On Existence and Uniqueness of Generalized Solutions for a Mixed–type Differential Equation

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Abstract

In this paper, we study a boundary value problem for a mixed–type differential equation. The existence and uniqueness of generalized solution is proved. The proof is based on an energy inequality and the density of the range of the operator generated by this problem.

Keywords: Energy inequality, Dual operator, Generalized solution

1. Introduction

Partial differential equations are the bases of almost all physical theorems. In the theory of sound in gases, liquids, and solids, in the investigations of elasticity, in optics, everywhere partial differential equations formulate basic laws of nature which can be checked against the experiments. Many problems of physical interest are also described by ordinary or partial differential equations- with appropriate initial or boundary conditions. These problems are usually formulated as initial-boundary value problems that seem to be mathematically more rigorous and physically realistic in applied and engineering sciences, for example, see (Maazzucato, 2003) and (Bouziani and Benouar, 2002). The energy inequality and the density of the range are particularly useful for proving the existence and uniqueness of generalized solutions, see for example (Bougoffa, 1999), (Bougoffa and Moulay, 2003) and (Shi, 1993).

In recent years, special equations of composite as well as mixed type have received many attention in several papers. Most of the papers were directed to parabolic-elliptic equations, and to hyperbolic-elliptic equations. Similarly, existence and uniqueness of generalized solutions for composite type was also discussed in (Tarig, 2009).

In this study we prove a priori estimates and derive from the existence and uniqueness of generalized solutions for mixed type equations where proof is based on an energy inequality and density of the range. First of all we introduce appropriate Sobolev spaces and investigate the corresponding linear problem, see (Adam, 1975).

Let Ω be a bounded domain in $\mathbb{R}^n$ and $x = (x_1 ... x_n)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$ and $Q = \Omega \times (0, T)$. Then we consider the following equation:

$$lu = \left(\frac{\partial^2}{\partial t^2} - \Delta\right)\left(\frac{\partial^2 u}{\partial t^2} + \Delta u\right) + \Delta\left(\frac{\partial u}{\partial u}\right) = f(t, x). \tag{1.1}$$

The initial conditions given by

$$u(0, x) = \frac{\partial u}{\partial t}(0, x) = \frac{\partial^2 u}{\partial t^2}(0, x) = 0, \ \forall x \in \Omega \tag{1.2}$$

the final condition:

$$u(T, x) = 0, \quad \forall x \in \Omega \tag{1.3}$$

and the boundary conditions:

$$u|_S = \frac{\partial u}{\partial r} = 0 \tag{1.4}$$

where $S = \Gamma \times (0, T)$, r is the unit exterior vector, and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Analogous to the problem (1.1) – (1.4), we consider its dual problem. First of all, we denote by the formal dual of the operator l which is defined with respect to the inner product in the space $L_2(Q)$ by using

$$(lu, v) = (u, l^{\star} v), \ \forall u, v \in C_0(Q) \tag{1.5}$$

where (., .) stands for the inner product in $L_2(Q)$. Then the problem (1.1)–(1.4) is transformed to the following the dual equations:

$$l^{\star} v = \left(\frac{\partial^2}{\partial t^2} - \triangle\right)\left(\frac{\partial^2 v}{\partial t^2} + \triangle v\right) - \triangle\left(\frac{\partial v}{\partial t}\right) = g(t, x) \tag{1.6}$$

with the initial condition:

$$v(0, x) = 0 \quad \forall x \in \Omega \tag{1.7}$$

the final conditions:

$$v(T, x) = \frac{\partial v}{\partial t}(T, x) = \frac{\partial^2 v}{\partial t^2}(T, x) = 0, \quad \forall x \in \Omega \tag{1.8}$$

and the boundary conditions:

$$v|_S = \frac{\partial v}{\partial t}|_S = 0. \tag{1.9}$$

2. Functional Spaces

In order to solve the new equation (1.6) we give the following notations. Now the set of functions belonging to $L_2(Q)$ and having all generalized derivatives up to order k, $k \geq 1$ (belonging to $L_2(Q)$) will be denoted by $H^k(Q)$. In particular, if $k = 0$, $H^k(Q) = L_2(Q)$ $\left(H^0(Q) = L_2(Q)\right)$. It is clear that $H^k(Q)$ is a linear space, and further Hilbert space with the following scalar product,

$$(f, g)_{H^k(Q)} = \sum_{|\alpha| \leq k}^{n} \int_Q D^{\alpha} f \, D^{\alpha} \bar{g} dx.$$

Further, suppose that a function $f(x)$ is defined in a region Q and the region Q' contains Q. A function $F(x)$ defined in $Q'$and coinciding $f(x)$ with in Q is called an extension of $f(x)$ into Q'. The related following theorem was proved in (Mikhailov, 1978).

Theorem (2.1):

Let Q, Q' be bounded regions such that $Q \subset Q'$ and $\partial Q \in C^k$. Then the function $f(x) \in H^k(Q)$ has an extension $F(x) \in H^k(Q')$ into Q' with compact support. Further satisfies

$$\|F\|_{H^k(Q')} \leq c \, \|f\|_{H^k(Q)}$$

where $c > 0$ depends only on Q and Q'.

Definition (2.1):

(i) Let be an $(n-1)$ dimension surface lying in $\bar{Q}$. We define subset of functions belonging to $C^k(\bar{Q})$ that vanish on the intersection Q of with some neighborhood of S . (Every function has its own neighborhood). Then the closure of $C_0^k|_S(\bar{Q})$ in the norm of $H^k(Q)$ is a subspace of $H^k(Q)$, this will be denoted by $H_0^k|_S(Q)$. If we set $S = \partial Q$, the subspace $H_0^k|_{\partial Q}$ will be denoted by $H_0^k|_S(Q)$ [the norm in $H_0^k|_S(Q)$ is the norm of $H^k(Q)$].

(ii) The closure of $C_c^1(\Omega)$ with respect to the norm in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. [$\bar{C}_c^1(\Omega) = H_0^1(\Omega)$]. So that $H_0^1(\Omega) = \left\{u \in H^1(\Omega), \ u|_\Gamma = 0, \Gamma = \partial\Omega \quad (\text{ boundary})\right\}$ or

$$H_0^1(\Omega) = \left\{u \in L_2(\Omega), \ \frac{\partial u}{\partial x_i} \in L_2(\Omega), \ u|_\Gamma = 0, \ \ 1 \leq i \leq n\right\}.$$

Then one can easily see that $H_0^1(\Omega) \subset H^1(\Omega)$. Now we let $H^{r,0}(Q_T)$ be the set of all functions $f(x, t)$ that together with their generalized derivatives $\frac{\partial^{\alpha_1 + \ldots + \alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ for all (nonnegative integers), $\alpha_1, \ldots, \alpha_n$, $\alpha_1 + \ldots + \alpha_n \leq r$, belong to $L_2(Q_T)$ where the integer $r \geq 1$. Similarly, the $H^{2s,s}(Q_T)$, denotes the set of all function $f(x, t)$ that together with the generalized derivatives $\frac{\partial^{\alpha_1 + \ldots + \alpha_n + \beta} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}}$ for all (nonnegative integers), $\alpha_1, \ldots, \alpha_n, \beta$, $\alpha_1 + \ldots + \alpha_n + 2\beta \leq 2s$, belong to $L_2(Q_T)$ where the integer $s \geq 1$. Then the space $H^{r,0}(Q_T)$ with $r = 0$ and the space $H^{2s,s}(Q_T)$ with $s = 0$ will be reduced to the space $H^{0,0}(Q_T) = L_2(Q_T)$.

Thus the domain $D(l)$ of the operator l is given by $D(l) = H_0^{4,4}(Q)$ The subspace $H^{4,4}(Q)$ of which consists of all the functions $u \in H^{4,4}(Q)$ satisfying the conditions (1.2)–(1.4).

Similarly, the domain of $l^\star$ is defined by $D(l^\star) = H_0^{4,4}(Q)$ which consists of all functions $v \in H_0^{4,4}(Q)$ satisfying the conditions (1.7)–(1.9).

Now, let $H_0^{-2,-2}(Q)$ be the dual space of $H_0^{2,2}(Q)$ with respect to canonical bilinear form (u, v) where $u \in H_0^{-2,-2}(Q)$ and $v \in H_0^{2,2}(Q)$, which is the extension by continuity of the bilinear form (u, v) where $u \in L_2(Q_T)$ and $v \in H_0^{2,2}(Q)$.

3. A priori Estimates

The solution of the problem (1.1)–(1.4) will be considered as a solution of the operational equation:

$$lu = f, \quad u \in D(l) \tag{3.1}$$

and the solution of the problem (1.6) – (1.9) as a solution of the following equation:

$$l^\star v = g, \quad v \in D(l^\star) \tag{3.2}$$

to solve the equation (3.1) for every $f \in H_0^{-2,-2}(Q)$, we construct an extension L of the operator l, whose range $L(R)$ coincides with $H_0^{-2,-2}(Q)$, thus L is invertible.

A function $u \in H_0^{2,2}(Q)$ will be considered as an element of $D(L)$, if the application:

$$v \to \varphi(u, v) = (u, l^\star v)$$

is a continuous linear functional on $H_0^{4,4}(Q)$ and dense within $H_0^{2,2}(Q)$ in which induced topology has been defined.

Therefore by virtue of the Riez's theorem, there exists a unique element $Lu \in H_0^{2,2}(Q)$ such that:

$$\varphi(u, v) = (Lu, v) \text{ for every } u \in D(L), \text{ and for every } v \in H_0^{2,2}(Q).$$

In the same manner, we can also construct, through the bilinear form:

$$\Psi(u, v) = (lu, v)$$

where the extension L^* of the operator l^*

Definition (3.1):

The solution of the operational equation:

$$Lu = f, \quad u \in D(L) \tag{3.3}$$

is called the generalized solution of problem (1.1)–(1.4), and similarly, the solution of the operational equation:

$$L^* u = g, \qquad v \in D(L^*) \tag{3.4}$$

is called generalized solution of the problem (1.6)–(1.9). Then we have the following theorem.

Theorem (3.2):

For problems (1.1)–(1.4) and (1.6)–(1.9) we have the following a priori estimates:

$$||u||_{2,2} \leq c\, ||u||_{-2,-2}\,, \quad \forall u \in D(L) \tag{3.5}$$

$$||v||_{2,2} \leq c^*\, ||L^* v||_{-2,-2}\,, \quad \forall v \in D(L^*) \tag{3.6}$$

where the constants $c > 0$ and $c^* > 0$ are independent of u and v.

Proof: Firstly we prove the inequality (3.5) for the function $u \in D(L)$.

In this case we have:

$$\phi(u, v) = (lu, v), \tag{3.7}$$

setting

$$Mu = (t - T)\frac{\partial u}{\partial t} + (t - T)u,$$

then we have

$$\int_Q luMudx = \int_Q f(x,t)Mudtdx.$$

On using the integration by parts into the domain and making use of the conditions (1.2)–(1.4) we obtain:

$$(lu, Mu) = 2\int_Q (T-t)\left(\frac{\partial u}{\partial t}\right)^2 dtdx + 2\int_Q (T-t)(\triangle u)^2 dtdx + \int_Q (T-t)^2 (\nabla u_t)^2 dtdx + \frac{1}{2}\int_Q (\nabla u)^2 dtdx \tag{3.8}$$

where $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$. For the function $u \in D(L)$ we have the following Poincare estimates:

$$\int_Q (u)^2 dtdx \leq 4T^2 \int_Q (u_t)^2 dtdx, \quad \forall u \in D(L)$$

$$\int_Q (u_t)^2 dtdx \leq 4T \int_Q (T-t)(u_{tt})^2 dtdx, \quad \forall u \in D(L)$$

$$\int_Q (\nabla u)^2 dtdx \leq 4T \int_Q (T-t)(\nabla u_t)^2 dtdx, \quad \forall u \in D(L). \tag{3.9}$$

Now apply $\varepsilon-$ inequality to the left hand side of (3.8) and using inequalities (3.9) we obtain (3.5) for $u \in D(L)$.

4. Solvability of the problem

Theorem (4.1):

For each function $f \in H_0^{-2,-2}(Q)$ [resp. $g \in H_0^{-2,-2}(Q)$] , there is a unique solution of the problem (1.1)–(1.4) [resp. (1.6) – (1.9)].

Proof:

In order to prove this theorem we shall first extend l to L such that $R(L) \in H_0^{-2,-2}(Q)$ so that L should be invertible, to accomplish this we need:

(i) Definition of extension l to L (using Riez's theorem).

(ii) Prove that is closed and dense in $H_0^{-2,-2}(Q)$ so that $R(L) = H_0^{-2,-2}(Q)$.

The uniqueness of the solution follow immediately on using the inequality (3.5). It also ensures the closure of the range set $R(L)$ of the operator L.

To prove that $\overline{R(L)}$ equals the space $H_0^{-2,-2}(Q)$, we need to obtain the inclusion $\overline{R(L)} \subseteq R(L)$, and $R(L) = H_0^{-2,-2}(Q)$.

Indeed, let $\{f_k\}_{k\in N}$ be a Cauchy sequence in the space $H_0^{-2,-2}(Q)$, which consists of element of set $R(L)$. Then it corresponds to a sequence $\{u_k\}_{k\in N} \subseteq D(L)$ such that:

$$Lu_k = f_k, \quad k \in N.$$

From the energy inequality (3.5) we have

$$\|u_k - u_n\| \leq c\,\|L(u_k - u_n)\| - c\,\|Lu_k - Lu_n\| = c\,\|f_k - f_n\| \leq \varepsilon.$$

Then $\{u_k\}_{k\in N}$ is also a Cauchy sequence in the space $H_0^{2,2}(Q)$, and converges to an element u in $H_0^{2,2}(Q)$. For all $k \in N$, the functional

$$v \to \varphi(u_k, v).$$

Thus if we define the element

$$Lu = f(f = \lim_{k\to\infty} f_k)$$

this means that $f \in R(L)$ and $\overline{R(L)} \subseteq R(L)$. Then $R(L)$ is closed.

Now we will prove that $R(L)$ is dense in the space $H_0^{-2,-2}(Q)$ when $u \in D(L)$. Therefore, we prove an equivalent result that

$$R(L)^{\perp} = \{0\}.$$

Let $v \in H_0^{-2,-2}(Q)$ be such that $(Lu, v) = 0, \forall u \in D(L)$. By virtue of the equality of the form (1.5), we have:

$$(u, l^{\star}v) = 0 \text{ for all } u \in D(L) \text{ and } v \in H_0^{-2,-2}(Q).$$

From the last equality, by virtue of the estimation (3.6) (energy inequality), we conclude that $v = 0$ in the space $H_0^{-2,-2}(Q)$ when $u \in D(L)$.

The second part of the theorem can be proved in a similar way by using the following multiplier

$$M^{*}v = -t^2 \frac{\partial v}{\partial t} = tv.$$

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