Plemelj Formula of Cauchy-Type Integral of Random Process with Second Order Moment

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Abstract

Under the condition of arc-wise smooth path of integration, the Plemelj formula of Cauchy-type integral on random process with second order moment is obtained.

Keywords: Cauchy-type integral, Second order moment, Arc-wise smooth path, Plemelj formula

1. Introduction

Let L be a simple, smooth and closed curve. It divides the complex plan into inter domain D^+ and outer domain D^-. (Ω, F, P) is a probability space, $g(\omega, \zeta)$ is a random process with second order moment on (Ω, F, P), which depends on parameters ζ on L. In [Wang, 2004], Wang Chuangrong gave the definition of random cauchy-type integral of $g(\omega, \zeta)$, and proved the existence of random singular integral on arc-wise smooth curve L. In [Wang, 2005], the author discussed some properties of random singular integral, proved that random singular integral operator was a linear bounded operator and gave the plemelj fomula of random cauchy-type integral on smooth curve L. In this paper, we continue to consider random singular integral of random process with second order moment, and we get the plemelj formula of general form when L is a an arc-wise smooth curve. It is well known that singular integral equation and boundary value problems of analytic function and random process are closely connected with many physical and engineering problems such as elastic mechanics, crack mechanics and aero-dynamics, ect. Therefore it is expected that the results of present paper will be applied in future.

2. Some Preliminaries

Lemma 1 Let L be an arc-wise smooth and closed curve, $f(t, \tau) \in H(L \times L)$. Let

$$g(\tau, z) = \frac{1}{2\pi i} \int_L \frac{f(t, \tau)}{t - z} dt,$$

Given $\tau \in L, z \in \overline{D^+}$ or $\tau \in L, z \in \overline{D^-}$, then $g(\tau, z) \in H$. when $z \in L$, $g(\tau, z)$ can be correspondingly understood by $g(\tau, z^+)$ and $g(\tau, z^-)$.

Proof We only proof that $g(\tau, z) \in H$ when $\tau \in L, z \in \overline{D^+}$. So it is sufficient to prove that: $g(\tau, z)$, as a function of one of its arguments, $\in H$ uniformly with respect to the other argument. For any $t_1, t_2, \tau_1, \tau_2 \in L$, we have

$$|f(t_1, \tau_1) - f(t_2, \tau_2)| \leq A|t_1 - t_2|^\alpha + B|\tau_1 - \tau_2|^\beta, \qquad (0 < \alpha, \beta < 1)$$

At first, if τ is fixed, we need to prove the following inequality

$$|g(\tau, z_1) - g(\tau, z_2)| \leq A_1|z_1 - z_2|^{\alpha_1}, \qquad (0 < \alpha_1 < 1)$$

holds for any $z_1, z_2 \in \overline{D^+}$, where A_1 is independent of τ. According to the proof of Privalov theorem [Lu, Jianke, 2004], we can know A_1 is really independent of τ.

Secondly, if z is fixed, we need to prove the following inequality

$$|g(\tau_1, z) - g(\tau_2, z)| \leq B_1|\tau_1 - \tau_2|^{\beta_1}, \qquad (0 < \beta_1 < 1)$$

holds for any $\tau_1, \tau_2 \in L$, where β_1 is independent of z, it can be proved by the proof of corollary 2 of theorem 1.3 [Hou Zongyi, 1990].

Lemma 2 Let L be an arc-wise smooth closed curve and $f(t,z) \in H$ when $t \in L$, $z \in T$, where T is a region containing L in its interior. Let

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(\tau,z)}{\tau - z}d\tau, \quad z \in T - L,$$

Then for any $t \in L$, we have

$$F^+(t) = \left(1 - \frac{\theta_0}{2\pi}\right) f(t,t) + \frac{1}{2\pi i}\int_L \frac{f(\tau,t)}{\tau - t}d\tau,$$

$$F^-(t) = -\frac{\theta_0}{2\pi} f(t,t) + \frac{1}{2\pi i}\int_L \frac{f(\tau,t)}{\tau - t}d\tau.$$

where θ_0 is the angle spanned by the two one-sided tangents at ζ_0 towards the positive side of L. If $f(t,z)$ is defined and fulfills the assumed condition only for z in one side of L (including L itself), then the conclusion is also valid for the boundary value of the same side.

The above lemma can be proved by the proof of theorem 1.4.2 [Lu Jianke, 2004].

Lemma 3 Let L be an arc-wise smooth and closed curve, $f(t,\tau) \in H(L \times L)$, let

$$g(\tau, t_0) = \frac{1}{2\pi i}\int_L \frac{f(t,\tau)}{t - t_0}dt,$$

then $g(\tau, t_0)$, as a function of τ, $\in H$ uniformly with respect to t_0, which can be proved by the proof of corollary 2 of theorem 1.3 [Hou Zongyi, 1990].

Lemma 4 Let L be an arc-wise smooth and closed curve, $f(t,\tau) \in H(L \times L)$, let

$$F(z) = \frac{1}{4\pi^2}\int_L \frac{d\tau}{\tau - z}\overline{\int_L \frac{f(t,\tau)}{t - z}dt},$$

then we get

$$\begin{aligned} F^+(t_0) = {} & \left(1 - \frac{\theta_0}{2\pi}\right)^2 \overline{f(t_0,t_0)} + \frac{i}{2\pi}\left(1 - \frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{f(t,t_0)}{t - t_0}dt} \\ & + \frac{1}{2\pi i}\left(1 - \frac{\theta_0}{2\pi}\right)\int_L \frac{\overline{f(t_0,\tau)}}{\tau - t_0}d\tau + \frac{1}{4\pi^2}\int_L \frac{1}{\tau - t_0}d\tau\overline{\int_L \frac{f(t,\tau)}{t - t_0}dt}, \end{aligned}$$

$$\begin{aligned} F^-(t_0) = {} & \left(-\frac{\theta_0}{2\pi}\right)^2 \overline{f(t_0,t_0)} + \frac{i}{2\pi}\left(-\frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{f(t,t_0)}{t - t_0}dt} \\ & + \frac{1}{2\pi i}\left(-\frac{\theta_0}{2\pi}\right)\int_L \frac{\overline{f(t_0,\tau)}}{\tau - t_0}d\tau + \frac{1}{4\pi^2}\int_L \frac{1}{\tau - t_0}d\tau\overline{\int_L \frac{f(t,\tau)}{t - t_0}dt}. \end{aligned}$$

Proof Denote

$$g(\tau, z) = \overline{\frac{1}{2\pi i}\int \frac{f(t,\tau)}{t - z}dt}, \qquad z\overline{\in}L,$$

thus, we have

$$F(z) = \frac{1}{2\pi i}\int_L \frac{g(\tau,z)}{\tau - z}d\tau, \qquad z\overline{\in}L,$$

By lemma 1 and lemma 2, we get

$$F^+(t_0) = \left(1 - \frac{\theta_0}{2\pi}\right) g(t_0, {t_0}^+) + \frac{1}{2\pi i}\int_L \frac{g(\tau, {t_0}^+)}{\tau - t_0}d\tau,$$

According to the plemelj fomula, we have

$$g(\tau, {t_0}^+) = \left(1 - \frac{\theta_0}{2\pi}\right)\overline{f(t_0,\tau)} + \overline{\frac{1}{2\pi i}\int_L \frac{f(t,\tau)}{t - t_0}dt},$$

$$g(t_0, {t_0}^+) = \left(1 - \frac{\theta_0}{2\pi}\right)\overline{f(t_0,t_0)} + \overline{\frac{1}{2\pi i}\int_L \frac{f(t,t_0)}{t - t_0}dt},$$

Therefore

$$
\begin{aligned}
F^{+}(t_0) =& \left(1-\frac{\theta_0}{2\pi}\right)g(t_0,t_0{}^{+})+\frac{1}{2\pi i}\int_L \frac{g(\tau,t_0{}^{+})}{\tau-t_0}d\tau \\
=& \left(1-\frac{\theta_0}{2\pi}\right)^2\overline{f(t_0,t_0)}+\frac{i}{2\pi}\left(1-\frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{f(t,t_0)}{t-t_0}dt} \\
&+\frac{1}{2\pi i}\int_L \frac{1}{\tau-t_0}\left[\left(1-\frac{\theta_0}{2\pi}\right)\overline{f(t_0,\tau)}+\overline{\frac{1}{2\pi i}\int_L \frac{f(t,\tau)}{t-t_0}dt}\right]d\tau \\
=& \left(1-\frac{\theta_0}{2\pi}\right)^2\overline{f(t_0,t_0)}+\frac{i}{2\pi}\left(1-\frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{f(t,t_0)}{t-t_0}dt} \\
&+\frac{1}{2\pi i}\left(1-\frac{\theta_0}{2\pi}\right)\int_L \frac{\overline{f(t_0,\tau)}}{\tau-t_0}d\tau+\frac{1}{4\pi^2}\int_L \frac{1}{\tau-t_0}d\tau\overline{\int_L \frac{f(t,\tau)}{t-t_0}dt}.
\end{aligned}
$$

3. Main Results

Let $R_g(\zeta,\zeta')$ be self-correlation function of $g(\omega,\zeta)$, namely

$$R_g(\zeta,\zeta') = E\left[g(\omega,\zeta)\overline{g(\omega,\zeta')}\right]$$

we have

Theorem Let $R_g(\zeta,\zeta')\in H(L\times L)$, and

$$F(\omega,z) = \frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-z}d\zeta,$$

Then the Plemelj formula

$$
\begin{aligned}
F^{+}(\omega,\zeta_0) =& \left(1-\frac{\theta_0}{2\pi}\right)g(\omega,\zeta_0)+\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-\zeta_0}d\zeta, \\
F^{-}(\omega,\zeta_0) =& -\frac{\theta_0}{2\pi}g(\omega,\zeta_0)+\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-\zeta_0}d\zeta,
\end{aligned}
$$

holds in the sense of mean square metric, where

$$F^{+}(\omega,\zeta_0) = \lim_{\substack{z\to\zeta_0\\ z\in D^+}}\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-z}d\zeta,\ F^{-}(\omega,\zeta_0) = \lim_{\substack{z\to\zeta_0\\ z\in D^-}}\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-z}d\zeta,$$

where θ_0 is the angle spanned by the two one-sided tangents at ζ_0 towards the positive side of L.

Proof Let $z\in D^+$,considering

$$
\begin{aligned}
&E\left\{\left|\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-z}d\zeta-\left[\left(1-\frac{\theta_0}{2\pi}\right)g(\omega,\zeta_0)+\frac{1}{2\pi i}\int_L \frac{g(\omega,\zeta)}{\zeta-\zeta_0}d\zeta\right]\right|^2\right\} \\
=&E\left\{\left|\frac{1}{2\pi i}\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)g(\omega,\zeta)d\zeta-\left(1-\frac{\theta_0}{2\pi}\right)g(\omega,\zeta_0)\right|^2\right\} \\
=&\frac{1}{2\pi i}\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)d\zeta\overline{\frac{1}{2\pi i}\int_L\left(\frac{1}{\zeta'-z}-\frac{1}{\zeta'-\zeta_0}\right)E[\overline{g(\omega,\zeta)}g(\omega,\zeta')]d\zeta'} \\
&-\frac{1}{2\pi i}\left(1-\frac{\theta_0}{2\pi}\right)\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)E[g(\omega,\zeta)\overline{g(\omega,\zeta_0)}]d\zeta \\
&+\frac{1}{2\pi i}\left(1-\frac{\theta_0}{2\pi}\right)\overline{\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)E[g(\omega,\zeta)\overline{g(\omega,\zeta_0)}]d\zeta} \\
&+\left(1-\frac{\theta_0}{2\pi}\right)^2 E[g(\omega,\zeta_0)\overline{g(\omega,\zeta_0)}] \\
=&\frac{1}{2\pi i}\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)d\zeta\overline{\frac{1}{2\pi i}\int_L\left(\frac{1}{\zeta'-z}-\frac{1}{\zeta'-\zeta_0}\right)R_g(\zeta',\zeta)d\zeta'} \\
&-\frac{1}{2\pi i}\left(1-\frac{\theta_0}{2\pi}\right)\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)R_g(\zeta,\zeta_0)d\zeta \\
&+\frac{1}{2\pi i}\left(1-\frac{\theta_0}{2\pi}\right)\overline{\int_L\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\zeta_0}\right)R_g(\zeta,\zeta_0)d\zeta}+\left(1-\frac{\theta_0}{2\pi}\right)^2 R_g(\zeta_0,\zeta_0) \\
\triangleq& I_1+I_2+I_3+I_4.
\end{aligned}
$$

And

$$\begin{aligned}
I_1 =& \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - z}d\zeta \overline{\int_L \frac{1}{\zeta' - z}R_g(\zeta', \zeta)d\zeta'} - \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - z}d\zeta \overline{\int_L \frac{1}{\zeta' - \zeta_0}R_g(\zeta', \zeta)d\zeta'} \\
&- \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - \zeta_0}d\zeta \overline{\int_L \frac{1}{\zeta' - z}R_g(\zeta', \zeta)d\zeta'} + \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - \zeta_0}d\zeta \overline{\int_L \frac{1}{\zeta' - \zeta_0}R_g(\zeta', \zeta)d\zeta'} \\
\triangleq& I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned}$$

By lemma 4, we have

$$\begin{aligned}
\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_{11} =& \left(1 - \frac{\theta_0}{2\pi}\right)^2 \overline{R_g(\zeta_0, \zeta_0)} + \frac{i}{2\pi}\left(1 - \frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{R_g(\zeta', \zeta_0)}{\zeta' - \zeta_0}d\zeta'} \\
&+ \frac{1}{2\pi i}\left(1 - \frac{\theta_0}{2\pi}\right)\int_L \frac{\overline{R_g(\zeta_0, \zeta)}}{\zeta - \zeta_0}d\zeta + \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - \zeta_0}d\zeta \overline{\int_L \frac{R_g(\zeta', \zeta)}{\zeta' - \zeta_0}d\zeta'},
\end{aligned}$$

And according to lemma 3 and Plemelj formula, we can get

$$\begin{aligned}
\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_{12} =& \frac{1}{2\pi i}\left(1 - \frac{\theta_0}{2\pi}\right)\overline{\int_L \frac{R_g(\zeta', \zeta_0)}{\zeta' - \zeta_0}d\zeta'} - \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - \zeta_0}d\zeta \overline{\int_L \frac{R_g(\zeta', \zeta)}{\zeta' - \zeta_0}d\zeta'}, \\
\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_{13} =& -\frac{1}{2\pi i}\left(1 - \frac{\theta_0}{2\pi}\right)\int_L \frac{\overline{R_g(\zeta_0, \zeta)}}{\zeta - \zeta_0}d\zeta - \frac{1}{4\pi^2}\int_L \frac{1}{\zeta - \zeta_0}d\zeta \overline{\int_L \frac{R_g(\zeta', \zeta)}{\zeta' - \zeta_0}d\zeta'},
\end{aligned}$$

Therefore,

$$\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_1 = \left(1 - \frac{\theta_0}{2\pi}\right)^2 \overline{R_g(\zeta_0, \zeta_0)}.$$

Similarly, we have

$$\begin{aligned}
\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_2 =& -\left(1 - \frac{\theta_0}{2\pi}\right)\left\{\left[\left(1 - \frac{\theta_0}{2\pi}\right)R_g(\zeta_0, \zeta_0) + \frac{1}{2\pi i}\int_L \frac{R_g(\zeta, \zeta_0)}{\zeta - \zeta_0}d\zeta\right] - \frac{1}{2\pi i}\int_L \frac{R_g(\zeta, \zeta_0)}{\zeta - \zeta_0}d\zeta\right\} \\
=& -\left(1 - \frac{\theta_0}{2\pi}\right)^2 R_g(\zeta_0, \zeta_0),
\end{aligned}$$

$$\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} I_3 = -\left(1 - \frac{\theta_0}{2\pi}\right)^2 \overline{R_g(\zeta_0, \zeta_0)},$$

Then, we obtain

$$\lim_{\substack{z\to\zeta_0 \\ z\in D^+}} E\left\{\left|\frac{1}{2\pi i}\int_L \frac{g(\omega, \zeta)}{\zeta - z}d\zeta - \left[\left(1 - \frac{\theta_0}{2\pi}\right)g(\omega, \zeta_0) + \frac{1}{2\pi i}\int_L \frac{g(\omega, \zeta)}{\zeta - \zeta_0}d\zeta\right]\right|^2\right\} = 0,$$

Analogously, we can prove that

$$\lim_{\substack{z\to\zeta_0 \\ z\in D^-}} E\left\{\left|\frac{1}{2\pi i}\int_L \frac{g(\omega, \zeta)}{\zeta - z}d\zeta - \left[-\frac{\theta_0}{2\pi}g(\omega, \zeta_0) + \frac{1}{2\pi i}\int_L \frac{g(\omega, \zeta)}{\zeta - \zeta_0}d\zeta\right]\right|^2\right\} = 0,$$

The proof is complete.

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