Tensor Product Of Zero-divisor Graphs With Finite Free Semilattices

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Abstract

Γ(SLₓ) is defined and has been investigated in (Toker, 2016). In this paper our main aim is to extend this study over Γ(SLₓ) to the tensor product. The diameter, radius, girth, domination number, independence number, clique number, chromatic number and chromatic index of Γ(SLₓ) ⊗ Γ(SLₓ) has been established. Moreover, we have determined when Γ(SLₓ) ⊗ Γ(SLₓ) is a perfect graph.

Keywords: Tensor product, finite free semilattice, zero-divisor graph, clique number, domination number, perfect graph

1. Introduction

Let G be a graph then edge set of G denoted by E(G) and vertex set of G denoted by V(G). Let G₁ and G₂ be graphs, tensor product of G₁ and G₂ has vertex set V(G₁) × V(G₂) and has edge set {(ui, v₁)(u₂, v₂) : u₁u₂ ∈ E(G₁) and v₁v₂ ∈ E(G₂)}, and it is denoted by G₁ ⊗ G₂. Let G₁ and G₂ be connected graphs then G₁ ⊗ G₂ is connected if and only if either G₁ or G₂ contains an odd cycle (Weichsel, 1962). Also it is clear that G₁ ⊗ G₂ ≃ G₂ ⊗ G₁.

Firstly zero-divisor graph on a commutative semigroup S with 0 was studied by Demeyer and his friends (DeMeyer et all., 2002; DeMeyer et all, 2005). Let the set of zero divisor elements in S is Z(S), the zero-divisor graph Γ(S) is defined as an undirected graph with vertices Z(S) \ {0} and the vertices x and y are adjacent with a single edge if and only if xy = 0. Always Γ(S) is a connected graph (DeMeyer et all., 2002).

Let X be a finite non-empty set. The free semilattice on a set X is the finite powerset of X except the empty set and operation is union of sets. We show it with SLₓ. Then SLₓ is a commutative semigroup of idempotents with the multiplication A · B = A ∪ B for all A, B ∈ SLₓ. In zero-divisor graph of SLₓ, any two distinct vertices A and B are adjacent with the rule A ∪ B = X. In a recent study, Γ(SLₓ) has been investigated in (Toker, 2016).

We know that if |X| ≥ 3 then Γ(SLₓ) contains an odd cycle (Toker, 2016). Let X₁ and X₂ be non-empty and finite sets and let Γ(SLₓ₁) be zero-divisor graph associated to SLₓ₁ and Γ(SLₓ₂) be zero-divisor graph associated to SLₓ₂. In this paper, without loss of generality we assume that |X₁| = n, |X₂| = m and we suppose that X₁ = {x₁, . . ., xₙ} and X₂ = {y₁, . . ., yₘ}. So if |X₁| ≥ 2, |X₂| ≥ 3 then Γ(SLₓ₁) ⊗ Γ(SLₓ₂) is connected graph and in this paper we have researched girth, diameter, radius, dominating number, clique number, chromatic number, chromatic index, independence number and perfectness of this graph.

For graph theory see (Gross & Yellen, 2004), and for semigroup theory (Howie, 1995).

2. Some Properties of Γ(SLₓ₁) ⊗ Γ(SLₓ₂)

Let G be a simple graph, the distance (length of the shortest path) between two vertices u, v in G is denoted by dG(u, v). In a connected simple graph the maximum distance (length of the shortest path) between v and any other vertex u in G is eccentricity of a vertex v, it is denoted by ecc(v), so that is

\[ ecc(v) = \max\{d_G(u, v) \mid u \in V(G)\}. \]

The diameter of G is defined by

\[ \max\{ecc(v) \mid v \in V(G)\} \]

and it is denoted by diam(G). Moreover radius of G is defined by

\[ \min\{ecc(v) \mid v \in V(G)\} \]

and it is denoted by rad(G). The girth of a graph is the length of a shortest cycle contained in the graph, and it is denoted by gr(G). If there is not any cycle in a graph, then its girth is defined to be infinity.
The degree (or valency) of a vertex of a graph is the number of edges incident to the vertex, with loops counted twice, degree of vertex \( v \in V(G) \) is denoted by \( \deg_G(v) \). Among all degrees, the maximum degree is denoted by \( \Delta(G) \) and the minimum degree is denoted by \( \delta(G) \). In a graph, the vertex of maximum degree is called delta-vertex and the set of delta-vertices of \( G \) denoted by \( \Delta_G \). In a graph, an independent set or stable set is a set of vertices in a graph, no two of which are adjacent. Independence number of \( G \) is denoted by \( \alpha(G) \) and it is defined by

\[
\alpha(G) = \max\{|I| \mid I \text{ is an independent set of } G \}.
\]

In a graph, a dominating set for a graph \( G \) is a subset \( D \) of \( V(G) \) such that every vertex not in \( D \) is adjacent to at least one member of \( D \). The domination number of \( G \) is the number of vertices in a smallest dominating set for \( G \), and it is denoted by \( \gamma(G) \), so dominating number of \( G \) is

\[
\gamma(G) = \min\{|D| \mid D \text{ is a dominating set of } G \}.
\]

The open neighbourhood of a vertex \( v \in V(G) \) is the set of vertices which are adjacent to \( v \) and it is denoted by \( N_G(v) \), the closed neighbourhood of \( v \) is \( N_G(v) \cup \{v\} \) and it is denoted by \( N_G[v] \). It is clear that \( |N_G[v]\cap D| \geq 1 \) for each dominating set \( D \), and for each \( v \in V(G) \).

In this section we mainly deal with some graph properties of \( \Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2}) \) namely diameter, radius, girth, domination number and independence number.

We use the notation \( \overline{A} = (X_i \setminus A) \) for all \( A \subseteq X_i \), \( i = 1, 2 \), and we use the notation \( d(u, v) \) instead of \( d_{G[S\cup L_X]}(u, v) \).

Moreover for convenience we use \( \Gamma_1 \otimes \Gamma_2 \) instead of \( \Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2}) \). Notice that, for \( u = (A_1, B_1) \), \( v = (A_2, B_2) \in V(\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})) \) there exists a single edge \( u - v \) in \( \Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2}) \) if and only if \( A_1 \supseteq \overline{A}_2 \) and \( B_1 \supseteq \overline{B}_2 \).

**Theorem 2.1**

(i) If \( |X_1| \geq 3 \) and \( |X_2| \geq 3 \) then \( \text{diam}(\Gamma_1 \otimes \Gamma_2) = 4 \).

(ii) If \( |X_1| = 2 \) and \( |X_2| \geq 3 \) then \( \text{diam}(\Gamma_1 \otimes \Gamma_2) = 5 \).

**Proof.** (i) Let \( |X_1| \geq 3 \), \( |X_2| \geq 3 \) and \( u = (A_1, B_1) \), \( v = (A_2, B_2) \in V(\Gamma_1 \otimes \Gamma_2) \). If \( A_1 \cup A_2 = X_1 \) and \( B_1 \cup B_2 = X_2 \) then \( d(u, v) = 1 \). It is clear that in other cases \( d(u, v) \geq 2 \). Second case is \( A_1 \cup A_2 \neq X_1 \) and \( B_1 \cup B_2 \neq X_2 \). In second case if \( A_1 \cap A_2 \neq \emptyset \) and \( B_1 \cap B_2 \neq \emptyset \), we take \( C_1 = A_1 \cap A_2 \), \( C_2 = B_1 \cap B_2 \) thus we have a path \((A_1, B_1) - (C_1, C_2) - (A_2, B_2) \) and \( d(u, v) = 2 \). In second case let \( A_1 \cap A_2 \neq \emptyset \) and \( B_1 \cap B_2 = \emptyset \). In this case since \( \overline{B}_1 \cup \overline{B}_2 = X_2 \) thus \( d(u, v) \geq 2 \). If \( A_1 \neq A_2 \) then \( A_1 \setminus A_2 \neq \emptyset \) or \( A_2 \setminus A_1 \neq \emptyset \). If \( A_1 \setminus A_2 \neq \emptyset \) then we have a path \((A_1, B_1) - (A_1, \overline{B}_1) - (A_1 \cup \overline{A}_2, \overline{B}_2) - (A_2, B_2) \) so \( d(u, v) = 3 \). If \( A_1 = A_2 \) and \( |A_1| = 1 \) then we have a path \((A_1, B_1) - (\overline{A}_1, B_1) - (A_1 \cup \overline{A}_2, \overline{B}_2) - (A_2, B_2) \) so \( d(u, v) \leq 4 \). Also, \((A_1, B)\) has adjacent form of \((C, \overline{C})\) where \( \overline{C} \subseteq B \) so \( d(u, v) \leq 3 \) then \( d(u, v) = 4 \). In second case if \( A_1 \cap A_2 = \emptyset \) and \( B_1 \cap B_2 \neq \emptyset \) is similar. In second case if \( A_1 \cap A_2 = \emptyset \) and \( B_1 \cap B_2 = \emptyset \) then we have a path \((A_1, B_1) - (A_1 \cup C, B_1) - (A_1 \cup C, \overline{B}_2) - (A_2, B_2) \) so \( d(u, v) = 3 \). Third case is \( A_1 \cup A_2 = X_1 \) and \( B_1 \cup B_2 \neq X_2 \). In third case if \( A_1 \cap A_2 \neq \emptyset \) and \( B_1 \cap B_2 \neq \emptyset \) then we have a path \((A_1, B_1) - (A_1 \cup \overline{A}_2, \overline{B}_2) - (A_2, B_2) \) so \( d(u, v) = 2 \). If \( A_1 \cap A_2 \neq \emptyset \) and \( B_1 \cap B_2 = \emptyset \) then we have a path \((A_1, B_1) - (A_1, \overline{B}_1) - (A_1 \cup \overline{A}_2, B_2) - (A_2, B_2) \) and since \( \overline{B}_1 \cup \overline{B}_2 = \emptyset \) so \( d(u, v) = 3 \). Let \( A_1 \cap A_2 = \emptyset \) and \( B_1 \cap B_2 \neq \emptyset \). In this case since \( \overline{A}_1 \cup \overline{A}_2 = X_1 \) then \( d(u, v) \geq 3 \). If \( B_1 \neq B_2 \) then \( B_1 \setminus B_2 \neq B_2 \setminus B_1 \). If \( B_1 \setminus B_2 \neq \emptyset \) then we have a path \((A_1, B_1) - (A_2, \overline{B}_1) \cup \overline{B}_2) - (A_1, \overline{B}_2) - (A_2, B_2) \) and if \( B_2 \setminus B_1 \neq \emptyset \) then we have a path \((A_1, B_1) - (A_1, \overline{B}_1) \cup \overline{B}_2) - (A_2, B_2) \) so \( d(u, v) \geq 3 \). Last case is \( A_1 \cup A_2 \neq X_1 \) and \( B_1 \cup B_2 = X_2 \) is similar with third case. Thus if \( |X_1| \geq 3 \) and \( |X_2| \geq 3 \) then \( \text{diam}(\Gamma_1 \otimes \Gamma_2) = 4 \).

(ii) Let \( |X_1| = 2 \), \( |X_2| \geq 3 \) and \( u = (A_1, B_1), v = (A_2, B_2) \in V(\Gamma_1 \otimes \Gamma_2) \). In here different case is \( A_1 \cup A_2 = X_1 \) and \( B_1 \cup B_2 \neq X_2 \) with \( B_1 = B_2 \) and \( |B_1| = 1 \), in other cases we have same results with i). This case we take 2-partition of \( \overline{B}_1 \), we say \( M \) and \( N \). We have a path \((A_1, B_1) - (A_2, \overline{B}_1) - (A_1, B_1 \cup M) - (A_2, B_1 \cup N) - (A_1, B_1) - (A_2, B_2) \) so \( d(u, v) \leq 5 \). \((A_1, B_1)\) has only one adjacent and it is \((A_2, \overline{B}_1)\) and \((A_2, B_1)\) has only one adjacent and it is \((A_1, B_1)\) they are different vertices and they are not adjacent, moreover \( d((A_2, \overline{B}_1), (A_1, B_1)) = 3 \), thus \( d(u, v) = 5 \). So if \( |X_1| = 2 \) and \( |X_2| \geq 3 \) then \( \text{diam}(\Gamma_1 \otimes \Gamma_2) = 5 \).
Theorem 2.2

(i) If $|X_1| \geq 3$ and $|X_2| \geq 3$ then $gr(\Gamma_1 \otimes \Gamma_2) = 3$.

(ii) If $|X_1| = 2, |X_2| = 3$ then $gr(\Gamma_1 \otimes \Gamma_2) = 6$ and if $|X_1| = 2, |X_2| \geq 4$ then $gr(\Gamma_1 \otimes \Gamma_2) = 4$.

(iii) If $|X_1| \geq 3$ and $|X_2| \geq 3$ then $rad(\Gamma_1 \otimes \Gamma_2) = 3$.

(iv) If $|X_1| = 2$ and $|X_2| \geq 3$ then $rad(\Gamma_1 \otimes \Gamma_2) = 4$.

Proof. (i) Let $|X_1| \geq 3, |X_2| \geq 3$ and $(A, B) \in V(\Gamma_1 \otimes \Gamma_2)$. Assume that $|A| \geq 2, |B| \geq 2$ so there exists 2–partition of A, we say $A_1$ and $A_2$ and there exists 2–partition of B, we say $B_1$ and $B_2$. Thus $(A, B)–(\bar{A} \cup A_1, \bar{B} \cup B_1)-(\bar{A} \cup A_2, \bar{B} \cup B_2)-(A, B)$ is a cycle. Let $|A| = 1, |B| \geq 2$ then there exists $\varnothing \neq C \subsetneq \bar{A}$ so we have a cycle $(A, B)-(\bar{A}, \bar{B} \cup B_1)-(\bar{A} \cup C, B)-(\bar{A}, \bar{B} \cup B_2)-(A, B)$. If $|A| \geq 2, |B| = 1$, we can find a cycle similar way. Moreover $\Gamma_1 \otimes \Gamma_2$ is simple graph and from its definition $gr(\Gamma_1 \otimes \Gamma_2) = 3$.

(ii) $\eta$:

If $|X_1| = 2, |X_2| = 3$ then $\Gamma_1 \otimes \Gamma_2$ is $\eta$ thus in this case $gr(\Gamma_1 \otimes \Gamma_2) = 6$. If $|X_1| = 2, |X_2| = m \geq 4$, $gr(\Gamma_1 \otimes \Gamma_2)$ can not be odd number since $|X_1| = 2$. So $gr(\Gamma_1 \otimes \Gamma_2) \geq 4$. Let $(A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, if $2 \leq |B| \leq m - 2$ then there exists $k \in B$ and 2–partition of B is E and F. So $(A, B)–(\bar{A}, \bar{B} \cup E)-(\bar{A}, \bar{B} \cup F)-(A, B)$ is a cycle. If $|B| = m - 1$ then without loss of generality we assume that $y_1, y_2 \in B$ then $(A, B)-(\bar{A}, \bar{B} \cup \{y_1, y_2\})-(A, B \setminus \{y_1\})-(\bar{A}, \bar{B} \cup \{y_1\})-(A, B)$ is a cycle. Thus if $|X_1| = 2, |X_2| \geq 4$ then $gr(\Gamma_1 \otimes \Gamma_2) = 4$.

(iii) Let $|X_1| \geq 3, |X_2| \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, we can determine $ecc(v)$. If $|A| \geq 2, |B| \geq 2$ then $ecc(v) \leq 3$ from proof of Theorem 2.1 (i) because let $u \in V(\Gamma_1 \otimes \Gamma_2)$, we found that if $|A| \geq 2, |B| \geq 2$ then $d(u, v) \leq 3$, so $ecc(v) \leq 3$. Moreover there exists $\varnothing \neq C \subsetneq B$, if we choose $u = (\bar{A}, C)$ so $d(u, v) = 3$ it follows that $ecc(v) = 3$. If $|A| = 1$ we choose $u = (A, \bar{B})$ and $d(u, v) = 4$. If $|B| = 1$ is similar. So $rad(\Gamma_1 \otimes \Gamma_2) = 3$.

(iv) Let $|X_1| = 2, |X_2| \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, we can determine $ecc(v)$. If $|A| = |B| = 1$ then $ecc(v) = 5$. If $|B| \geq 2$ it is clear that $ecc(v) \leq 4$, if we choose $u = (A, \bar{B})$ then $d(u, v) = 4$, so $ecc(v) = 4$. Thus $rad(\Gamma_1 \otimes \Gamma_2) = 4$.

Theorem 2.3 If $|X_1| = n \geq 2, |X_2| = m \geq 3$ then $\gamma(\Gamma_1 \otimes \Gamma_2) = n.m$

Proof. Let $|X_1| = n \geq 2, |X_2| = m \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, D be a dominating set for $\Gamma_1 \otimes \Gamma_2$. It is clear that $deg_{\Gamma_1 \otimes \Gamma_2}(v) = deg_{\Gamma_1}(A).deg_{\Gamma_2}(B)$ so if $|A| = |B| = 1$ then $deg_{\Gamma_1 \otimes \Gamma_2}(v) = 1$ and adjacent of $v$ is only $u = (\bar{A}, \bar{B})$ so $N_{\Gamma_1 \otimes \Gamma_2}(v) = \{u, v\}$. Since $N_{\Gamma_1 \otimes \Gamma_2}(\{v\}) \cap D \geq 1$, each $v \in D$ or $u \in D$. Let $x_i, x_k \in X_1$ and $y_j, y_l \in X_2$. Moreover since $|X_2| \geq 3$ if $(x_i, y_j) \neq (x_k, y_l)$ then $N_{\Gamma_1 \otimes \Gamma_2}(\{x_i, y_j\}) \cap N_{\Gamma_1 \otimes \Gamma_2}(\{x_k, y_l\}) = \varnothing$.

Thus $\gamma(\Gamma_1 \otimes \Gamma_2) \geq n.m$. If we choose $D = \{u = (A, B); u \in V(\Gamma_1 \otimes \Gamma_2)$ and $|A| = n - 1, |B| = m - 1\}$, it is easy to see that $|D| = n.m$ and D is a dominating set for $\Gamma_1 \otimes \Gamma_2$. It follows that $\gamma(\Gamma_1 \otimes \Gamma_2) = n.m$.

Theorem 2.4 If $|X_1| = n \geq 2, |X_2| = m \geq 3$ then

$$\alpha(\Gamma_1 \otimes \Gamma_2) = \frac{(2^n - 2)(2^m - 2)}{2}$$

Proof. Let $x_i \in X_1$ and $C = \{(A_i, B_j) : \varnothing \neq A_i \subseteq X_1 \setminus \{x_i\}, B_j \in V(\Gamma_2)\}$ and $D = \{(X_1 \setminus A_i, X_2 \setminus B_j) : (A_i, B_j) \in C\}$. It is clear that $C \cap D = \varnothing$ and $|C| = |D| = \frac{(2^n - 2)(2^m - 2)}{2}$ thus $C \cup D = V(\Gamma_1 \otimes \Gamma_2)$. Let I be an independence set of $\Gamma_1 \otimes \Gamma_2$, then from the pigeonhole principle $|I| \leq \frac{(2^n - 2)(2^m - 2)}{2}$, and $C$ is an independence set for $\Gamma_1 \otimes \Gamma_2$, moreover $|C| = \frac{(2^n - 2)(2^m - 2)}{2}$.
3. Perfectness of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$

Let $G$ be a graph. Clique is the each of the maximal complete subgraphs of $G$. The number of all the vertices in any clique of $G$ is clique number and it is denoted by $\omega(G)$. The chromatic number of a graph $G$ is the smallest number of colors needed to color the vertices of $G$ so that no two adjacent vertices share the same color and it is denoted by $\chi(G)$. It is well-known that

$$\chi(G) \geq \omega(G) \quad (1)$$

for any graph $G$ (Chartrand & Zhang, 2009). Let $V' \subseteq V(G)$, then induced subgraph $G' = (V', E')$ is a subgraph of $G$ such that $E'$ consists of those edges whose endpoints are in $V'$. For each induced subgraph $H$ of $G$, if $\chi(H) = \omega(H)$, then $G$ is called a perfect graph.

The complement or inverse of a graph $G$ is a graph on the same vertices such that two distinct vertices are adjacent if and only if they are not adjacent in $G$, the complement of $G$ is denoted by $G'$.

A graph $G$ is called Berge if no induced subgraph of $G$ is an odd cycle of length at least five or the complement of one. The edges are called adjacent if they share a common end vertex. An edge coloring of a graph is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. The minimum required number of colours for an edge colouring of $G$ is called the chromatic index of $G$ and it is denoted by $\bar{\chi}(G)$. Vizing gave a fundamental theorem for that, for any graph $G$, we have

$$\Delta(G) \leq \bar{\chi}(G) \leq \Delta(G) + 1$$

(Vizing, 1964). Graph $G$ is called class-1 if $\Delta(G) = \bar{\chi}(G)$ and called class-2 if $\bar{\chi}(G) = \Delta(G) + 1$.

The core of a graph $G$ is defined to be the largest induced subgraph of $G$ such that each edge in core is part of a cycle and it is denoted by $G_\infty$. Finally, let $M$ be a subset of $E(G)$ for a graph $G$, if there is no two edges in $M$ which are adjacent then $M$ is called a matching.

**Conjecture 3.1** Let $G$ and $H$ be graphs then $\chi(G \otimes H) = \min\{\chi(G), \chi(H)\}$. (Hedetniemi, 1966)

**Theorem 3.2** If $|X_1| = n \geq 2$ and $|X_2| = m \geq 3$ then

$$\omega(\Gamma_1 \otimes \Gamma_2) = \chi(\Gamma_1 \otimes \Gamma_2) = \min\{n, m\}.$$
vertices in $Q_k$, there is no two adjacent vertices have same colour, and so $\chi(G) \leq n$. Thus $\chi(G \otimes G) = \min\{n, m\}$. If $n \geq m$ is similar. So conjecture holds for $G_1 \otimes G_2$.

Lemma 3.3 A graph is perfect if and only if it is Berge (Chudnovsky et all., 2006).

Therefore, a graph $G$ is perfect if and only if neither $G$ nor $G'$ contains an odd cycle of length at least 5 as an induced subgraph.

Theorem 3.4 If $|X_1| = 2, |X_2| \geq 3$ then $G_1 \otimes G_2$ is perfect graph but if $|X_1| \geq 3, |X_2| \geq 3$ then $G_1 \otimes G_2$ is not perfect graph.

Proof. We assume that $X_1 = \{x_1, \ldots, x_m\}$ and $X_2 = \{y_1, \ldots, y_n\}$. Let $|X_1| = 2$ and $|X_2| \geq 3$. It is clear that there is not any odd cycle at least 5 as induced subgraph of $G_1 \otimes G_2$ since $|X_1| = 2$. Let $G = (G_1 \otimes G_2)'$. For $k \geq 3$ we assume that there is an induced subgraph of $G$ which is cycle with $2k - 1$ vertices, say

$$C_1 \cup C_2 \cup \cdots \cup C_{2k-1}$$.

Without loss of generality assume that $C_1 = (x_1, B_1)$. Then $C_2 = (x_2, B_2)$ and $C_{2k-2} = (x_{2k-2}, B_{2k-2})$ because $C_1$ and $C_3$ adjacent vertices in $G_1 \otimes G_2$ and $C_1$ and $C_{2k-2}$ adjacent vertices in $G_1 \otimes G_2$. Thus $C_2 = (x_1, B_2)$ and $C_{2k-1} = (x_1, B_{2k-1})$ because $C_2$ and $C_{2k-2}$ adjacent vertices in $G_1 \otimes G_2$ and $3 \leq |X_2|$. But in this case $C_2$ and $C_2k-1$ adjacent vertices in $G$ which is a contradiction. So if $|X_1| = 2, |X_2| \geq 3$ then $G_1 \otimes G_2$ is perfect graph. Let $|X_1| \geq 3, |X_2| \geq 3$ and $Y_1 = X_1 \setminus \{x_1, x_2, x_3\}$, $Y_2 = X_2 \setminus \{y_1, y_2, y_3\}$. Let $H = \{(x_1, x_2) \cup y_1, (y_2, y_3) \cup y_2\} - \{(x_1, x_2) \cup y_1, (y_2, y_3) \cup y_2\}$. So $H$ is cycle of length of 5 which subinduced graph of $G_1 \otimes G_2$. Thus if $|X_1| \geq 3, |X_2| \geq 3$ then $G_1 \otimes G_2$ is not perfect graph.

Lemma 3.5 Consider the graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set. Suppose that $E_1$ is a matching such that no edge has both endvertices in $N_{G_1}[\mathcal{N}_{G_2}]$. If the union graph $G = G_1 \cup G_2$ has maximum degree $\Delta(G) = \Delta(G_2) + 1$ then $G$ is class-1 (Machado & Figueiredo, 2010).

Theorem 3.6 If $|X_1| = m \geq 2, |X_3| = n \geq 3$ then

$$\chi'(G_1 \otimes G_2) = \chi'(G_1), \chi'(G_2) = (2^{n-1} - 1)(2^{m-1} - 1)$$.

Proof. Let $(u, v) \in V(G_1 \otimes G_2)$, if $|u| \geq 2, |v| \geq 2$ or if $|u| = 1, |v| \geq 2$ or $|u| \geq 2, |v| = 1$ then $(u, v)$ is in core at graph from Theorem 2.2. Let $B = \{(x_i, y_j) - (x_i, x_j) \setminus (y_i) : x_i \in X_1, y_j \in X_2, G_1 = (V(G_1 \otimes G_2), B)\}$ and $G_2 = (V(G_1 \otimes G_2), E(G_1 \otimes G_2))$. So $B$ is a matching such that no edge has both endvertices in $N_{G_1}[\mathcal{N}_{G_2}]$. Also $(G_1 \otimes G_2) = G_1 \cup G_2$ and $\Delta(G_1 \otimes G_2) = \Delta(G_2) + 1$ so from Lemma 3.5, $G_1 \otimes G_2$ is class-1.
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