Absolute Valued Algebras with Strongly One Sided Unit

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Received: October 30, 2016 Accepted: December 5, 2016 Online Published: December 30, 2016

doi:10.5539/jmr.v9n1p32 URL: http://dx.doi.org/10.5539/jmr.v9n1p32

Abstract

We classify the absolute valued algebras with strongly left unit of dimension $\leq 4$. Also we prove that every 8-dimensional absolute valued algebra with strongly left unit contain a 4-dimensional subalgebra, next we determine the form of theirs algebras by the duplication process.

Keywords: absolute valued algebra, strongly left unit, duplication process

Mathematics Subject Classification: 17A35, 17A36

1. Introduction

The absolute valued algebras are introduced by Ostrowski 1918. It’s the normed algebra $A$ such that $||xy|| = ||x|| ||y||$ for all $x, y$ in $A$. For an element $a$ in an algebra $A$, we denote by $L_a : A \rightarrow A x \mapsto ax$ and $R_a : A \rightarrow A x \mapsto xa$. The algebra is called division if and only if $R_e$ and $L_a$ are bijective for all $a$ in $A$. We denote by $O$ the orthogonal group of linear isometries of Euclidean space $H$. We recall $O^*$ the subgroup of proper linear isometries and $O^-$ the subset of improper linear isometries. Let $A$ be an absolute valued algebra with unit, then $A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (Urbanik & Wright, 1960). The absolute valued algebras with left unit satisfying to $(x^2, x^2, x^2) = 0$, for all $x \in A$ is classified in (Diankha & all, 2013$_2$). These algebras are finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, *\mathbb{C}, H, *H, *H(i, 1), O, *O, *O(i, 1), O$ or $O(i)$ and the element $e$ satisfy to $L_e = R_e^2 = I_A$. The algebras $\mathbb{H}_i, O_i$ (Diankha & all, 2013$_1$), satisfy to $L_e = R_e^2 = I_A$ and not satisfy to $(x^2, x^2, x^2) = 0$. In this paper we give a classification of the absolute valued algebras with strongly left unit of dimension $\leq 4$. We proves that if $A$ is 8-dimensional absolute valued algebra with strongly left unit, then $A$ contain a 4-dimensional subalgebra and $A$ is obtained by the duplication process. Otherwise $A$ is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$ with $\varphi, \psi : \mathbb{H} \rightarrow \mathbb{H}$ are linear isometries such that $\varphi(1) = 1$ and $(\varphi, \psi)^2 = (\varphi, \psi)$. The algebras $\mathbb{R}, \mathbb{C}, *\mathbb{C}, H, *H, *H(i, 1), O, *O, O_i, *O(i, 1), O$ are absolute valued algebras with strongly left unit. This list is completed by new algebras.

2. Preliminary

In this section we recall the some interest results:

Theorem 1 The finite-dimensional absolute valued real algebras with a left unit are precisely those of the form $\mathbb{A}_\varphi$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $\varphi$ is an isometric of the euclidian space $\mathbb{A}$ fixes 1, and $\mathbb{A}_\varphi$ denotes the absolute-valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := \varphi(x)$. Moreover, given linear isometries $\varphi, \psi : \mathbb{A} \rightarrow \mathbb{A}$ fixing 1, the algebras $\mathbb{A}_\varphi$ and $\mathbb{A}_\psi$ are isomorphic if and only if there exists an algebra automorphism $\psi$ of $\mathbb{A}$ satisfying $\phi = \psi \circ \varphi \circ \psi^{-1}$ (Rochdi, 2003).

Lemma 1 Let $A$ be an absolute valued algebra with strongly left unit. The following equalities hold for all $x \in A$.

1. $[(xe)x]e = x(xe)$
2. $[x(xe)]e = (xe)x$
3. $[xe, x] = \langle e, x \rangle [e, x - xe]$
   If, moreover, $x$ is orthogonal to $e$, then
4. $[xe, x] = 0$
5. $(xe)x^2 = 2 \langle e, x^2 \rangle x - ||x||^2 xe$
6. $(xe)^2 = 2 \langle e, x^2 \rangle e - x^2$
7. $x^2 x = -||x||^2 xe$ (Chandik & Rochdi, 2008).
The group $G_2$ acts transitively on the sphere $S(\text{Im}(\mathbb{O})) := S^6$, that is the mapping $G_2 \to S^6 \Phi \mapsto \Phi(i)$ is surjective (Postnikov, 1985).

Let $A$ be one of the unital absolute valued algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ of dimension $m$. Consider the caley dickson product $\odot$ in $\mathbb{A} \times \mathbb{A}$, we define on the space $\mathbb{A} \times \mathbb{A}$ the product

$$(x, y) \star (x', y') = (f_1(x), f(x)) \odot (g_1(x'), g(y')).$$

With $f_1, g_1, f, g$ be linear isometries of $\mathbb{A}$ and $f_1(1) = g_1(1) = 1$. We obtain a $2m$-dimensional absolute valued real algebra $\mathbb{A} \times \mathbb{A}(f_1, g_1)$. The process is called duplication process. Note that the algebra is left unit if $g_1 = g = I_{\mathbb{A}}$ and this case we not the algebra by $\mathbb{A} \times \mathbb{A}(f_1, g_1)$. We have the following result (Calderon & all, 2011):

**Theorem 2** Let $A$ be an $8$-dimensional absolute valued algebra, then the following are equivalent:

1. $A$ contains a $4$-dimensional subalgebra.
2. $A$ is obtained by the duplication process.
3. $\text{Aut}(A)$ contains a reflexion.

**Lemma 2** Let $I^+ = \{f \in O^+ : f \text{ involutive} \}$, $I^- = \{f \in O^- : f \text{ involutive} \}$, $I^+_1 = \{f \in I^+ : f(1) = 1 \}$ and $I^-_1 = \{f \in I^- : f(1) = 1 \}$. We have:

1. $O^+ = \{T_{ab} : a, b \in S(\mathbb{H})\}$
2. $O^- = \{T_{ab} \circ \sigma_{\mathbb{H}} : a, b \in S(\mathbb{H})\} := O^+ \circ \sigma_{\mathbb{H}}$
3. $O^+_1 = \{T_{ab} : a \in S(\mathbb{H})\}$
4. $O^-_1 = \{T_{ab} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\} := O^+_1 \circ \sigma_{\mathbb{H}}$
5. $I^+ = \{\pm I_3\} \cup \{T_{ab} : a, b \in S(\text{Im}(\mathbb{H}))\}$
6. $I^- = \{\pm T_{aa} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\}$
7. $I^+_1 = \{I_4\} \cup \{T_{aa} : a \in S(\text{Im}(\mathbb{H}))\}$
8. $I^-_1 = \{\sigma_{\mathbb{H}}\} \cup \{T_{aa} \circ \sigma_{\mathbb{H}} : a \in S(\text{Im}(\mathbb{H}))\} := I^+_1 \circ \sigma_{\mathbb{H}}$ (Diankha & all, 2013).

**Corollary 1** Let $A$ be an absolute valued algebra with left unit satisfying to $(x^p, x^q, x^r) = 0$ with $\{p, q, r\} \in \{1, 2\}$. Then $A$ contains a strongly left unit.

**Proof.** Lemma 1 (Diankha & all, 2013) and proof of Proposition 4.8 (Chandik & Rochdi, 2008).

The converse of Corollary 1 is false, an effect the algebra $A := \mathbb{O}_i$ is an absolute valued algebra with strongly left unit and $A$ not satisfy to $(x^p, x^q, x^r) = 0$.

3. **Absolute Valued Algebras with Strongly Left Unit**

**Definition 1** An element $e \in A$ is called strongly left unit, if it’s left unit and square root of right unit ($L_e = \mathbb{R}_e^2 = I_{\mathbb{A}}$).

**Theorem 3** Let $A$ be an absolute valued algebra with strongly left unit. Then $A$ is finite dimensional. Moreover if $\text{dim}(A) \leq 4$, then $A$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}(i, 1)$ or $\mathbb{H}(i, 1)$.

**Proof.** The algebra $A$ is left unit, hence $A$ is left division (Rodriguez, 2004). Morover the assertion $\mathbb{R}_e^2 = I_{\mathbb{A}}$ imply that $A$ is right division, then $A$ is finite dimensional. Also $A$ is of the form $\mathbb{A}_e$, with $\varphi$ a linear isometric fixed $1$ and $\mathbb{A}_e \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (Theorem 1). If $\text{dim}(A) \leq 2$, it’s clear that $A$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{C}$.

Assume now $\text{dim}(A) \geq 4$, then the assertion $\mathbb{R}_e^2 = I_{\mathbb{A}}$ imply:

$$x = (x \odot 1) \odot 1 = \varphi^2(x).$$

Then $\varphi$ is an involutive linear isometric $\varphi^2 = I_{\mathbb{A}}$.

If $\text{dim}(A) = 4$, we have:

$$\varphi \in I^+_1 \cup I^-_1 = \{I_4\} \cup \{T_{aa} : a \in S(\text{Im}(\mathbb{H}))\} \cup \{\sigma_{\mathbb{H}}\} \cup \{T_{aa} \circ \sigma_{\mathbb{H}} : a \in S(\text{Im}(\mathbb{H}))\}$$

(Lemma 2).
Let $A$ be an 8-dimensional absolute valued algebra with strongly left unit. Then $A$ contains a four-dimensional subalgebra.

If $dim(A) = 4$, the last result can be obtained so by using the identity $R^2_i = I_3$ and the principal isotopes of $H$: $H(a, 1), \ast H(a, 1)$, where $a \in S(H)$. For the first isotope $e = \overline{a}$, and for the second isotope $e = a$.

For all alternative algebra $A$, Artin’s theorem (Schafer, 1996) shows that for any $x, y \in A$, the set $\{x, y, \overline{x}, \overline{y}\}$ is contained in an associative subalgebra of $A$. We note by $T(x) = x + \overline{x}$ the face of $x \in A$ and we have $x^2 - T(x)x + \|x\|^2e = 0$ for all $x \in A$. As $A$ is real alternative quadratic algebra, we have $A = \mathbb{R} \oplus Im(A)$ (Frobenius decomposition) and their exist a unique linear form $\lambda : A \rightarrow \mathbb{R}$ such that $\lambda(1) = 1$, $ker(\lambda) = Im(A)$ and $< x, y > = \lambda(\overline{x}y) = \lambda(\overline{y}x)$ for all $x, y \in A$ (Koecher & Remmert, 1991). Otherwise for all $x, y \in Im(A)$ we have $xy + yx = -2 < x, y > e$ $(\ast)$ and the identity $xy - 2 \lambda(xy)x - \|x\|^2\overline{y}$ for all $x, y \in Im(A)$ is called the triple product identity (TPI).

In 8-dimensional, by the duplication process we recover theirs algebras.

**Theorem 4** Let $A$ be an 8-dimensional absolute valued algebra with strongly left unit. Then $A$ contains a four-dimensional subalgebra.

**Proof.** We have $O = \mathbb{R} \oplus Im(O)$ and their exist a unique linear form $\lambda : O \rightarrow \mathbb{R}$ such that $\lambda(1) = 1$ and $ker(\lambda) = Im(O)$.

Let $u = e^1, \omega = 0 = \omega < 1, u = \omega < 1, \omega \phi^2(1), \omega^0(u) = < 1, \omega \phi^2(u) >$. Then we have $\omega^0(1^2) = 1^2$, for all $n \in \mathbb{N}$. The algebra $A$ is of the form $O_8$ with $\Phi(1) = 1$ and $\Phi^2 = I_2$ (Theorem 1 and Theorem 3). Otherwise we have $i \circ i = \Phi(i)i$ and $i \circ 1 = \Phi(i)$. Using the equality $(\ast)$ we have $i\Phi(i) + \Phi(i)i = -2 < i, \Phi(i) > 1$. Also using Lemma 1 ($\ast$), we have $\Phi[\Phi(i)i] = \Phi(i)i$. Using the TPI we have

$$\Phi(i)i\Phi(i) = 2\lambda[\Phi(i)i][\Phi(i) + i = -2 < i, \Phi(i) > \Phi(i) + i. \ \ (\ast)$$

and

$$i\Phi(i)i = 2\lambda[\Phi(i)i] + \Phi(i) = -2 < i, \Phi(i) > i + \Phi(i). \ \ (\ast)$$

Hence we have the products,

$$\Phi(i) \circ 1 = \Phi^2(i) = i.$$  

$$\Phi(i) \circ i = \Phi^2(i)i = -1.$$  

$$\Phi(i) \circ \Phi(i) = \Phi^2(i)\Phi(i) = i\Phi(i) = -2 < i, \Phi(i) > 1 - \Phi(i)i. \ \ (\ast)$$

$$\Phi(i) \circ \Phi(i)i = \Phi^2(i)\Phi(i)i = i\Phi(i)i = -2 < i, \Phi(i) > i + \Phi(i). \ \ (\ast)$$

$$\Phi(i)i \circ 1 = \Phi[\Phi(i)i] = \Phi(i)i.$$  

$$\Phi(i)i \circ i = \Phi[\Phi(i)i]i = \Phi(i)i^2 = -\Phi(i).$$  

$$\Phi(i)i \circ \Phi(i) = \Phi[\Phi(i)i][\Phi(i) = \Phi(i)\Phi(i) = -2 < i, \Phi(i) > \Phi(i) + i. \ \ (\ast)$$

$$\Phi(i)i \circ \Phi(i)i = \Phi[\Phi(i)i][\Phi(i)i = (\Phi(i)i)^2 = T[\Phi(i)i][\Phi(i)i] = 1.$$
Then the algebra $A$ contains a four-dimensional sub-algebra. 

**Theorem 5** Let $A$ be an 8-dimensional absolute valued algebra with strongly left unit. Then $A$ is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi,\psi)}$ where $(\varphi,\psi)$ are linear isometries of $\mathbb{H}$ belong to $S_1 \cup S_2 \cup S_3 \cup S_4$ with:

$$
S_1 = \{ I_H \} \times \{ \pm I_H, T_{a,b} : a,b \in S^2, \pm T_{c,d} \circ \sigma_H : c \in S^3 \} \\
S_2 = \{ \sigma_H \} \times \{ \pm I_H, T_{a,b} : a,b \in S^2, \pm T_{c,d} \circ \sigma_H : c \in S^3 \} \\
S_3 = \{ T_{a,b} : a \in S^2 \} \times \{ \pm I_H, T_{b,c} : b,c \in S^2, \pm T_{d,e} \circ \sigma_H : d \in S^3 \} \\
S_4 = \{ T_{a,b} \circ \sigma_H : a \in S^2 \} \times \{ \pm I_H, T_{b,c} : b,c \in S^2, \pm T_{d,e} \circ \sigma_H : d \in S^3 \}.
$$

**Proof.** Using the Theorem 2 and Theorem 4, the algebra $A$ is obtained by the duplication process. It’s clear that the algebra $A$ is of the form $\mathbb{H} \times \mathbb{H}_{(\varphi,\psi)}$, with $\varphi(1) = 1$. The linear isometric $(\varphi,\psi)$ is involutive, then $\varphi^2 = \psi^2 = I_A$. We have $\varphi \in I_1^+ \cup I_1^-$ and $\psi \in I^+ \cup I^-$. Then the lemma 2 gives the result.

**Problem 1** In dimension 8, it will be interesting to specify these algebras by reducing the isomorphism classes.

**Acknowledgements**

I thank the reviewers for the relevant remarks and suggestions.

**References**


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