\zeta(x)\zeta(x)dx + \int_{\check{\Omega}} (\phi(x) - \phi(y))\zeta(y)\zeta(y)dy \\
&\quad + \int_{\check{\Omega}} \phi(x)\zeta(y)\zeta(y)[\nabla \cdot (\frac{\mathbf{u}}{\phi})\Delta t_c + O((\Delta t_c)^2)]dy\} \\
&= \frac{1}{2\Delta t_c}\{\int_{\check{\Omega}\backslash\Omega} \phi(y)\zeta(y)\zeta(y)dy - \int_{\Omega\backslash\check{\Omega}} \phi(x)\zeta(x)\zeta(x)dx\} \\
&\quad + \frac{1}{2\Delta t_c}\{\int_{\check{\Omega}} (\phi(x) - \phi(y))\zeta(y)\zeta(y)dy + \int_{\check{\Omega}} \phi(x)\zeta(y)\zeta(y)[\nabla \cdot (\frac{\mathbf{u}}{\phi})\Delta t_c + O((\Delta t_c)^2)]dy\} \\
&= T_1 + T_2.
\end{aligned} \tag{32}$$

From the boundary condition $\mathbf{u} \cdot \nu = 0$, we can get meas$\{\check{\Omega}\backslash\Omega\} = O((\Delta t_c)^2)$, meas$\{\Omega\backslash\check{\Omega}\} = O((\Delta t_c)^2)$. If the partition satisfies $\Delta t_c = O(h_c^3)$, then T_1 is bounded by $|T_1| \le K\|\phi^{1/2}\zeta\|_{L^\infty}^2 \Delta t_c \le K h_c^{-3} \Delta t_c(\phi\zeta, \zeta) \le K(\phi\zeta, \zeta)$. Similarly, T_2 is bounded by $|T_2| \le K(\phi\zeta, \zeta)$. Then,

$$\frac{1}{2\Delta t_c}\{(\phi\check{\zeta}, \check{\zeta}) - (\phi\zeta, \zeta)\} \le K(\phi\zeta, \zeta). \tag{33}$$

Estimate the other terms on the left-hand side of (30),

$$(\mathbf{D}(E\mathbf{u}_h^n)\nabla\zeta^n, \nabla\zeta^n) = \|\mathbf{D}^{1/2}(E\mathbf{u}_h^n)\nabla\zeta^n\|_0^2. \tag{34}$$

$$(\tilde{q}\zeta^n, \zeta^n) = \|\tilde{q}^{1/2}\zeta^n\|_0^2. \tag{35}$$

The right-hand side of (30) is considered as follows,

$$\left|(\sigma^n, \zeta^n)\right| \le K\{\Delta t_c \|\frac{\partial^2 c}{\partial \tau^2}\|_{L^2(t^{n-1}, t^n; L^2)}^2 + \|\zeta^n\|^2\}. \tag{36}$$

$$\left|(\phi \frac{\check{\xi}^{n-1} - \xi^{n-1}}{\Delta t_c}, \zeta^n)\right| \le K\{\|\nabla\xi^{n-1}\|^2 + \|\zeta^n\|^2\} \le K\{h^{2l} + \|\zeta^n\|^2\}. \tag{37}$$

$$\left|(\mathbf{D}(E\mathbf{u}_h^n)\nabla\xi^n, \nabla\zeta^n)\right| \le \frac{1}{3}\|\mathbf{D}^{1/2}(E\mathbf{u}_h^n)\nabla\zeta^n\|_0^2 + Kh^{2l}. \tag{38}$$

$$\begin{aligned}
\left|(\phi \frac{\check{F}_h^{n-1} - \hat{F}_h^{n-1}}{\Delta t_c}, \zeta^n)\right| &\le K\|\nabla \bar{c}_h^{n-1}\|_\infty \|E(\mathbf{u} - \mathbf{u}_h^n)\|_0 \|\zeta^n\|_0 \\
&\le K\{h_c^{2(l+1)} + h_p^{2(k+1)} + (\Delta t_c)^2 + \|\zeta_m\|^2 + \|\zeta_{m-1}\|^2 + \|\zeta^n\|^2\}.
\end{aligned} \tag{39}$$

$$\begin{aligned}
\left|(\phi \frac{\check{\zeta}^{n-1} - \hat{\zeta}^{n-1}}{\Delta t_c}, \zeta^n)\right| &\le K\|\nabla\zeta^{n-1}\| \|E(\mathbf{u} - \mathbf{u}_h^n)\|_0 \|\zeta^n\|_\infty \\
&\le K h_c^{-5/2} \|E(\mathbf{u} - \mathbf{u}_h^n)\|_0 \|\zeta^{n-1}\|_0 \|\zeta^n\|_0 \\
&\le K\{h_c^{2(l+1)} + h_p^{2(k+1)} + \|\zeta_m\|^2 + \|\zeta_{m-1}\|^2 + \|\zeta^{n-1}\|^2\}.
\end{aligned} \tag{40}$$

An induction hypothesis is used

$$\sup_{0 \le n \le L-1} h_c^{-5/2}\|\zeta^n\|_0 \le K. \tag{41}$$

The last term is discussed.

$$\begin{aligned}
&\left|([\mathbf{D}(\mathbf{u}^n) - \mathbf{D}(E\mathbf{u}_h^n)]\nabla c^n, \nabla\zeta)\right| \\
&= \int_\Omega \int_0^1 [\frac{\partial \mathbf{D}}{\partial \mathbf{u}}(\theta\mathbf{u}^n - (1-\theta)E\mathbf{u}_h^n) - \frac{\partial \mathbf{D}}{\partial \mathbf{u}}(E\mathbf{u}_h^n)]d\theta(\mathbf{u}^n - E\mathbf{u}_h^n)\nabla c^n \cdot \nabla\zeta^n dX \\
&\quad + \int_\Omega \frac{\partial \mathbf{D}}{\partial \mathbf{u}}(E\mathbf{u}_h^n)(\mathbf{u}^n - E\mathbf{u}_h^n)\nabla c^n \cdot \nabla\zeta^n dX = W_1 + W_2.
\end{aligned} \tag{42}$$

Suppose that $\frac{\partial^2\mathbf{D}}{\partial\mathbf{u}^2}$ is bounded, we have

$$\begin{aligned}\left|W_1\right| &\le K\Big|\frac{\partial^2\mathbf{D}}{\partial\mathbf{u}^2}\Big|\|\mathbf{u}^n - E\mathbf{u}_h^n\|_0^2\|\nabla\zeta\|_\infty \\ &\le Kh_c^{-5/2}\{(\Delta t_c)^2 + h_c^{2(l+1)} + h_p^{2(k+1)} + \|\zeta_m\|^2 + \|\zeta_{m-1}\|^2\}\|\zeta^n\|_0 \\ &\le K\{(\Delta t_c)^2 + h_c^{2(l+1)} + h_p^{2(k+1)} + \|\zeta_m\|^2 + \|\zeta_{m-1}\|^2\},\end{aligned} \tag{43a}$$

and

$$\left|W_2\right| \le \frac{1}{3}\|\mathbf{D}^{1/2}(E\mathbf{u}_h^n)\nabla\zeta^n\|_0^2 + K\int_\Omega \mathbf{D}(E\mathbf{u}_h^n)^{-1}\Big|\frac{\partial\mathbf{D}}{\partial\mathbf{u}}(E\mathbf{u}_h^n)\Big|^2|\mathbf{u}^n - E\mathbf{u}_h^n|^2|\nabla c^n|^2 dX. \tag{43b}$$

Later it is shown that $\mathbf{D}^{-1}|\frac{\partial\mathbf{D}}{\partial\mathbf{u}}|^2$ is bounded. For simplicity, we suppose that $\mathbf{u}$ is oriented in x-direction, since the rotation transformation of coordinates will not affect the size of $|\frac{\partial\mathbf{D}}{\partial\mathbf{u}}|^2$. Then,

$$\mathbf{D} = \mathbf{D}_m + |\mathbf{u}|^\beta\begin{pmatrix}\alpha_l & 0 & 0\\ 0 & \alpha_t & 0\\ 0 & 0 & \alpha_t\end{pmatrix}, \qquad \mathbf{D}^{-1} \le |\mathbf{u}|^{-\beta}\begin{pmatrix}\alpha_l^{-1} & 0 & 0\\ 0 & \alpha_t^{-1} & 0\\ 0 & 0 & \alpha_t^{-1}\end{pmatrix},$$

$$\frac{\partial\mathbf{D}}{\partial u_x} = |\mathbf{u}|^{\beta-1}\begin{pmatrix}\alpha_l & 0 & 0\\ 0 & \alpha_t & 0\\ 0 & 0 & \alpha_t\end{pmatrix}, \qquad \frac{\partial\mathbf{D}}{\partial u_y} = |\mathbf{u}|^{\beta-1}\begin{pmatrix}0 & \alpha_l-\alpha_t & 0\\ \alpha_l-\alpha_t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\frac{\partial\mathbf{D}}{\partial u_z} = |\mathbf{u}|^{\beta-1}\begin{pmatrix}0 & 0 & \alpha_l-\alpha_t\\ 0 & 0 & 0\\ \alpha_l-\alpha_t & 0 & 0\end{pmatrix},$$

$$\begin{aligned}\Big|\mathbf{D}^{-1}(\frac{\partial\mathbf{D}}{\partial\mathbf{u}})^2\Big| \le |\mathbf{u}|^{\beta-2}\Bigg\{&\beta^2\begin{pmatrix}\alpha_l & 0 & 0\\ 0 & \alpha_t & 0\\ 0 & 0 & \alpha_t\end{pmatrix} + \begin{pmatrix}\alpha_l^{-1}(\alpha_l-\alpha_t)^2 & 0 & 0\\ 0 & \alpha_t^{-1}(\alpha_l-\alpha_t)^2 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &+\begin{pmatrix}\alpha_l^{-1}(\alpha_l-\alpha_t)^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \alpha_t^{-1}(\alpha_l-\alpha_t)^2\end{pmatrix}\Bigg\}.\end{aligned}$$

When the constraint conditions hold,

$$\frac{\alpha_l^2}{\alpha_t} \le \alpha^* < \infty, \quad \frac{\alpha_t^2}{\alpha_l} \le \alpha^* < \infty, \quad \beta \ge 2, \tag{43c}$$

then $\mathbf{D}^{-1}|\frac{\partial\mathbf{D}}{\partial\mathbf{u}}|^2$ is bounded. Now $\frac{\partial^2\mathbf{D}}{\partial\mathbf{u}^2}$ is argued. Noting that

$$\frac{\partial^2\mathbf{D}}{\partial u_x^2} = \beta(\beta-1)|\mathbf{u}|^{\beta-2}\begin{pmatrix}\alpha_l & 0 & 0\\ 0 & \alpha_t & 0\\ 0 & 0 & \alpha_t\end{pmatrix}, \qquad \frac{\partial^2\mathbf{D}}{\partial u_y^2} = 2|\mathbf{u}|^{\beta-2}\begin{pmatrix}\alpha_t & 0 & 0\\ 0 & \alpha_l 0 & 0\\ 0 & 0 & \alpha_t\end{pmatrix},$$

$$\frac{\partial^2\mathbf{D}}{\partial u_z^2} = 2|\mathbf{u}|^{\beta-2}\begin{pmatrix}\alpha_l & 0 & 0\\ 0 & \alpha_t & 0\\ 0 & 0 & \alpha_t\end{pmatrix}, \qquad \frac{\partial^2\mathbf{D}}{\partial u_x\partial u_y} = \beta|\mathbf{u}|^{\beta-2}\begin{pmatrix}0 & \alpha_l-\alpha_t & 0\\ \alpha_l-\alpha_t & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \cdots$$

we can get that $\frac{\partial^2\mathbf{D}}{\partial\mathbf{u}^2}$ is bounded for $\beta \ge 2$. In a similar discussion, W_2 is estimated by the right-hand side expression of (43a).

For (30), using the estimates of (31)-(43), we have

$$\begin{aligned}&\frac{1}{2\Delta t_c}\{\|\phi^{1/2}\zeta^n\|_0^2 - \|\phi^{1/2}\zeta^{n-1}\|_0^2\} + \|\mathbf{D}^{1/2}(E\mathbf{u}_h^n)\nabla\zeta^n\|_0^2 + \|\tilde{q}^{1/2}\zeta^n\|_0^2\\ &\le K\{\Delta t_c\|\frac{\partial^2 c}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;L^2)} + (\Delta t_c)^2 + (\Delta t_p^1)^3 + (\Delta t_p)^4 + h_c^{2l} + h_p^{2(k+1)}\\ &\quad + \|\zeta_{m-1}\|^2 + \|\zeta_m\|^2 + \|\zeta^{n-1}\|^2 + \|\zeta^n\|^2\}.\end{aligned} \tag{44}$$

Multiplying both sides of (44) by $2\Delta t$, summing them over $n(1 \le n \le L)$, and applying the discrete Gronwall inequality, we have

$$\|\zeta^L\|_0^2 \le K\{h_c^{2l} + h_p^{2(k+1)} + (\Delta t_c)^2 + (\Delta t_p^1)^3 + (\Delta t_p)^4\}. \tag{45}$$

It remains to testify the induction hypothesis (41). As $l \geq 3$, and the partition parameters satisfy

$$\Delta t_c = O(h_c^{5/2}), \quad h_p^{k+1} = O(h_c^{5/2}), \tag{46}$$

then it is easy to check the proof of (41).

The following statement is concluded by using (17), (14), (45) and (25).

Theorem. Suppose that the problem of (1)-(4) is regular (R), and positive semi-definite (C). And suppose that the index of M_h, $l \geq 3$, and (46) holds. Adopting the schemes of (28) and (29), we have the following estimates,

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(H(\text{div}))} + \|p - p_h\|_{L^\infty(L^2)} + \|c - c_h\|_{L^\infty(L^2)} \leq K\{h_p^{k+1} + h_c^l + \Delta t_c + (\Delta t_p^1)^{3/2} + (\Delta t_p)^2\}, \tag{47}$$

where K is a constant dependent on p, c and their derivatives, independent of the parameters Δt_c, h_p, h_c.

4. Conclusions and Discussions

In the present paper, we discuss a mixed element-characteristic finite element method to solve three-dimensional incompressible miscible positive semi-definite displacement problem in porous media. In §1 Introduction, we state the mathematical model, physical background and some related international research studies. In §2, we define some notations and partitions, then construct a composite computational scheme of mixed element-characteristic finite element. In §3, we introduce an induction hypothesis and use theoretical techniques of priori estimates, then show convergence analysis of the present scheme. In conclusion, the numerical scheme and its theoretical analysis can solve actual applications such as well-known positive semi-definite displacement problem efficiently.

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