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Tensor Product Of Zero-divisor Graphs With Finite Free Semilattices

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Abstract

$\Gamma(SL_X)$ is defined and has been investigated in (Toker, 2016). In this paper our main aim is to extend this study over $\Gamma(SL_X)$ to the tensor product. The diameter, radius, girth, domination number, independence number, clique number, chromatic number and chromatic index of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$ has been established. Moreover, we have determined when $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$ is a perfect graph.

Keywords: Tensor product, finite free semilattice, zero-divisor graph, clique number, domination number, perfect graph

1. Introduction

Let G be a graph then edge set of G denoted by $E(G)$ and vertex set of G denoted by $V(G)$. Let G_1 and G_2 be graphs, tensor product of G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and has edge set $\{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$, and it is denoted by $G_1 \otimes G_2$. Let G_1 and G_2 be connected graphs then $G_1 \otimes G_2$ is connected if and only if either G_1 or G_2 contains an odd cycle (Weichsel, 1962). Also it is clear that $G_1 \otimes G_2 \simeq G_2 \otimes G_1$.

Firstly zero-divisor graph on a commutative semigroup S with 0 was studied by Demeyer and his friends (DeMeyer et all., 2002; DeMeyer et all, 2005). Let the set of zero divisor elements in S is $Z(S)$, the zero-divisor graph $\Gamma(S)$ is defined as an undirected graph with vertices $Z(S) \setminus \{0\}$ and the vertices x and y are adjacent with a single edge if and only if $xy = 0$. Always $\Gamma(S)$ is a connected graph (DeMeyer et all., 2002).

Let X be a finite non-empty set. The free semilattice on a set X is the finite powerset of X except the empty set and operation is union of sets. We show it with SL_X. Then SL_X is a commutative semigroup of idempotents with the multiplication $A \cdot B = A \cup B$ for all $A, B \in SL_X$. In zero-divisor graph of SL_X, any two distinct vertices A and B are adjacent with the rule $A \cup B = X$. In a recent study, $\Gamma(SL_X)$ has been investigated in (Toker, 2016).

We know that if $|X| \geq 3$ then $\Gamma(SL_X)$ contains an odd cycle (Toker, 2016). Let X_1 and X_2 be non-empty and finite sets and let $\Gamma(SL_{X_1})$ be zero-divisor graph associated to SL_{X_1} and $\Gamma(SL_{X_2})$ be zero-divisor graph associated to SL_{X_2}. In this paper, without loss of generality we assume that $|X_1| = n$, $|X_2| = m$ and we suppose that $X_1 = \{x_1, \ldots, x_n\}$ and $X_2 = \{y_1, \ldots, y_m\}$. So if $|X_1| \geq 2$, $|X_2| \geq 3$ then $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$ is connected graph and in this paper we have researched girth, diameter, radius, dominating number, clique number, chromatic number, chromatic index, independence number and perfectness of this graph.

For graph theory see (Gross & Yellen, 2004), and for semigroup theory (Howie, 1995).

2. Some Properties of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$

Let G be a simple graph, the distance (length of the shortest path) between two vertices u, v in G is denoted by $d_G(u, v)$. In a connected simple graph the maximum distance (lenght of the shortest path) between v and any other vertex u in G is eccentricity of a vertex v,it is denoted by $\text{ecc}(v)$, so that is

$$\text{ecc}(v) = \max\{d_G(u, v) \mid u \in V(G)\}.$$

The diameter of G is defined by

$$\max\{\text{ecc}(v) \mid v \in V(G)\}$$

and it is denioted by $\text{diam}(G)$. Moreover radius of G is defined by

$$\min\{\text{ecc}(v) \mid v \in V(G)\}$$

and it is denoted by $\text{rad}(G)$. The girth of a graph is the length of a shortest cycle contained in the graph, and it is denoted by $\text{gr}(G)$. If there is not any cycle in a graph, then its girth is defined to be infinity.

The degree (or valency) of a vertex of a graph is the number of edges incident to the vertex, with loops counted twice, degree of vertex $v \in V(G)$ is denoted by $\deg_G(v)$. Among all degrees, the maximum degree is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. In a graph, the vertex of maximum degree is called delta-vertex and the set of delta-vertices of G denoted by Λ_G. In a graph, an independent set or stable set is a set of vertices in a graph, no two of which are adjacent. Independence number of G is denoted by $\alpha(G)$ and it is defined by

$$\alpha(G) = \max\{|I| \mid I \text{ is an independent set of } G\}.$$

In a graph, a dominating set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. The domination number of G is the number of vertices in a smallest dominating set for G, and it is denoted by $\gamma(G)$, so dominating number of G is

$$\gamma(G) = \min\{|D| \mid D \text{ is a dominating set of } G\}.$$

The open neighbourhood of a vertex $v \in V(G)$ is the set of vertices which are adjacent to v and it is denoted by $N_G(v)$, the closed neighbourhood of v is $N_G(v) \cup \{v\}$ and it is denoted by $N_G[v]$. It is clear that $|N_G[v] \cap D| \geq 1$ for each dominating set D, and for each $v \in V(G)$.

In this section we mainly deal with some graph properties of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$ namely diameter, radius, girth, domination number and independence number.

We use the notation $\overline{A} = (X_i \setminus A)$ for all $A \subseteq X_i$ $(i = 1, 2)$, and we use the notation $d(u, v)$ instead of $d_{\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})}(u, v)$. Moreover for convenience we use $\Gamma_1 \otimes \Gamma_2$ instead of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$. Notice that, for $u = (A_1, B_1), v = (A_2, B_2) \in V(\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2}))$ there exists a single edge $u - v$ in $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$ if and only if $A_1 \supseteq \overline{A_2}$ and $B_1 \supseteq \overline{B_2}$.

Theorem 2.1

(*i*) *If* $|X_1| \geq 3$ *and* $|X_2| \geq 3$ *then* $diam(\Gamma_1 \otimes \Gamma_2) = 4$.

(*ii*) *If* $|X_1| = 2$ *and* $|X_2| \geq 3$ *then* $diam(\Gamma_1 \otimes \Gamma_2) = 5$.

Proof. (i) Let $|X_1| \geq 3$, $|X_2| \geq 3$ and $u = (A_1, B_1), v = (A_2, B_2) \in V(\Gamma_1 \otimes \Gamma_2)$. If $A_1 \cup A_2 = X_1$ and $B_1 \cup B_2 = X_2$ then $d(u, v) = 1$. It is clear that in other cases $d(u, v) \geq 2$. Second case is $A_1 \cup A_2 \neq X_1$ and $B_1 \cup B_2 \neq X_2$. In second case if $A_1 \cap A_2 \neq \varnothing$ and $B_1 \cap B_2 \neq \varnothing$, we take $C_1 = \overline{A_1} \cup \overline{A_2}$, $C_2 = \overline{B_1} \cup \overline{B_2}$ thus we have a path $(A_1, B_1) - (C_1, C_2) - (A_2, B_2)$ and $d(u, v) = 2$. In second case let $A_1 \cap A_2 \neq \varnothing$ and $B_1 \cap B_2 = \varnothing$. In this case since $\overline{B_1} \cup \overline{B_2} = X_2$ thus $d(u, v) \neq 2$. If $A_1 \neq A_2$ then $A_1 \setminus A_2 \neq \varnothing$ or $A_2 \setminus A_1 \neq \varnothing$. If $A_1 \setminus A_2 \neq \varnothing$ then we have a path $(A_1, B_1) - (\overline{A_1} \cup A_2, \overline{B_1}) - (\overline{A_2}, \overline{B_2}) - (A_2, B_2)$ and if $A_2 \setminus A_1 \neq \varnothing$ then we have a path $(A_1, B_1) - (\overline{A_1}, \overline{B_1}) - (A_1 \cup \overline{A_2}, \overline{B_2}) - (A_2, B_2)$ so $d(u, v) = 3$. If $A_1 = A_2$ and $|A_1| \geq 2$ we take any 2−partition of A_1, say C and D. We have a path $(A_1, B_1) - (\overline{A_1} \cup C, \overline{B_1}) - (\overline{A_1} \cup D, \overline{B_2}) - (A_2, B_2)$ so $d(u, v) = 3$. If $A_1 = A_2$ and $|A_1| = 1$ then we have a path $(A_1, B_1) - (\overline{A_1}, \overline{B_1}) - (A_1, B_1 \cup B_2) - (\overline{A_1}, \overline{B_2}) - (A_2, B_2)$ so $d(u, v) \leq 4$. Also (A_1, B) has adjacent form of $(\overline{A_1}, C)$ where $\overline{C} \subseteq B$ so $d(u, v) \neq 3$ then $d(u, v) = 4$. In second case if $A_1 \cap A_2 = \varnothing$ and $B_1 \cap B_2 \neq \varnothing$ is similar. In second case if $A_1 \cap A_2 = \varnothing$ and $B_1 \cap B_2 = \varnothing$ then we have a path $(A_1, B_1) - (\overline{A_1}, \overline{B_1}) - (\overline{A_2}, \overline{B_2}) - (A_2, B_2)$ and since $\overline{A_1} \cup \overline{A_2} = X_1$ and $\overline{B_1} \cup \overline{B_2} = X_2$ so $d(u, v) = 3$. Third case is $A_1 \cup A_2 = X_1$ and $B_1 \cup B_2 \neq X_2$. In third case if $A_1 \cap A_2 \neq \varnothing$ and $B_1 \cap B_2 \neq \varnothing$ then we have a path $(A_1, B_1) - (\overline{A_1} \cup \overline{A_2}, \overline{B_1} \cup \overline{B_2}) - (A_2, B_2)$, so $d(u, v) = 2$. If $A_1 \cap A_2 \neq \varnothing$ and $B_1 \cap B_2 = \varnothing$ then we have a path $(A_1, B_1) - (A_2, \overline{B_1}) - (A_1, \overline{B_2}) - (A_2, B_2)$ and since $\overline{B_1} \cup \overline{B_2} = X_2$ so $d(u, v) = 3$. Let $A_1 \cap A_2 = \varnothing$ and $B_1 \cap B_2 \neq \varnothing$, in this case since $\overline{A_1} \cup \overline{A_2} = X_1$ thus $d(u, v) \geq 3$. If $B_1 \neq B_2$, so $B_1 \setminus B_2 \neq \varnothing$ or $B_2 \setminus B_1 \neq \varnothing$. If $B_1 \setminus B_2 \neq \varnothing$ then we have a path $(A_1, B_1) - (A_2, \overline{B_1} \cup B_2) - (A_1, \overline{B_2}) - (A_2, B_2)$ and if $B_2 \setminus B_1 \neq \varnothing$ then we have a path $(A_1, B_1) - (A_2, \overline{B_1}) - (A_1, B_1 \cup \overline{B_2}) - (A_2, B_2)$, so $d(u, v) = 3$. If $B_1 = B_2$ and $|B_1| \geq 2$, we take 2−partition of B_1, we say E and F, then we have a path $(A_1, B_1) - (A_2, \overline{B_1} \cup E) - (A_1, \overline{B_1} \cup F) - (A_2, B_2)$ so $d(u, v) = 3$. If $B_1 = B_2$ and $|B_1| = 1$ in this case $|A_1| \neq 1$ or $|A_2| \neq 1$ since $A_1 \cup A_2 = X_1$ and $|X_1| \geq 3$, if $|A_1| \neq 1$ then there exists $\varnothing \neq C \subsetneq A_1$, and we have path $(A_1, B_1) - (A_2 \cup C, \overline{B_1}) - (\overline{C}, B_1) - (A_1, \overline{B_1}) - (A_2, B_2)$ and adjacent of (A, B_1) is $(D, \overline{B_1})$ where $\overline{D} \subseteq A$ so $d(u, v) = 4$ and if $|A_2| \neq 1$ is similar. In third case if $A_1 \cap A_2 = \varnothing$ and $B_1 \cap B_2 = \varnothing$ then we have a path $(A_1, B_1) - (A_2, \overline{B_1}) - (A_1, \overline{B_2}) - (A_2, B_2)$ so $d(u, v) = 3$. Last case is $A_1 \cup A_2 \neq X_1$ and $B_1 \cup B_2 = X_2$ is similar with third case. Thus if $|X_1| \geq 3$ and $|X_2| \geq 3$ then $diam(\Gamma_1 \otimes \Gamma_2) = 4$.

(ii) Let $|X_1| = 2$, $|X_2| \geq 3$ and $u = (A_1, B_1), v = (A_2, B_2) \in V(\Gamma_1 \otimes \Gamma_2)$. In here different case is $A_1 \cup A_2 = X_1$ and $B_1 \cup B_2 \neq X_2$ with $B_1 = B_2$ and $|B_1| = 1$, in other cases we have same results with i). This case we take 2−partition of $\overline{B_1}$, we say M and N. We have a path $(A_1, B_1) - (A_2, \overline{B_1}) - (A_1, B_1 \cup M) - (A_2, B_1 \cup N) - (A_1, \overline{B_1}) - (A_2, B_2)$ so $d(u, v) \leq 5$. (A_1, B_1) has only one adjacent and it is $(A_2, \overline{B_1})$ and (A_2, B_1) has only one adjacent and it is $(A_1, \overline{B_1})$ and they are different vertices and they are not adjacent, moreover $d((A_2, \overline{B_1}), (A_1, \overline{B_1})) = 3$, thus $d(u, v) = 5$. So if $|X_1| = 2$ and $|X_2| \geq 3$ then $diam(\Gamma_1 \otimes \Gamma_2) = 5$.

Theorem 2.2

(i) If $|X_1| \geq 3$ and $|X_2| \geq 3$ then $gr(\Gamma_1 \otimes \Gamma_2) = 3$.

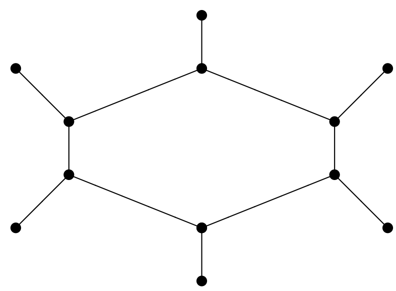
(ii) If $|X_1| = 2, |X_2| = 3$ then $gr(\Gamma_1 \otimes \Gamma_2) = 6$ and if $|X_1| = 2, |X_2| \geq 4$ then $gr(\Gamma_1 \otimes \Gamma_2) = 4$.

(iii) If $|X_1| \geq 3$ and $|X_2| \geq 3$ then $rad(\Gamma_1 \otimes \Gamma_2) = 3$.

(iv) If $|X_1| = 2$ and $|X_2| \geq 3$ then $rad(\Gamma_1 \otimes \Gamma_2) = 4$.

Proof. (i) Let $|X_1| \geq 3$, $|X_2| \geq 3$ and $(A, B) \in V(\Gamma_1 \otimes \Gamma_2)$. Assume that $|A| \geq 2$, $|B| \geq 2$ so there exists 2−partition of A, we say A_1 and A_2 and there exists 2−partition of B, we say B_1 and B_2. Thus $(A, B)-(\overline{A}\cup A_1, \overline{B}\cup B_1)-(\overline{A}\cup A_2, \overline{B}\cup B_2)-(A, B)$ is a cycle. Let $|A| = 1, |B| \geq 2$ then there exists $\varnothing \neq C \subsetneq \overline{A}$ so we have a cycle $(A, B)-(\overline{A}, \overline{B}\cup B_1)-(A\cup C, B)-(\overline{A}, \overline{B}\cup B_2)-(A, B)$. If $|A| \geq 2, |B| = 1$, we can find a cycle similar way. Moreover $\Gamma_1\otimes\Gamma_2$ is simple graph and from its definition $gr(\Gamma_1\otimes\Gamma_2) = 3$.
(ii)

η :



If $|X_1| = 2, |X_2| = 3$ then $\Gamma_1 \otimes \Gamma_2$ is η thus in this case $gr(\Gamma_1 \otimes \Gamma_2) = 6$. If $|X_1| = 2, |X_2| = m \geq 4$, $gr(\Gamma_1 \otimes \Gamma_2)$ can not be odd number since $|X_1| = 2$. So $gr(\Gamma_1 \otimes \Gamma_2) \geq 4$. Let $(A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, if $2 \leq |B| \leq m - 2$ then there exists $k \in \overline{B}$ and 2−partition of B is E and F. So $(A, B) - (\overline{A}, \overline{B} \cup E) - (A, \{k\}) - (\overline{A}, \overline{B} \cup F) - (A, B)$ is a cycle. If $|B| = m - 1$ then without loss of generality we assume that $y_1, y_2 \in B$ then $(A, B) - (\overline{A}, \overline{B} \cup \{y_1, y_2\}) - (A, B \setminus \{y_1\}) - (\overline{A}, \overline{B} \cup \{y_1\}) - (A, B)$ is a cycle. Thus if $|X_1| = 2, |X_2| \geq 4$ then $gr(\Gamma_1 \otimes \Gamma_2) = 4$.

(iii) Let $|X_1| \geq 3$, $|X_2| \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, we can determine $ecc(v)$. If $|A| \geq 2$, $|B| \geq 2$ then $ecc(v) \leq 3$ from proof of Theorem 2.1 (i) because let $u \in V(\Gamma_1 \otimes \Gamma_2)$, we found that if $|A| \geq 2$, $|B| \geq 2$ then $d(u, v) \leq 3$, so $ecc(v) \leq 3$. Moreover there exists $\varnothing \neq C \subsetneq B$, if we choose $u =(\overline{A}, C)$ so $d(u, v) = 3$ it follows that $ecc(v) = 3$. If $|A| = 1$ we choose $u = (A, \overline{B})$ and $d(u, v) = 4$. If $|B| = 1$ is similar. So $rad(\Gamma_1 \otimes \Gamma_2) = 3$.

(iv) Let $|X_1| = 2$, $|X_2| \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, we can determine $ecc(v)$. If $|A| = |B| = 1$ then $ecc(v) = 5$. If $|B| \geq 2$ it is clear that $ecc(v) \leq 4$, if we choose $u = (A, \overline{B})$ then $d(u, v) = 4$, so $ecc(v) = 4$. Thus $rad(\Gamma_1 \otimes \Gamma_2) = 4$.

Theorem 2.3 *If* $|X_1| = n \geq 2$*,* $|X_2| = m \geq 3$ *then* $\gamma(\Gamma_1 \otimes \Gamma_2) = n.m$

Proof. Let $|X_1| = n \geq 2$, $|X_2| = m \geq 3$ and $v = (A, B) \in V(\Gamma_1 \otimes \Gamma_2)$, D be a dominating set for $\Gamma_1 \otimes \Gamma_2$. It is clear that $\deg_{(\Gamma_1\otimes\Gamma_2)}(v) = \deg_{\Gamma_1}(A).\deg_{\Gamma_2}(B)$ so if $|A| = |B| = 1$ then $\deg_{(\Gamma_1\otimes\Gamma_2)}(v) = 1$ and adjacent of v is only $u = (\overline{A}, \overline{B})$ so $N_{(\Gamma_1\otimes\Gamma_2)}[v] = \{u, v\}$. Since $|N_{(\Gamma_1\otimes\Gamma_2)}[\{v\}] \cap D| \geq 1$, either $v \in D$ or $u \in D$. Let $x_i, x_k \in X_1$ and $y_j, y_l \in X_2$. Moreover since $|X_2| \geq 3$ if $(\{x_i\}, \{y_j\}) \neq (\{x_k\}, \{y_l\})$ then

$$N_{(\Gamma_1\otimes\Gamma_2)}[(\{x_i\}, \{y_j\})] \cap N_{(\Gamma_1\otimes\Gamma_2)}[(\{x_k\}, \{y_l\})] = \varnothing.$$

Thus $\gamma(\Gamma_1 \otimes \Gamma_2) \geq n.m$. If we choose $D = \{u = (A, B); u \in V(\Gamma_1 \otimes \Gamma_2)$ and $|A| = n - 1, |B| = m - 1\}$, it is easy to see that $|D| = n.m$ and D is a dominating set for $\Gamma_1 \otimes \Gamma_2$. It follows that $\gamma(\Gamma_1 \otimes \Gamma_2) = n.m$

Theorem 2.4 *If* $|X_1| = n \geq 2$*,* $|X_2| = m \geq 3$ *then*

$$\alpha(\Gamma_1 \otimes \Gamma_2) = \frac{(2^n - 2)(2^m - 2)}{2}$$

Proof. Let $x_i \in X_1$ and $C = \{(A_i, B_j) : \varnothing \neq A_i \subseteq X_1 \setminus \{x_i\}, B_j \in V(\Gamma_2)\}$ and $D = \{(X_1 \setminus A_i, X_2 \setminus B_j) : (A_i, B_j) \in C\}$. It is clear that $C \cap D = \varnothing$ and $|C| = |D| = \frac{(2^n-2)(2^m-2)}{2}$ thus $C \cup D = V(\Gamma_1 \otimes \Gamma_2)$. Let I be an independence set of $\Gamma_1 \otimes \Gamma_2$, then from the pigeonhole principle $|I| \leq \frac{(2^n-2)(2^m-2)}{2}$, and C is an independence set for $\Gamma_1 \otimes \Gamma_2$, moreover $|C| = \frac{(2^n-2)(2^m-2)}{2}$.

3. Perfectness of $\Gamma(SL_{X_1}) \otimes \Gamma(SL_{X_2})$

Let G be a graph. Clique is the each of the maximal complete subgraphs of G. The number of all the vertices in any clique of G is clique number and it is denoted by $\omega(G)$. The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color and it is denoted by $\chi(G)$. It is well-known that

$$\chi(G) \geq \omega(G) \tag{1}$$

for any graph G (Chartrand & Zhang, 2009). Let $V' \subseteq V(G)$, then induced subgraph $G' = (V', E')$ is a subgraph of G such that E' consists of those edges whose endpoints are in V'. For each induced subgraph H of G, if $\chi(H) = \omega(H)$, then G is called a perfect graph.

The complement or inverse of a graph G is a graph on the same vertices such that two distinct vertices are adjacent if and only if they are not adjacent in G, the complement of G is denoted by G^c.

A graph G is called Berge if no induced subgraph of G is an odd cycle of length at least five or the complement of one.

The edges are called adjacent if they share a common end vertex. An edge coloring of a graph is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. The minimum required number of colours for and edge colouring of G is called the chromatic index of G and it is denoted by $\chi'(G)$. Vizing gave a fundamental theorem for that, for any graph G, we have

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

(Vizing, 1964). Graph G is called class-1 if $\Delta(G) = \chi'(G)$ and called class-2 if $\chi'(G) = \Delta(G) + 1$.

The core of a graph G is defined to be the largest induced subgraph of G such that each edge in core is part of a cycle and it is denoted by G_Δ. Finally, let M be a subset of $E(G)$ for a graph G, if there is no two edges in M which are adjacent then M is called a matching.

Conjecture 3.1 *Let G and H be graphs then* $\chi(G \otimes H) = min\{\chi(G), \chi(H)\}$. (Hedetniemi, 1966)

Theorem 3.2 *If* $|X_1| = n \geq 2$ *and* $|X_2| = m \geq 3$ *then*

$$\omega(\Gamma_1 \otimes \Gamma_2) = \chi(\Gamma_1 \otimes \Gamma_2) = \min\{n, m\}.$$

Proof. Let $X_1 = \{x_1, \ldots, x_n\}$ and $X_2 = \{y_1, \ldots, y_m\}$ with $n \geq 2$, $m \geq 3$. We assume that K is one the maximal complete subgraph of $\Gamma_1 \otimes \Gamma_2$ and

$$V(K) = \{(A_{i_1}, B_{j_1}), (A_{i_2}, B_{j_2}), \ldots, (A_{i_k}, B_{j_k})\}.$$

It is clear that for $1 \leq p \neq q \leq k$, $A_{i_p} \neq A_{i_q}$, $B_{j_p} \neq B_{j_q}$ and graph of spanned by the vertices $V_1 = \{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ is complete subgraph of Γ_1 and graph of spanned by the vertices $V_2 = \{B_{i_1}, B_{i_2}, \ldots, B_{i_k}\}$ is complete subgraph of Γ_2. It follows that $k \leq \min\{n, m\}$ from (Toker, 2016). Assume that $m \geq n$ so if we choose

$$V(\Pi) = \{((x_2, x_3, \ldots, x_n), (y_2, y_3, \ldots, y_m)), ((x_1, x_3, \ldots, x_n),$$

$$(y_1, y_3, \ldots, y_m)), \ldots, ((x_1, x_2, x_3, \ldots, x_{n-1}), (y_1, y_2, \ldots, y_{n-1}, y_{n+1}, \ldots y_m))\}$$

and if $n \geq m$ is similar. Let Π be spanned graph by $V(\Pi)$. It is easy to see that Π is $\min\{n, m\}$ vertices complete subgraph of $\Gamma_1 \otimes \Gamma_2$ and so $\omega(\Gamma_1 \otimes \Gamma_2) = \min\{n, m\}$.

Thus $\chi(\Gamma_1 \otimes \Gamma_2) \geq \min\{n, m\}$ from equation (1). Assume that $m \geq n$ and $\forall 1 \leq i \leq n$; $A_i = X_1 \setminus \{i\}$. In addition let

$$\begin{aligned}
Q_1 &= \{(B, v) \mid \varnothing \neq B \subseteq A_1 \text{ and } v \in V(\Gamma_2)\}, \\
Q_2 &= \{(B, v) \mid \varnothing \neq B \subseteq A_2 \text{ and } (B, v) \notin Q_1 \text{ and } v \in V(\Gamma_2)\}, \\
&\vdots \\
Q_n &= \{(B, v) \mid \varnothing \neq B \subseteq A_n \text{ and } (B, v) \notin \bigcup_{i=1}^{n-1} Q_i \text{ and } v \in V(\Gamma_2)\}.
\end{aligned}$$

It is clear if $(u, v) \in V(\Gamma_1 \otimes \Gamma_2)$ then $(u, v) \in Q_s$ $(1 \leq s \leq n)$ for unique s, moreover $\bigcup_{i=1}^{n} Q_i = V(\Gamma_1 \otimes \Gamma_2)$ and $Q_i \cap Q_j = \varnothing$ for $1 \leq i \neq j \leq n$. For each $1 \leq k \leq n$ if we choose a different colour for each Q_k and assign the chosen colour to the all vertices in Q_k, there is no two adjacent vertices have same colour, and so $\chi(\Gamma_1 \otimes \Gamma_2) \leq n$. Thus $\chi(\Gamma_1 \otimes \Gamma_2) = \min\{n, m\}$. If $n \geq m$ is similar. So conjecture holds for $\Gamma_1 \otimes \Gamma_2$.

Lemma 3.3 *A graph is perfect if and only if it is Berge* (Chudnovsky et all., 2006).

Therefore, a graph G is perfect if and only if neither G nor G^c contains an odd cycle of length at least 5 as an induced subgraph.

Theorem 3.4 *If* $|X_1| = 2, |X_2| \geq 3$ *then* $\Gamma_1 \otimes \Gamma_2$ *is perfect graph but if* $|X_1| \geq 3, |X_2| \geq 3$ *then* $\Gamma_1 \otimes \Gamma_2$ *is not perfect graph.*

Proof. We assume that $X_1 = \{x_1, \ldots, x_n\}$ and $X_2 = \{y_1, \ldots, y_m\}$. Let $|X_1| = 2$ and $|X_2| \geq 3$. It is clear that there is not any odd cycle at least 5 as induced subgraph of $\Gamma_1 \otimes \Gamma_2$ since $|X_1| = 2$. Let $G = (\Gamma_1 \otimes \Gamma_2)^c$. For $k \geq 3$ we assume that there is an induced subgraph of G which is cycle with $2k - 1$ vertices, say

$$C_1 - C_2 - \cdots - C_{2k-1} - C_1.$$

Without loss of generality assume that $C_1 = (x_1, B_1)$. Then $C_3 = (x_2, B_3)$ and $C_{2k-2} = (x_2, B_{2k-2})$ because C_1 and C_3 adjacent vertices in $\Gamma_1 \otimes \Gamma_2$ and C_1 and C_{2k-2} adjacent vertices in $\Gamma_1 \otimes \Gamma_2$. Thus $C_2 = (x_1, B_2)$ and $C_{2k-1} = (x_1, B_{2k-1})$ because C_2 and C_{2k-2} adjacent vertices in $\Gamma_1 \otimes \Gamma_2$ and C_3 and C_{2k-1} adjacent vertices in $\Gamma_1 \otimes \Gamma_2$. But in this case C_2 and C_{2k-1} adjacent vertices in G which is a contradiction. So if $|X_1| = 2, |X_2| \geq 3$ then $\Gamma_1 \otimes \Gamma_2$ is perfect graph. Let $|X_1| \geq 3, |X_2| \geq 3$ and $Y_1 = X_1 \setminus \{x_1, x_2, x_3\}$, $Y_2 = X_2 \setminus \{y_1, y_2, y_3\}$. Let $H = (\{x_1, x_2\} \cup Y_1, \{y_1, y_2\} \cup Y_2) - (\{x_3\} \cup Y_1, \{y_2, y_3\} \cup Y_2) - (\{x_1, x_2\} \cup Y_1, \{y_1, y_3\} \cup Y_2) - (\{x_1, x_3\} \cup Y_1, \{y_2\} \cup Y_2) - (\{x_2, x_3\} \cup Y_1, \{y_1, y_3\} \cup Y_2) - (\{x_1, x_2\} \cup Y_1, \{y_1, y_2\} \cup Y_2)$. So H is cycle of length of 5 which subinduced graph of $\Gamma_1 \otimes \Gamma_2$. Thus if $|X_1| \geq 3, |X_2| \geq 3$ then $\Gamma_1 \otimes \Gamma_2$ is not perfect graph.

Lemma 3.5 *Consider the graphs* $G_1 = (V, E_1)$ *and* $G_2 = (V, E_2)$ *with the same vertex set. Suppose that* E_1 *is a matching such that no edge has both endvertices in* $N_{G_2}[\Lambda_{G_2}]$*. If the union graph* $G = G_1 \cup G_2$ *has maximum degree* $\Delta(G) = \Delta(G_2) + 1$ *then* G *is class*-1 (Machado & Figueiredo, 2010).

Theorem 3.6 *If* $|X_1| = n \geq 2$, $|X_2| = m \geq 3$ *then*

$$\chi'(\Gamma_1 \otimes \Gamma_2) = \chi'(\Gamma_1).\chi'(\Gamma_2) = (2^{n-1} - 1).(2^{m-1} - 1)$$

Proof. Let $(u, v) \in V(\Gamma_1 \otimes \Gamma_2)$, if $|u| \geq 2, |v| \geq 2$ or if $|u| = 1, |v| \geq 2$ or $|u| \geq 2, |v| = 1$ then (u, v) is in core at graph from Theorem 2.2. Let $B = \{(\{x_i\}, \{y_j\}) - (X_1 \setminus \{x_i\}, X_2 \setminus \{y_j\}) : x_i \in X_1, y_j \in X_2\}$, $G_1 = (V(\Gamma_1 \otimes \Gamma_2), B))$ and $G_2 = (V(\Gamma_1 \otimes \Gamma_2), E(\Gamma_1 \otimes \Gamma_2)_\Delta)$. So B is a matching such that no edge has both endvertices in $N_{G_2}[\Lambda_{G_2}]$. Also $(\Gamma_1 \otimes \Gamma_2) = G_1 \cup G_2$ and $\Delta(\Gamma_1 \otimes \Gamma_2) = \Delta(G_2) + 1$ so from Lemma 3.5, $\Gamma_1 \otimes \Gamma_2$ is class-1.

References

Chartrand, G., & Zhang, P. (2009). *Chromatic Graph Theory*. Charpman & Hall/CRC, London.

Chudnovsky, M., Robertson, N., Seymour, P., & Thomas, R. (2006). The strong perfect graph theorem. *Ann. Math.*, *164*, 51-229. http://dx.doi.org/10.4007/annals.2006.164.51

DeMeyer, F., & DeMeyer, L. (2005). Zero divisor graphs of semigroups. *J. Algebra*, *283*, 190-198. http://dx.doi.org/10.1016/j.jalgebra.2004.08.028

DeMeyer, F., McKenzie, T., & Schneider., K. (2002). The zero-divisor graph of a commutative semigroup. *Semigroup Forum*, *65*, 206-214. http://dx.doi.org/10.1007/s002330010128

Gross, J. L., & Yellen, J. (2004). *Handbook of Graph Theory*. Charpman & Hall/CRC, London.

Howie, J. M. (1995). *Fundamentals of Semigroup Theory*. Oxford University Press, New York.

Hedetniemi, S. (1966). *Homomorphisms of graphs and automata*. Technical Report, 03105-44-T, University of Michigan.

Machado£ R. C. S., & Figueiredo de, C. M. H. (2010). Decompositions for edge-coloring join graphs and cobipartite graphs. *Discrete Applied Mathematics*, *158*, 1336-1342. http://dx.doi.org/10.1016/j.dam.2009.01.009

Toker, K. (2016). On the zero-divisor graphs of finite free semilattices. *Turkish Journal of Mathematics*, *40*(4), 824-831. http://dx.doi.org/10.3906/mat-1508-38

Vizing, V. G. (1964). On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.*, *3*, 25-30.

Weichsel, P. M. (1962). The Kronecker product of graphs. *Proc. Amer. Math. Soc.*, *13*, 47-52. http://dx.doi.org/10.2307/2033769

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