

Existence of Positive Solutions of Fisher-KPP Equations in Locally Spatially Variational Habitat with Hybrid Dispersal

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Abstract

The current paper investigate the persistence of positive solutions of KPP type evolution equations with random/nonlocal dispersal in locally spatially inhomogeneous habitat. By the constructions of super/sub solutions and comparison principle, we prove that such an equation has a unique globally stable positive stationary solution.

Keywords: KPP equations, random dispersal, nonlocal dispersal, localized spatial inhomogeneity, positive stationary solution, principal eigenvalue, sub-solution, super-solution, comparison principle

Mathematics subject classification. 35K57, 45G10, 58D20, 92D25.

1. Introduction

The current paper is concerned with persistence of species in locally spatially variational environments or habitat,

$$u_t(t, x) = \mathcal{A}u(t, x) + u(t, x)f(x, u(t, x)), \quad x \in \mathbb{R}^N \quad (1.1)$$

where $(\mathcal{A}u)(t, x) = v_1 \Delta u + v_2 \left[\int_{\mathbb{R}^N} \kappa(y - x)u(t, y)dy - u(t, x) \right]$, $v_1, v_2 \geq 0$ and $v_1 + v_2 > 0$.

If $v_2 = 0$, then (1.1) is the classical reaction diffusion equation, so called random dispersal equation, (without loss of generality, let $v_1 = 1$)

$$u_t(t, x) = \Delta u(t, x) + u(t, x)f(x, u(t, x)), \quad x \in \mathbb{R}^N \quad (1.2)$$

which is broadly used to model the population dynamics of many species in unbounded environments, where $u(t, x)$ is the population density of the species at time t and location x , Δu characterizes the internal interaction of the organisms, and $f(x, u)$ represents the growth rate of the population, which satisfies that $f(x, u) < 0$ for $u \gg 1$ and $\partial_u f(x, u) < 0$ for $u \geq 0$ (see Aronson & Weinberger, 1957; Aronson & Weinberger, 1978; Cantrell & Cosner, 2003; Fife, 1979; Fife & Peletier, 1977; Fisher, 1937; Kolmogorov, Petrowsky, & Poscunov, 1937; Murray, 1989; Shigesada & Kawasaki, 1997; Skellam, 1951; Weinberger, 1982; Weinberger, 2002; Zhao, 2003; etc.).

If $v_1 = 0$, then (1.1) is so called nonlocal dispersal equation, (without loss of generality, let $v_2 = 1$)

$$u_t(t, x) = \int_{\mathbb{R}^N} \kappa(y - x)u(t, y)dy - u(t, x) + u(t, x)f(x, u(t, x)), \quad x \in \mathbb{R}^N, \quad (1.3)$$

where $\kappa(\cdot)$ is a smooth convolution kernel supported on a ball centered at the origin (that is, there is a $\delta_0 > 0$ such that $\kappa(z) > 0$ if $\|z\| < \delta_0$, $\kappa(z) = 0$ if $\|z\| \geq \delta_0$, where $\|\cdot\|$ denotes the norm in \mathbb{R}^N and δ_0 represents the nonlocal dispersal distance), $\int_{\mathbb{R}^N} \kappa(z)dz = 1$, and $f(\cdot, \cdot)$ is of the same property as f in (1.2) (see Bates & Zhao, 2007; Chasseigne, Chaves, & Rossi, 2006; Cortazar, Coville, Elgueta, & Martinez, 2007; Cousens, Dytham, & Law, 2008; Fife, 2003; Grinfeld, Hines, Hutson, Mischaikow, & Vickers, 2005; Hutson, Martinez, Mischaikow & Vickers, 2003; Lee, Hoopes, Diehl, Gilliland, Huxel, Leaver, McCain, Umbanhowar, & Mogilner, 2001; Levin, Muller-Landau, Nathan, & Chave, 2003; etc.).

When using (1.2) to model the population dynamics of a species, it is assumed that the underlying environment is continuous and the dispersal of cells or organisms are based on the hypothesis that the movement of the dispersing species can be described as a random walk in which there is no correlation between steps. However, dispersal of large organisms often involves mechanisms that may introduce correlations in movements. To model the population dynamics of such species in the case that the underlying environment is continuous, the nonlocal dispersal equation (1.3) is often used. This paper propose to study a mixed dispersal strategy, that is, a hybrid of random and non-local dispersal. We assume that a fraction of individuals in the population adopt random dispersal, while the rest fraction assumes non-local dispersal. Some research has been done on the hybrid dispersal in the spatially periodic habitat (see Kao, Lou & Shen, 2010; Kao, Lou & Shen, 2012; and Zhang, 2013). Our main goal is to study how the hybrid dispersal affects the persistence of a

single species and how the hybrid dispersal strategies will evolve in spatially locally inhomogeneous environment (see H1 and H2).

Since the seminal works by Fisher (Fisher, 1937) and Kolmogorov, Petrowsky, Piscunov (Kolmogorov, Petrowsky, & Poscunov, 1937) on the following special case of (1.2),

$$\frac{\partial}{\partial t}u(t, x) = \frac{\partial^2}{\partial x^2}u(t, x) + u(t, x)(1 - u(t, x)), \quad x \in \mathbb{R}. \quad (1.4)$$

A great deal of research has been carried out toward the spatial spreading dynamics of (1.2) and (1.3) with $f(\cdot, \cdot)$ being independent of the space variable or periodic in the space variable, which reflects the spatial periodicity of the media. We refer to (Aronson & Weinberger, 1957; Aronson & Weinberger, 1978; Berestycki, Hamel, & Nadinashvili, 2010; Kametaka, 1976; Liang & Zhao, 2007; Liang, Yi, & Zhao, 2006; Sattinger, 1976; Uchiyama, 1978; Weinberger, 1982; etc.) for the study of (1.2) in the case that $f(x, u)$ is independent of x and refer to (Berestycki, Hamel, & Nadinashvili, 2005; Berestycki, Hamel, & Roques, 2005; Freidlin & Gärtner, 1979; Hamel, 2008; Hudson & Zinner, 1995; Nadin, 2009; Nolen, Rudd, & Xin, 2005; Nolen & Xin, 2005; Weinberger, 2002; etc.) for the study of (1.2) in the case that $f(x, u)$ is periodic in x ; refer to (Coville & Dupaigne, 2005; Coville, Dávila, & Martínez, 2008; Li, Sun, & Wang, 2010; etc.) for the study of (1.3) in the case that $f(x, u)$ is independent of x and refer to (Hetzer, Shen, & Zhang, 2013; Shen & Zhang, 2012; etc.) for the study of (1.3) in the case that $f(x, u)$ is periodic in x and refer to (Berestycki & Nadin, 2016; Berestycki, Jin, & Silvestre, 2016; Kong & Shen, 2011; Kong & Shen, 2014; Nolen, Roquejoffre, & Ryzhik; Shen, 2011; etc.) for the study of (1.2) and/or (1.3) in the case that $f(t, x, u)$ is temporally and/or spatially heterogeneous.

For example, consider (1.2) and assume that $f(x + p_i \mathbf{e}_i, u) = f(x, u)$ for $i = 1, 2, \dots, N$, where p_i ($i = 1, 2, \dots, N$) are positive constants and

$$\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{iN}), \quad \delta_{ij} = 1 \text{ if } i = j \text{ and } 0 \text{ if } i \neq j.$$

If the principal eigenvalue of the following eigenvalue problem associated to the linearized equation of (1.2) at $u = 0$,

$$\begin{cases} \Delta u(x) + f(x, 0)u(x) = \lambda u(x), & x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.5)$$

is positive, then (1.2) has a unique positive stationary solution $u^*(\cdot)$ with $u^*(\cdot + p_i \mathbf{e}_i) = u^*(\cdot)$.

In this paper, we consider (1.1) in the case that the growth rates depend on the space variable, but only when it is in some bounded subset of the underlying habitat, which reflects the localized spatial inhomogeneity of the media. More precisely, let $\mathcal{H} = \mathbb{R}^N$, we assume

(H1) $f : \mathcal{H} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function, $f(x, u) < 0$ for all $(x, u) \in \mathcal{H} \times \mathbb{R}^+$ with $u \geq \beta_0$ for some $\beta_0 > 0$, and $\partial_u f(x, u) < 0$ for all $(x, u) \in \mathcal{H} \times \mathbb{R}^+$.

(H2) $f(x, u) = f^0(u)$ for some C^2 function $f^0 : \mathbb{R} \rightarrow \mathbb{R}$ and all $(x, u) \in \mathcal{H} \times \mathbb{R}$ with $\|x\| \geq L_0$ for some $L_0 > 0$, and $f^0(0) > 0$.

Assume (H1) and (H2). Then (1.1) has the following limit equations as $\|x\| \rightarrow \infty$,

$$u_t(t, x) = \mathcal{A}u(t, x) + u(t, x)f^0(u(t, x)), \quad x \in \mathbb{R}^N. \quad (1.6)$$

Equations (1.6) will play an important role in the study of (1.1). Equations (1.6) has a unique positive constant stationary solution u^0 . We introduce some standing notations and then state the main results of the paper.

Let $p = (p_1, p_2, \dots, p_N)$ with $p_i > 0$ for $i = 1, 2, \dots, N$. We define the Banach spaces X_p by

$$X_p = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u(\cdot + p_i \mathbf{e}_i) = u(\cdot), \quad i = 1, \dots, N\} \quad (1.7)$$

with norm $\|u\|_{X_p} = \max_{x \in \mathbb{R}^N} |u(x)|$.

Let

$$X_p^+ = \{u \in X_p \mid u(x) \geq 0 \quad \forall x \in \mathcal{H}\} \quad (1.8)$$

and

$$X_p^{++} = \{u \in X_p \mid u(x) > 0 \quad \forall x \in \mathcal{H}\}. \quad (1.9)$$

We define X by

$$X = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is uniformly continuous and bounded}\} \quad (1.10)$$

with norm $\|u\|_X = \sup_{x \in \mathbb{R}^N} |u(x)|$.

Let

$$X^+ = \{u \in X_i \mid u(x) \geq 0 \quad \forall x \in \mathcal{H}\} \quad (1.11)$$

and

$$X^{++} = \{u \in X_i^+ \mid \inf_{x \in \mathcal{H}} u(x) > 0\}. \quad (1.12)$$

Without occurring confusion, we may write $\|\cdot\|_{X_p}$ and $\|\cdot\|_X$ as $\|\cdot\|$.

Assume (H1). By general semigroup theory (see Henry, 1981; Pazy, 1983), for any $u_0 \in X$, (1.2) has a unique local solution $u(t, \cdot; u_0)$ with $u(0, \cdot; u_0) = u_0(\cdot)$. Moreover, if $u_0 \in X^+$, then $u(t, \cdot; u_0)$ exist and $u(t, \cdot; u_0) \in X^+$ for all $t \geq 0$ (see Proposition 2.2).

Our objective is to explore the spatial spreading dynamics of (1.1) with localized spatial inhomogeneity. The main results of this paper are stated in the following two theorems.

Theorem 1.1 (Positive stationary solutions). *Assume (H1) and (H2). Equation (1.2) has a unique stationary solution $u = u^*(\cdot) \in X^{++}$. Moreover,*

$$\lim_{r \rightarrow \infty} \sup_{x \in \mathcal{H}, \|x\| \geq r} |u^*(x) - u^0| = 0,$$

where $u^0 > 0$ is such that $f^0(u^0) = 0$.

Theorem 1.2 (Stability). *Assume (H1) and (H2). For any $u_0 \in X^{++}$,*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot; u_0) - u^*(\cdot)\|_X = 0.$$

The rest of the paper is organized as follows. In section 2, we present some preliminary materials to be used in later sections. Section 3 is devoted to the study of positive stationary solutions of (1.1). Theorem 1.1 and Theorem 1.2 are proved in this section.

2. Preliminary

In this section, we present some preliminary materials to be used in later sections, including some basic properties of solutions of (1.1); principal eigenvalue theories for spatially periodic dispersal operators with random, and nonlocal; and spatial spreading dynamics of KPP equations in spatially periodic media.

2.1 Basic Properties of KPP Equations

In this subsection, we present some basic properties of solutions of (1.1), including comparison principle, global existence, convergence in open compact topology, and decreasing of the so called part metric along the solutions. Throughout this subsection, we assume (H1).

Let X be as in (1.10). For given $u_0 \in X$, let $u(t, \cdot; u_0)$ be the (local) solution of (1.2) with $u(0, \cdot; u_0) = u_0(\cdot)$.

Let X^+ and X^{++} be as in (1.11) and (1.12). For given $u, v \in X$, we define

$$u \leq v \ (u \geq v) \quad \text{if } v - u \in X^+ \ (u - v \in X^+) \quad (2.1)$$

and

$$u \ll v \ (u \gg v) \quad \text{if } v - u \in X^{++} \ (u - v \in X^{++}). \quad (2.2)$$

For given continuous and bounded function $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$, it is called a *super-solution* (*sub-solution*) of (1.1) on $[0, T]$ if

$$u_t(t, x) \geq (\leq) \mathcal{A}u(t, x) + u(t, x)f(x, u(t, x)) \quad \forall (t, x) \in (0, T) \times \mathbb{R}^N.$$

Proposition 2.1 (Comparison principle). *Assume (H1).*

- (1) *Suppose that $u^1(t, x)$ and $u^2(t, x)$ are sub- and super-solutions of (1.1) on $[0, T]$ with $u^1(0, \cdot) \leq u^2(0, \cdot)$. Then $u^1(t, \cdot) \leq u^2(t, \cdot)$ for $t \in (0, T)$. Moreover, if $u^1(0, \cdot) \neq u^2(0, \cdot)$, then $u^1(t, x) < u^2(t, x)$ for $x \in \mathcal{H}$, and $t \in (0, T)$.*
- (2) *If $u_{01}, u_{02} \in X$ and $u_{01} \leq u_{02}$, $u_{01} \neq u_{02}$, then $u(t, x; u_{01}) < u(t, x; u_{02})$ for all $x \in \mathcal{H}$ and $t > 0$ at which both $u(t, \cdot; u_{01})$ and $u(t, \cdot; u_{02})$ exist.*
- (3) *If $u_{01}, u_{02} \in X$ and $u_{01} \ll u_{02}$, then $u(t, \cdot; u_{01}) \ll u(t, \cdot; u_{02})$ for $t > 0$ at which both $u(t, \cdot; u_{01})$ and $u(t, \cdot; u_{02})$ exist.*

Proof. (1) The case $v_2 = 0$ follows from comparison principle for parabolic equations. The case $v_1 = 0$ follows from (Shen & Zhang, 2010) [Propositions 2.1 and 2.2].

(2) follows from (1).

(3) We provide a proof for the case $v_1 = 0$. Other cases can be proved similarly. Take any $T > 0$ such that both $u(t, \cdot; u_{01})$ and $u(t, \cdot; u_{02})$ exist on $[0, T]$. It suffices to prove that $u(t, \cdot; u_{02}) \gg u(t, \cdot; u_{01})$ for $t \in [0, T]$. To this end, let $w(t, x) = u(t, x; u_{02}) - u(t, x; u_{01})$. Then $w(t, x)$ satisfies the following equation,

$$w_t(t, x) = \int_{\mathbb{R}^N} \kappa(y - x)w(t, y)dy - w(t, x) + a(t, x)w(t, x),$$

where

$$a(t, x) = f(x, u(t, x; u_{02})) + u(t, x; u_{01}) \int_0^1 \partial_u f(x, su(t, x; u_{02}) + (1-s)u(t, x; u_{01}))ds.$$

Let $M > 0$ be such that $M \geq \sup_{x \in \mathbb{R}^N, t \in [0, T]} (1 - a(t, x))$ and $\tilde{w}(t, x) = e^{Mt} w(t, x)$. Then $\tilde{w}(t, x)$ satisfies

$$\tilde{w}_t(t, x) = \int_{\mathbb{R}^N} \kappa(y - x)\tilde{w}(t, y)dy + [M - 1 + a(t, x)]\tilde{w}(t, x).$$

Let $\mathcal{K} : X \rightarrow X$ be defined by

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}^N} \kappa(y - x)u(y)dy \quad \text{for } u \in X. \quad (2.3)$$

Then \mathcal{K} generates an analytic semigroup on X and

$$\tilde{w}(t, \cdot) = e^{\mathcal{K}t}(u_{02} - u_{01}) + \int_0^t e^{\mathcal{K}(t-\tau)}(M - 1 + a(\tau, \cdot))\tilde{w}(\tau, \cdot)d\tau.$$

Observe that $e^{\mathcal{K}t}u_0 \geq 0$ for any $u_0 \in X^+$ and $t \geq 0$ and $e^{\mathcal{K}t}u_0 \gg 0$ for any $u_0 \in X^{++}$ and $t \geq 0$. Observe also that $u_{02} - u_{01} \in X^{++}$. By (2), $\tilde{w}(\tau, \cdot) \geq 0$ and hence $(M - 1 + a(\tau, \cdot))\tilde{w}(\tau, \cdot) \geq 0$ for $\tau \in [0, T]$. It then follows that $\tilde{w}(t, \cdot) \gg 0$ and then $w(t, \cdot) \gg 0$ (i.e. $u(t, \cdot; u_{02}) \gg u(t, \cdot; u_{01})$) for $t \in [0, T]$. \square

Proposition 2.2. Assume (H1). For any given $u(t, \cdot; u_0)$ exists for all $t \geq 0$.

Proof. Let $u_0 \in X^+$ be given. There is $M \gg 1$ such that $0 \leq u_0(x) \leq M$ and $f(x, M) < 0$ for all $x \in \mathcal{H}$. Then by Proposition 2.1,

$$0 \leq u(t, \cdot; u_0) \leq M$$

for any $t > 0$ at which $u(t, \cdot; u_0)$ exists. It is then not difficult to prove that for any $T > 0$ such that $u(t, \cdot; u_0)$ exists on $(0, T)$, $\lim_{t \rightarrow T} u(t, \cdot; u_0)$ exists in X . This implies that $u(t, \cdot; u_0)$ exists and $u(t, \cdot; u_0) \geq 0$ for all $t \geq 0$. \square

For given $u, v \in X^{++}$, define

$$\rho(u, v) = \inf\{\ln \alpha \mid \frac{1}{\alpha}u \leq v \leq \alpha u, \alpha \geq 1\}.$$

Observe that $\rho(u, v)$ is well defined and there is $\alpha \geq 1$ such that $\rho(u, v) = \ln \alpha$. Moreover, $\rho(u, v) = \rho(v, u)$ and $\rho(u, v) = 0$ iff $u \equiv v$. In literature, $\rho(u, v)$ is called the *part metric* between u and v .

Proposition 2.3. For given $u_0, v_0 \in X^{++}$ with $u_0 \neq v_0$, $\rho(u(t, \cdot; u_0), u(t, \cdot; v_0))$ is non-increasing in $t \in (0, \infty)$.

Proof. It can be proved by similar argument in (Kong & Shen, 2011) [Proposition 3.3]. For completeness, we provide a proof here.

First, note that there is $\alpha^* > 1$ such that $\rho(u_0, v_0) = \ln \alpha^*$ and $\frac{1}{\alpha^*}u_0 \leq v_0 \leq \alpha^*u_0$. By Proposition 2.1,

$$u(t, \cdot; v_0) \leq u(t, \cdot; \alpha^*u_0) \quad \text{for } t > 0.$$

Let $v(t, x) = \alpha^*u(t, x; u_0)$. Then

$$\begin{aligned} v_t(t, x) &= \mathcal{A}v(t, x) + v(t, x)f(x, u(t, x; u_0)) \\ &= \mathcal{A}v(t, x) + v(t, x)f(x, v(t, x)) + v(t, x)f(x, u(t, x; u_0)) - v(t, x)f(x, v(t, x)) \\ &> \mathcal{A}v(t, x) + v(t, x)f(x, v(t, x)). \end{aligned}$$

This together with Proposition 2.1 implies that

$$u(t, \cdot; \alpha^*u_0) \leq \alpha^*u(t, \cdot; u_0) \quad \text{for } t > 0$$

and then

$$u(t, \cdot; v_0) \leq \alpha^*u(t, \cdot; u_0) \quad \text{for } t > 0.$$

Similarly, it can be proved that

$$\frac{1}{\alpha^*}u(t, \cdot; u_0) \leq u(t, \cdot; v_0) \quad \text{for } t > 0.$$

It then follows that

$$\rho(u(t, \cdot; u_0), u(t, \cdot; v_0)) \leq \rho(u_0, v_0) \quad \forall t > 0$$

and hence

$$\rho(u(t_2, \cdot; u_0), u(t_2, \cdot; v_0)) \leq \rho(u(t_1, \cdot; u_0), u(t_1, \cdot; v_0)) \quad \forall 0 \leq t_1 < t_2.$$

\square

2.2 Principal Eigenvalues of Spatially Periodic Dispersal Operators

In this subsection, we present some principal eigenvalue theories for spatially periodic dispersal operators with hybrid dispersals.

Let $p = (p_1, p_2, \dots, p_N)$ with $p_i > 0$ for $i = 1, 2, \dots, N$ and X_p be as in (1.7). We will denote \mathcal{I} as an identity map on the Banach space under consideration. For given $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, $a \in X_p$, consider the following eigenvalue problems,

$$\begin{cases} Ou(x) = \lambda u(x), & x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e}_i) = u(x), & x \in \mathbb{R}^N, \end{cases} \quad (2.4)$$

where

$$Ou(x) := \nu_1 \Delta u(x) + \nu_2 \int_{\mathbb{R}^N} e^{-\mu(y-x)\xi} \kappa(y-x) u(y) dy - 2\mu \nu_2 \xi \cdot \nabla u(x) + (a(x) + \nu_1 \mu^2 - \nu_2) u(x) \quad (2.5)$$

and $O : \mathcal{D}(O) \subset X_p \rightarrow X_p$.

Observe that if $\nu_1 = 1$, and $\nu_2 = 0$,

$$(Ou)(x) = \Delta u(x) - 2\mu \xi \cdot \nabla u(x) + (a(x) + \mu^2) u(x) \quad \forall u \in \mathcal{D}(O) \subset X_p, \quad (2.6)$$

If $\nu_1 = 0$, and $\nu_2 = 1$,

$$(Ou)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\xi} \kappa(y-x) u(y) dy - u(x) + a(x) u(x) \quad \forall u \in \mathcal{D}(O) \subset X_p \quad (2.7)$$

Let $\sigma(O)$ be the spectrum of O .

Definition 2.1. Let $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$ be given. A real number $\lambda(\mu, \xi, a) \in \mathbb{R}$ is called the principal eigenvalue of O if it is an isolated algebraic simple eigenvalue of O with a positive eigenfunction and for any $\lambda \in \sigma(O) \setminus \{\lambda(\mu, \xi, a)\}$, $\operatorname{Re} \lambda < \lambda(\mu, \xi, a)$.

For given $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$, let

$$\lambda^0(\mu, \xi, a) = \sup\{\operatorname{Re} \mu \mid \mu \in \sigma(O)\}. \quad (2.8)$$

Observe that for any $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$, O generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ in X_p and moreover, $T(t)$ is strongly positive (that is, $T(t)u_0 \geq 0$ for any $t \geq 0$ and $u_0 \in X_p^+$ and $T(t)u_0 \gg 0$ for any $t > 0$ and $u_0 \in X_p^+ \setminus \{0\}$). Then by (Meyer-Nieberg, 1991) [Proposition 4.1.1], $r(T(t)) \in \sigma(T(t))$ for any $t > 0$, where $r(T(t))$ is the spectral radius of $T(t)$. Hence by the spectral mapping theorem (see Chicone & Latushkin, 1999; [Theorem 2.7]), $\lambda^0(\mu, \xi, a) \in \sigma(O)$. Observe also that $\lambda^0(0, \xi, a)$ are independent of $\xi \in S^{N-1}$. We may then put

$$\lambda^0(a) = \lambda^0(0, \xi, a).$$

It is well known that the principal eigenvalue $\lambda(\mu, \xi, a)$ in (2.6) exist for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$ and

$$\lambda(\mu, \xi, a) = \lambda^0(\mu, \xi, a).$$

The principal eigenvalue of O in (2.7) may not exist (see Shen & Zhang, 2010 for examples). If the principal eigenvalue $\lambda(\mu, \xi, a)$ exists in (2.7), then

$$\lambda(\mu, \xi, a) = \lambda^0(\mu, \xi, a).$$

Regarding the existence of principal eigenvalue of O in (2.7), the following proposition is proved in (Shen & Zhang, 2010; Shen & Zhang, 2012).

Proposition 2.4 (Existence of principal eigenvalue). (1) If $a \in C^N(\mathbb{R}^N, \mathbb{R}) \cap X_p$ and the partial derivatives of $a(x)$ up to order $N-1$ are zero at some x_0 satisfying that $a(x_0) = \max_{x \in \mathbb{R}^N} a(x)$, then the principal eigenvalue $\lambda(\mu, \xi, a)$ of O exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$.

(2) If $a(x)$ satisfies that $\max_{x \in \mathbb{R}^N} a(x) - \min_{x \in \mathbb{R}^N} a(x) < \inf_{\xi \in S^{N-1}} \int_{z \cdot \xi \leq 0} k(z) dz$, then the principal eigenvalue $\lambda(\mu, \xi, a)$ of O exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$.

Proof. (1) It follows from (Shen & Zhang, 2010) [Theorem B].

(2) It follows from (Shen & Zhang, 2012) [Theorem B']. □

Let \hat{a} be the average of $a(\cdot)$, that is,

$$\hat{a} = \frac{1}{|D|} \int_D a(x) dx \quad \text{for} \quad (2.9)$$

where

$$D = [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N] \cap \mathcal{H} \quad (2.10)$$

and

$$|D| = p_1 \times p_2 \times \cdots \times p_N \quad \text{for} \quad (2.11)$$

By Proposition 2.4 (2), $\lambda(\mu, \xi, \hat{a})$ exists for all $\mu \in \mathbb{R}$ and $\xi \in S^{N-1}$. The following proposition shows a relation between $\lambda^0(\mu, \xi)$ and $\lambda^0(\mu, \xi, \hat{a})$.

Proposition 2.5 (Influence of spatial variation). *For given $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$, there holds*

$$\lambda^0(\mu, \xi) \geq \lambda^0(\mu, \xi, \hat{a}).$$

Proof. It follow from (Hetzer, Shen, & Zhang, 2013) [Theorem 2.1]. □

We remark that $\lambda(\mu, \xi, \hat{a}) (= \lambda^0(\mu, \xi, \hat{a}))$ have the following explicit expressions,

$$\lambda(\mu, \xi, \hat{a}) = v_1 \mu^2 + v_2 \left(\int_{\mathbb{R}^N} e^{-\mu z \cdot \xi} \kappa(z) dz - 1 \right) + \hat{a} \quad (2.12)$$

2.3 KPP Equations in Spatially Periodic Media

In this subsection, we recall some spatial spreading dynamics of KPP equations in spatially periodic media.

Consider

$$u_t(t, x) = \mathcal{A}u(t, x) + u(t, x)g(x, u(t, x)), \quad x \in \mathbb{R}^N, \quad (2.13)$$

where $g(\cdot, \cdot)$ are periodic in the first variable and monostable in the second variable. More precisely, we assume

(P1) $g : \mathcal{H} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function, $g(x + p_l \mathbf{e}_l, u) = g(x, u)$, where $p_l > 0$ and $g(x, u) < 0$ for all $(x, u) \in \mathcal{H} \times \mathbb{R}^+$ with $u \geq \alpha_0$ for some $\alpha_0 > 0$ and $\partial_u g(x, u) < 0$ for all $(x, u) \in \mathcal{H} \times \mathbb{R}^+$.

(P2) $\lambda^0(g(\cdot, 0)) > 0$.

Assume (P1). Similarly, by general semigroup theory, for any $u_0 \in X$, (2.13) has a unique (local) solution $u(t, \cdot; u_0, g(\cdot, \cdot)) \in X$ with initial data $u_0(\cdot)$. Moreover, if $u_0 \in X_p$, then $u(t, \cdot; u_0, g(\cdot, \cdot)) \in X_p$ for any $t > 0$ at which $u(t, \cdot; u_0, g(\cdot, \cdot))$ exists. By Proposition 2.1, if $u_0 \in X^+$, then $u(t, \cdot; u_0, g(\cdot, \cdot))$ exists and $u(t, \cdot; u_0, g(\cdot, \cdot)) \in X^+$ for all $t > 0$.

Proposition 2.6 (Spatially periodic positive stationary solution). *Assume (P1) and (P2). Then (2.13) has a unique spatially periodic stationary solution $u^*(\cdot; g(\cdot, \cdot)) \in X_p^{++}$ which is globally asymptotically stable with respect to perturbations in $X_p^+ \setminus \{0\}$.*

Proof. It follows from (Zhao, 1996) [Theorem 2.3] and (Shen, & Zhang, 2012) [Theorem C]. □

Let $\hat{g}(u)$ be the spatial average of $g(x, u)$, that is,

$$\hat{g}(u) = \frac{1}{|D|} \int_D g(x, u) dx \quad \text{for} \quad (2.14)$$

where D , $|D|$ is as in (2.10).

Assume

(P3) $\hat{g}(0) > 0$.

Observe that $\lambda(\hat{g}(0)) = \hat{g}(0)$. Then by Proposition 2.5, (P3) implies (P2).

3. Positive Stationary Solutions

In this section, we explore the existence of positive stationary solutions of (1.1), and prove Theorem 1.1 and 1.2.

Throughout this section, we assume (H1) and (H2). We first prove some lemmas.

Lemma 3.1. For any $\epsilon > 0$, there are $p = (p_1, p_2, \dots, p_N) \in \mathbb{N}^N$ and $h \in X_p \cap C^N(\mathcal{H}, \mathbb{R})$ such that

$$f(x, 0) \geq h(x) \quad \text{for } x \in \mathcal{H},$$

$$\hat{h} \geq f^0(0) - \epsilon \quad (\text{hence } \lambda^0(h(\cdot)) \geq f^0(0) - \epsilon),$$

and the partial derivatives of $h(x)$ up to order $N - 1$ are zero at some $x_0 \in \mathcal{H}$ with $h(x_0) = \max_{x \in \mathcal{H}} h(x)$, where \hat{h} is the average of $h(\cdot)$ (see (2.9) for the definition).

Proof. By (H2), there is $L_0 > 0$ such that $f(x, 0) = f^0(0)$ for $x \in \mathcal{H}$ with $\|x\| \geq L_0$. Let $M_0 = \inf_{x \in \mathcal{H}} f(x, 0)$. Let $h_0 : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $h_0(s) = 1$ for $|s| \leq 1$ and $h_0(s) = 0$ for $|s| \geq 2$. For any $p = (p_1, p_2, \dots, p_N) \in \mathbb{N}^N$ with $p_j > 4L_0$, let $h \in X_p \cap C^N(\mathcal{H}, \mathbb{R})$ be such that

$$h(x) = f^0(0) - h_0\left(\frac{\|x\|^2}{L_0^2}\right)(f^0(0) - M_0) \quad \text{for } x \in \left[-\frac{p_1}{2}, \frac{p_1}{2}\right] \times \left[-\frac{p_2}{2}, \frac{p_2}{2}\right] \times \dots \times \left[-\frac{p_N}{2}, \frac{p_N}{2}\right] \cap \mathcal{H}.$$

Then

$$f(x, 0) \geq h(x) \quad \forall x \in \mathcal{H}.$$

It is clear that the partial derivatives of $h(x)$ up to order $N - 1$ are zero at some $x_0 \in \mathcal{H}$ with $h(x_0) = \max_{x \in \mathcal{H}} h(x) (= f^0(0))$. For given $\epsilon > 0$, choosing $p_j \gg 1$, we have

$$\hat{h} > f^0(0) - \epsilon.$$

By Proposition 2.5, $\lambda^0(h(\cdot)) \geq \lambda^0(\hat{h}) = \hat{h}$ and hence

$$\lambda^0(h(\cdot)) \geq f^0(0) - \epsilon.$$

The lemma is thus proved. \square

Lemma 3.2. Suppose that $\tilde{u}^* : \mathbb{R}^N \rightarrow [\sigma_0, M_0]$ is Lebesgue measurable, where σ_0 and M_0 are two positive constants. If

$$\Delta \tilde{u}^*(x) + \int_{\mathbb{R}^N} \kappa(y - x) \tilde{u}^*(y) dy - \tilde{u}^*(x) + \tilde{u}^*(x) \tilde{f}(x, \tilde{u}^*(x)) = 0 \quad \forall x \in \mathbb{R}^N,$$

where $\tilde{f}(x, u) = f(x, u)$ or $f^0(u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$, then $\tilde{u}^*(\cdot) \in X^{++}$.

Proof. We prove the case that $\tilde{f}(x, u) = f(x, u)$. The case that $\tilde{f}(x, u) = f^0(u)$ can be proved similarly.

Let $h^*(x) = \Delta \tilde{u}^*(x) + \int_{\mathbb{R}^N} \kappa(y - x) \tilde{u}^*(y) dy$ for $x \in \mathbb{R}^N$. Then $h^*(\cdot)$ is C^1 and has bounded first order partial derivatives. Let

$$F(x, \alpha) = h^*(x) - \alpha + \alpha f(x, \alpha) \quad \forall x \in \mathbb{R}^N, \alpha \in \mathbb{R}.$$

Then $F : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and $F(x, \tilde{u}^*(x)) = 0$ for each $x \in \mathbb{R}^N$. If $\alpha^* > 0$ is such that $F(x, \alpha^*) = 0$, then

$$-1 + f(x, \alpha^*) = -\frac{h^*(x)}{\alpha^*} < 0$$

and hence

$$\partial_\alpha F(x, \alpha^*) = -1 + f(x, \alpha^*) + \alpha^* \partial_u f(x, \alpha^*) < 0.$$

By Implicit Function Theorem, $\tilde{u}^*(x)$ is C^1 in x . Moreover,

$$\frac{\partial \tilde{u}^*(x)}{\partial x_j} = \frac{\frac{\partial h^*(x)}{\partial x_j}}{-1 + f(x, \tilde{u}^*(x)) + \partial_u f(x, \tilde{u}^*(x)) \tilde{u}^*(x)} \quad \forall x \in \mathbb{R}^N, 1 \leq j \leq N.$$

Therefore, \tilde{u}^* has bounded first order partial derivatives. It then follows that $\tilde{u}^*(x)$ is uniformly continuous in $x \in \mathbb{R}^N$ and then $\tilde{u}^* \in X^{++}$. \square

Lemma 3.3. Suppose that $u^*(\cdot) \in X^{++}$ and $u = u^*(\cdot)$ is a stationary solution of (1). Then

$$u^*(x) \rightarrow u^0 \quad \text{as } \|x\| \rightarrow \infty.$$

Proof. Assume that $u^*(x) \not\rightarrow u^0$ as $\|x\| \rightarrow \infty$. Then there are $\epsilon_0 > 0$ and $x_n \in \mathbb{R}^N$ such that $\|x_n\| \rightarrow \infty$ and

$$|u^*(x_n) - u^0| \geq \epsilon_0 \quad \text{for } n = 1, 2, \dots$$

By the uniform continuity of $u^*(x)$ in $x \in \mathbb{R}^N$, without loss of generality, we may assume that there is a continuous function $\tilde{u}^* : \mathbb{R}^N \rightarrow [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$u(x + x_n) \rightarrow \tilde{u}^*(x)$$

as $n \rightarrow \infty$ uniformly in x on bounded sets. By the Lebesgue Dominated Convergence Theorem, we have

$$\Delta \tilde{u}^*(x) + \int_{\mathbb{R}^N} \kappa(y - x) \tilde{u}^*(y) dy - \tilde{u}^*(x) + \tilde{u}^*(x) f^0(\tilde{u}^*(x)) = 0 \quad \forall x \in \mathbb{R}^N.$$

By Lemma 3.2, $\tilde{u}^* \in X^{++}$. By Proposition 2.6 again, we have $\tilde{u}^*(x) \equiv u^0$ and then $u^*(x_n) \rightarrow u^0$ as $n \rightarrow \infty$. This is a contradiction. Therefore $u^*(x) \rightarrow u^0$ as $\|x\| \rightarrow \infty$. \square

Lemma 3.4. *There is $u^- \in X^{++}$ such that for any $\delta > 0$ sufficiently small, $u(t, x; \delta u^-)$ is increasing in $t > 0$ and $u^{-*,\delta} \in X^{++}$, where $u^{-*,\delta}(x) = \lim_{t \rightarrow \infty} u(t, x; \delta u^-)$, and hence $u = u^{-*,\delta}(\cdot)$ is a stationary solution of (1.1) in X^{++} .*

Proof. Let $M^* > 0$ be such that $f(x, M^*) < 0$. Let $\epsilon > 0$ be such that

$$f^0(0) - \epsilon > 0.$$

By Lemma 3.1, there are $p \in \mathbb{N}^N$ and $h(\cdot) \in X_p \cap C^N(\mathcal{H}, \mathbb{R})$ such that

$$f(x, 0) \geq h(x), \quad \text{and } \hat{h} \geq f^0(0) - \epsilon (> 0).$$

Moreover, the partial derivatives of $h(x)$ up to order $N - 1$ are zero at some $x_0 \in \mathcal{H}$ with $h(x_0) = \max_{x \in \mathcal{H}} h(x)$. Let u^- be the positive principal eigenfunction of \mathcal{O} with $a(\cdot) = h(\cdot)$ and $\|u^-\| = 1$. It is not difficult to verify that $u = \delta u^-$ is a sub-solution of (1.1) for any $\delta > 0$ sufficiently small. It then follows that for any $\delta > 0$ sufficiently small,

$$\delta u^-(\cdot) \leq u(t_1, \cdot; \delta u^-) \leq u(t, \cdot; \delta u^-) \quad \forall 0 < t_1 < t.$$

This implies that there is a Lebesgue measurable function $u^{-*,\delta} : \mathcal{H} \rightarrow [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$\lim_{t \rightarrow \infty} u(t, x; \delta u^-) = u^{-*,\delta}(x) \quad \forall x \in \mathcal{H}.$$

By Lemma 3.2, $u^{-*,\delta} \in X^{++}$. Therefore $u^{-*,\delta} \in X^{++}$ and $u = u^{-*,\delta}(\cdot)$ is a stationary solution of (1.1) in X^{++} . \square

Lemma 3.5. *Let $M \gg 1$ be such that $f(x, M) < 0$ for $x \in \mathcal{H}$. Then $\lim_{t \rightarrow \infty} u(t, x; u_0)$ exists for every $x \in \mathcal{H}$, where $u_0(x) \equiv M$. Moreover, $u^{+*,M}(\cdot) \in X^{++}$, where $u^{+*,M}(x) := \lim_{t \rightarrow \infty} u(t, x; u_0)$, and hence $u = u^{+*,M}(\cdot)$ is a stationary solution of (1.1) in X^{++} .*

Proof. For any $M > 1$ with $f(x, M) < 0$ for all $x \in \mathcal{H}$, $u = M$ is a super-solution of (1). Hence

$$u(t, \cdot; M) \leq u(t_1, \cdot; M) \leq M \quad \forall 0 \leq t_1 < t.$$

It then follows that $\lim_{t \rightarrow \infty} u(t, x; M)$ exists for all $x \in \mathbb{R}^N$. Let $u^{+*,M}(x) = \lim_{t \rightarrow \infty} u(t, x; M)$. We have $u^{+*,M}(x) \geq u^{-*,\delta}(x)$ for $0 < \delta \ll 1$. By the similar arguments as in Lemma 3.4, $u^{+*,M} \in X^{++}$ and $u = u^{+*,M}(\cdot)$ is a stationary solution of (1.1) in X^{++} . \square

Proof of Theorem 1.1. (1) First, by Lemmas 3.4 and 3.5, (1) has stationary solutions in X^{++} . We claim that stationary solution of (1) in X^{++} is unique. In fact, suppose that $u^{1,*}$ and $u^{2,*}$ are two stationary solutions of (1) in X^{++} . Assume that $u^{1,*} \neq u^{2,*}$. Then there is $\alpha^* > 1$ such that $\rho(u^{1,*}, u^{2,*}) = \ln \alpha^* > 0$. Note that

$$\frac{1}{\alpha^*} u^{1,*} \leq u^{2,*} \leq \alpha^* u^{1,*}.$$

By Lemma 3.3, $\lim_{\|x\| \rightarrow \infty} u^{1,*}(x) = u^0$ and $\lim_{\|x\| \rightarrow \infty} u^{2,*}(x) = u^0$. This implies that there is $\epsilon > 0$ such that

$$\frac{1}{\alpha^* - \epsilon} u^{1,*}(x) \leq u^{2,*}(x) \leq (\alpha^* - \epsilon) u^{1,*}(x) \quad \text{for } \|x\| \gg 1.$$

By Proposition 2.1 and the arguments in Proposition 2.3,

$$\frac{1}{\alpha^*} u^{1,*}(x) < u^{2,*}(x) < \alpha^* u^{1,*}(x) \quad \forall x \in \mathbb{R}^N.$$

It then follows that for $0 < \epsilon \ll 1$,

$$\frac{1}{\alpha^* - \epsilon} u^{1,*}(x) \leq u^{2,*}(x) \leq (\alpha^* - \epsilon) u^{1,*}(x) \quad \forall x \in \mathbb{R}^N$$

and then $\rho(u^{1,*}, u^{2,*}) \leq \ln(\alpha^* - \epsilon)$, this is a contradiction. Therefore $u^{1,*} = u^{2,*}$ and (1.1) has a unique stationary solution u^* in X^{++} . \square

Proof of Theorem 1.2. For any $u_0 \in X^{++}$, there is $\delta > 0$ sufficiently small and $M > 0$ sufficiently large such that $\delta u^- \leq u_0 \leq M$ and $u = \delta u^-$ is a sub-solution of (1) (u^- is as in Lemma 3.4) and $u = M$ is a super-solution of (1.1). Then

$$\delta u^- \leq u(t, \cdot; \delta u^-) \leq u(t, \cdot; u_0) \leq u(t, \cdot; M) \leq M \quad \forall t \geq 0.$$

By Theorem 1.1, Lemmas 3.4 and 3.5, and Dini's Theorem,

$$u(t, x; \delta u^-) < u^*(x) < u(t, x; M) \quad \forall t > 0, x \in \mathcal{H}$$

and

$$\lim_{t \rightarrow \infty} u(t, x; \delta u^-) = \lim_{t \rightarrow \infty} u(t, x; M) = u^*(x)$$

uniformly in x on bounded sets. It then follows that

$$\lim_{t \rightarrow \infty} u(t, x; u_0) = u^*(x)$$

uniformly in x on bounded sets.

We claim that $\|u(t, \cdot; u_0) - u^*(\cdot)\| \rightarrow 0$ as $t \rightarrow \infty$. Assume the claim is not true. Then there are $\epsilon_0 > 0$, $t_n \rightarrow \infty$, and x_n with $\|x_n\| \rightarrow \infty$ such that

$$|u(t_n, x_n; u_0) - u^*(x_n)| \geq \epsilon_0 \quad \forall n \in \mathbb{N}.$$

Then by Lemma 3.3,

$$|u(t_n, x_n; u_0) - u^0| \geq \frac{\epsilon_0}{2} \quad \forall n \gg 1.$$

Let $\tilde{\delta} > 0$ and $\tilde{M} > 0$ be such that

$$\tilde{\delta} \leq u(t, \cdot; u_0) \leq \tilde{M} \quad \forall t \geq 0.$$

For any $\epsilon > 0$, let $T > 0$ be such that

$$|u(T, \cdot; \tilde{\delta}, f^0(\cdot)) - u^0| < \epsilon, \quad |u(T, \cdot; \tilde{M}, f^0(\cdot)) - u^0| < \epsilon. \quad (3.1)$$

Observe that

$$\tilde{\delta} \leq u(t_n - T, x_n + x; u_0) \leq \tilde{M}$$

and

$$u(t_n, x_n + \cdot; u_0) = u(T, x_n + \cdot; u(t_n - T, \cdot; u_0)) = u(T, \cdot; u(t_n - T, \cdot + x_n; u_0), f(\cdot + x_n, \cdot))$$

for $n \gg 1$. Then

$$u(T, \cdot; \tilde{\delta}, f(\cdot + x_n)) \leq u(t_n, x_n + \cdot; u_0) \leq u(T, \cdot; \tilde{M}, f(\cdot + x_n, \cdot)). \quad (3.2)$$

Observe also that $f(x + x_n, u) \rightarrow f^0(u)$ as $n \rightarrow \infty$ uniformly in (x, u) on bounded sets. Then

$$u(T, x; \tilde{\delta}, f(\cdot + x_n, \cdot)) \rightarrow u(T, x; \tilde{\delta}, f^0(\cdot))$$

and

$$u(T, x; \tilde{M}, f(\cdot + x_n, \cdot)) \rightarrow u(T, x; \tilde{M}, f^0(\cdot))$$

as $n \rightarrow \infty$ uniformly in x on bounded sets. This together with (3.1) implies that

$$|u(T, 0; \tilde{\delta}, f(\cdot + x_n, \cdot)) - u^0| < 2\epsilon, \quad |u(T, 0; \tilde{M}, f(\cdot + x_n, \cdot)) - u^0| < 2\epsilon \quad \text{for } n \gg 1$$

and then by (3.2),

$$|u(t_n, x_n; u_0) - u^0| < 2\epsilon \quad \text{for } n \gg 1.$$

Hence $\lim_{n \rightarrow \infty} u(t_n, x_n; u_0) = u^0$, which is a contradiction. Therefore $\|u(t, \cdot; u_0) - u^*(\cdot)\| \rightarrow 0$ as $t \rightarrow \infty$.

□

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