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Some Results on FGS-modules

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Abstract

Let R be a commutative ring, with a unity $1 \neq 0$ and M a unitary left R-module. In this paper we give some properties of an FGS-module. After that we give others important characterizations. Indeed, we first show that M is a local FGS-module if and only if it is of finite representation type. Secondly, we show that M is a prime FGS-module if and only if it is a serial type module and of finite length if and only if it is a finite representation type module.
Keywords: Hopfian, finitely generated, finite representation type, local

1. Introduction

Let R be a commutative ring with $1 \neq 0$ as unity and M a left module over R. The category $\sigma[M]$ is a full subcategory of R-Mod. Its objects are all submodules of a M-generated module(Wisbauer, R., 1985). we call that a module M is Hopfian if every epimorphism of M is an automorphism of M. For a commutative ring any finitely generated module is Hopfian but Hopfian module is not always finitely generated(Ba, A. & Diankha, O., 2013). Therefore we characterize the modules for which every Hopfian object of $\sigma[M]$ is finitely generated. These modules are called FGS-modules.
An object N of $\sigma[M]$ is said to be *coherent* if it is finitely generated and every finitely generated submodule of K is finitely presented. If any submodule of a module M is an intersection of maximal submodule then, M is called *co-semismple*. A module M is said *good* if $M/Rad(M)$ is co-semisimple where $Rad(M)$ is the Jacobson radical of M. A module is *uniserial* if its submodules are linearly ordered by inclusion. A module is said *serial(resp. semisimple)* if it is direct sum of uniserial(resp. simple) modules. A module is said *serial type* if every object of $\sigma[M]$ is direct sum of uniserial modules of finite length. A module M is said to be prime module if for any submodule N of M, we have $Ann(N) = Ann(M)$. A module of finite length is *finite representation type* if there exists, in $\sigma[M]$, only many non-isomorphic finitely generated indecomposable modules. A ring R is said to be S-ring if any Hopfian module over R is noetherian.

2. Some Properties of FGS-module

In this part we give some preliminaries results which we will use in this paper.

Proposition 1 *Let R be a commutative ring and M a finitely generated prime FGS-module. Then, M is simple.*

Proof. Since M is finitely generated.(Wisbauer, R., 1991) that $\sigma[M] = R/Ann(M)$-Mod i.e any object of $\sigma[M]$ is a module over $R/Ann(M)$. As M is an FGS-module then $R/Ann(M)$ is also an FGS-ring. It results from(Gueye, C. T. & Sangharé, M.,2004)that $R/Ann(M)$ is an artinian ring. We know that any finitely generated module over an artinian ring is artinian. Hence, M is an artinian module. Therefore, there exists a minimal submodule in M. Let N_1 be that minimal submodule and the following diagram:

$$\begin{tikzcd} g : R \arrow[r] \arrow[d] & N_1 \\ R/Ann(N_1) \arrow[ur, equal] & \end{tikzcd}$$

We can see that $R/Ann(N_1) \simeq N_1$. Hence, $R/Ann(N_1)$ is a field.
Let $g : R \longrightarrow M$ be an epimorphism. Therefore, $M \simeq R/Ann(M)$. As M is a prime module(i.e $Ann(M) = Ann(N_1)$), then M is simple.

Corollary 2 *Let R be a commutative ring and M a finitely generated, faithful and prime FGS-module then, R is a field.*

Proof. We have seen in proposition 1 that $R/Ann(N_1)$ is simple. Since M is prime then, $R/Ann(N_1) = R/Ann(M)$ is simple. As M is faithful then, R is a field.

Proposition 2 *Let R be a commutative ring and M a prime and finitely generated FGS-module. Then:*

(1) *Every submodule of an object N of $\sigma[M]$ is maximal;*
(2) *There exists a finite number of submodules in N.*
Proof. (1) It follows from the proposition 1 that M is simple, hence semisimple. Therefore, every module of $\sigma[M]$ is semisimple (Wisbauer, R., 1991). Let N be an objet of $\sigma[M]$ and $\{K_j\}_J$ a family of submodules of N. Let's assume $L = \bigoplus_{j\in J} N/K_j$. $L \in \sigma[M]$ because $\sigma[M]$ is closed under direct sum. As, any object of $\sigma[M]$ is semisimple then, L is semisimple too. Hence, for every $j \in J$, N/K_j is simple.Thus, for every $j \in J$, K_j is maximal.
(2) Let's suppose $L = \bigoplus_{j\in J} N/K_j$. We have already seen that for any $j \in J$ N/K_j is simple. And it is obvious to see that, for all $j \in J$, N/K_j is Hopfian and fully invariant. Then $L = \bigoplus_{j\in J} N/K_j$ is Hopfian. As M is an FGS-module, then L is finitely generated. Thus, J is finite.

Lemma 1 *If M is a local module then, M is finitely generated.*

Proof. It follows from 21.6 of (Wisbauer, R., 1991) and the definition of local module.

Proposition 3 *Let R be commutative ring and M a local FGS-module. Then, for every object N of $\sigma[M]$, the following statements are equivalent:*
(a) *N is finitely generated;*
(b) *N is noetherian;*
(c) *N is artinian;*
(d) *N is of finite length.*

Proof. Let M be a local module. By lemma 1, M is finitely generated. Hence $\sigma[M] = R/Ann(M)$-Mod i.e every object of $\sigma[M]$ is a $R/Ann(M)$-module. Since, M is an FGS-module then, $R/Ann(M)$ is an FGS-ring. Hence, $R/Ann(M)$ is an artinian ring. It results from 15.21 of (Anderson, F. W. & Fuller, K., 1973) that (a), (b), (c) and (d) are equivalent.

Lemma 2 (Anderson, F.W & Fuller, K., 1973) *R is noetherian iff every finitely generated R-module is finitely presented.*

Proposition 4 *Let M be a local FGS-module. Then, M is a coherent module in $\sigma[M]$.*

Proof. We have already seen that M is finitely generated. Hence, $M \simeq R/Ann(M)$. It results from the proposition 1 that $R/Ann(M)$ is artinian. It is well known that any artinian ring is noetherian, hence $R/Ann(M)$ is a noetherian ring. Let N be a finitely generated submodule of M. N is also module over $R/Ann(M)$. It follows from the lemma 2 that N is finitely presented. Thus, M is coherent.

Proposition 5 *Let M be a local FGS-module, then M is a good module and so is every module of $\sigma[M]$.*

Proof. As M is local then, $M/Rad(M)$ is simple hence semisimple. It is well know that any semisimple module is co-semisimple, hence $M/Rad(M)$ is co-semisimple. By referring to 23.3 of (Wisbauer, R., 1991), M is a good module. Let N be an object of $\sigma[M]$. N is a module over $R/Ann(M)$. We have seen that $M \simeq R/Ann(M)$, hence $R/Ann(M)$ is good ring. It results from 23.7 of (Wisbauer, R., 1991) that N is a good module.

3. Results

Lemma 3 *If M is an FGS-module then, there exists a finite number of non-isomorphic simple modules in $\sigma[M]$.*

Proof. It results from proposition 2 of (Ba, A. & Diankha, O., 2013).

Theorem 1 *Let R be a commutative ring and M a local module then, the following assertions are equivalent:*
(1) *M is an FGS-module;*
(2) *M is of finite representation type.*

Proof. (1) $\Rightarrow$ (2) By the proof of proposition 1 $M \simeq R/Ann(M)$ is artinian. Since M is of finitely generated, then is of finite length. It results from the lemma 3 that M is of finite representation type.
(2) $\Rightarrow$ (1) We have already seen that $M \simeq R/Ann(M)$. As M is a finite representation type, then it is of finite length and it follows from theorem 3.1 of (Diankha, O., 2007) that M is an I-module. Hence $R/Ann(M)$ is an I-ring. It results theorem 9 of (Kaidi, A. & Sanghare, M., 1965) that $R/Ann(M)$ is a S-ring.
Let N be an Hopfian object of $\sigma[M]$. Since $R/Ann(M)$ is S-ring then N is noetherian. Any noetherian module of an artinian ring is finitely generated. Thus M is an FGS-module.

Theorem 2 *Let R be a commutative ring and M a prime module. Then, the following assertions are equivalent:*
(1) *M is an FGS-module;*
(2) *M is a serial type and of finite length;*
(3) *M is of finite representation type.* *Proof.* (1) $\Rightarrow$ (2) Assume that M is an FGS-module. It follows from proposition 1 that M is a simple module. Hence it of finite length and semisimple. Let $N = \oplus_{i\in I} N_i$ be an element of $\sigma[M]$. Since N is a semisimple module, then N_i is a simple module for all $i \in I$. It is uniserial and of finite length. Therefore M is of serial

type.
(2) ⇒ (3) It follows from 55.14 of (Wisbauer, R., 1991) that M is of finite representation type.
(3) ⇒ (1) It results from theorem 1.

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