Super Lehmer-3 Mean Labeling

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Abstract

Let f:V(G)→{1,2,…,p+q} be an injective function. The induced edge labeling \( f^*(e=uv) \) is defined by
\[
\frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2}
\]
(or \( \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \)), then \( f \) is called Super Lehmer-3 mean labeling, if \( \{ f(V(G)) \} \cup \{ f(e)/e \in E(G) \} =\{1,2,3,\ldots,p+q\} \). A graph which admits Super Lehmer-3 Mean labeling is called Super Lehmer-3 Mean graph.

In this paper we prove that Path, Comb, Ladder, Crown are Super Lehmer-3 mean graphs.

Keywords: graph, Lehmer-3 mean graph, Super Lehmer-3 mean graph, Path, Comb, Ladder, Kite, Crown.

1. Introduction

A graph considered here are finite, undirected and simple. The vertex set and the edge set of a graph is denoted by \( V(G) \) and \( E(G) \) respectively. Lehmer mean is another type of generalized mean. A path of length \( n \) is denoted by \( P_n \). For standard terminology and notations we follow Harary (1988) and for the detailed survey of graph labeling we follow J.A. Gallian (2010). S.Somasundaram, S.S Sandhya and R.Ponraj introduced the concept of Harmonic Mean Labeling of Graphs in (Somasundaram, Ponraj, & Sandhya) and its basic results was proved in (Somasundaram, Ponraj, & Sandhya). We will provide a brief summary of other in formations which are necessary for our present investigation.

\textbf{Definition 1.1}

A graph \( G=(V,E) \) with \( p \) vertices and \( q \) edges is called \textbf{Lehmer-3 mean graph}. If it is possible to label vertices \( x \in V \) with distinct labels \( f(x) \) from \( 1, 2, 3,\ldots,q+1 \) in such a way that when each edge \( e=uv \) is labeled with \( f(e=uv)=\frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \) (or \( \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \)), then the edge labels are distinct. In this case “f” is called Lehmer-3 mean labeling of \( G \).

\textbf{Definition 1.2}

A Path \( P_n \) is obtained by joining \( u_i \) to the consecutive vertices \( u_{i+1} \) for \( 1\leq i \leq n \)

\textbf{Definition 1.3}

A graph obtained by joining a single pendant edge to each vertex of a path is called a \textbf{comb}.

\textbf{Definition 1.4}

A product graph \( P_n \times P_n \) is called a planar grid \( P_2 \times P_n \) is called a \textbf{Ladder}.

\textbf{Definition 1.5}

\textbf{Crown} is a graph obtained by joining a single pendant edge to each vertex of a cycle.

\textbf{Definition 1.6}

The \textbf{corona} of two graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \circ G_2 \) formed from one copy of \( G_1 \) and \(|V(G_1)| \) copies of \( G_2 \), where the \( i^{th} \) vertex of \( G_1 \) is adjacent to every vertex in the \( i^{th} \) copy of \( G_2 \).
2. Main Results

**Theorem: 2.1**

A Path $P_n$ is a Super Lehmer-3 mean graph.

**Proof:**

Let $P_n$ be a Path $v_1, v_2, \ldots, v_n$ with edge set $E=\{v_iv_{i+1}/1 \leq i \leq n-1\}$

Define a function $f: V(P_n) \rightarrow \{1,2,\ldots,p+q\}$ by

$$f(v_i) = 2i - 1; \ 1 \leq i \leq n.$$ 

Then the induced edge labels are

$$f^*(v_iv_{i+1})=2i; \ 1 \leq i \leq n-1$$

Therefore $f(V(P_n))U{f(e)} = \{1,2,3,\ldots,p+q\}$

Hence $P_n$ is a Super Lehmer-3 mean graph

**Example: 2.2**

A Super Lehmer-3 mean labeling of $P_6$ is given below.

![Figure 1](http://jmr.ccsenet.org)

**Theorem: 2.3**

$(P_n \circ K_1)$ is a Super Lehmer-3 mean graph.

**Proof:**

Let $G$ be a Comb obtained from a path $P_n = v_1, v_2, \ldots, v_n$ by joining the vertex $v_i$ to $u_i$ where $1 \leq i \leq n$ and hence the edge set is

$$E=\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_{i+1}/1 \leq i \leq n\}$$

Define a function $f: V(G) \rightarrow \{1,2,\ldots,p+q\}$ by

$$f(u_i) = 4i - 3; \ 1 \leq i \leq n$$

$$f(v_i) = 4i - 1; \ 1 \leq i \leq n$$

Thus the edges are labeled with

$$f^*(u_iu_{i+1})=4i; \ 1 \leq i \leq n-1$$

$$f^*(u_iv_{i+1})=4i-2; \ 1 \leq i \leq n$$

Therefore $f(V(G))U \{f(e)/e \in E(G)\} = \{1,2,3,\ldots,p+q\}$

Thus $f$ is a Super Lehmer-3 mean graph

**Example: 2.4**

A Super Lehmer-3 mean labeling of $P_6 \circ K_1$ is drawn above

![Figure 2](http://jmr.ccsenet.org)
Theorem: 2.5
A Ladder is a Super Lehmer-3 mean graph.

Proof:
Let G be a ladder L_n obtained from a path P_n=v_1,v_2,......v_n and u_1,u_2,......u_n joining u_i to v_i and u_i to u_{i+1}, v_i to v_{i+1}.
Define a function f:V(G)→{1,2,......p+q} by
\[ f(v_i) = v_1 + (3i-3); \quad 2 \leq i \leq n, \]
where \( v_1 \) denote the first vertex label of path \( v_i \)
Thus we get distinct edge labels.
Therefore \( f(V(G) \cup \{f(e)/e \in E(G)\}) = \{1,2,3,......p+q\} \)
Hence \( f \) is a Super Lehmer-3 mean graph.

Example: 2.6
\( L_3 \) is a Super Lehmer-3 mean graph.

\[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12 \\
13 & 14 & 15 \\
16 & 17 & 18 \\
19 & 20 & 21 \\
22 & 23 \\
\end{array} \]

Figure 3.

Theorem: 2.7
Let G be a graph obtained by identifying a pendant vertex \( P_n \) and an end vertex \( C_3 \). Then G admits a Super Lehmer-3 mean labeling.

Proof:
Let \( P_n \) be a path \( u_1,u_2,......u_n \) and uvw be a cycle \( C_3 \). Identify u with \( u_0 \). Then the resultant graph is G.
Define a function \( f: V(G) \rightarrow \{1,2,......p+q\} \) by
\[ f(u_i) = 2i-1; \quad 1 \leq i \leq n \]
\[ f(v) = 2n+1 \]
\[ f(w) = 2n+4 \]
Thus the edges are labeled with
\[ f^*(u_iu_{i+1}) = 2i; \quad 1 \leq i \leq n-1 \]
\[ f^*(u_0v) = 2n \]
\[ f^*(u_0w) = 2n+2 \]
\[ f^*(vw) = 2n+3 \]
hence by the above labeling pattern \( \{f(V(G) \cup \{f(e)/e \in E(G)\}) = \{1,2,3,......p+q\} \)
Thus G admits a Super Lehmer-3 mean labeling.

Example: 2.8
A Super Lehmer-3 mean labeling of G when n=6 is given below
**Theorem: 2.9**

$C_n \oplus K_1$ is a Super Lehmer-3 mean graph

**Proof:**

Let $u_1, u_2, u_3, \ldots, u_n, u_1$ be a cycle of $n$ vertices. Add a new vertex $v_i$ such that $v_i$ is adjacent to $u_i$, $1 \leq i \leq n$. Then define a function $f: V(C_n \oplus K_1) \rightarrow \{1, 2, \ldots, p+q\}$ by

- $f(u_1) = 3$
- $f(u_i) = 4i - 3; \ 2 \leq i \leq n - 1$
- $f(u_n) = 4n - 2$
- $f(v_1) = 1$
- $f(v_i) = 4i - 1; \ 2 \leq i \leq n - 1$
- $f(v_n) = 4n - 1$

Then the edges are labeled with

- $f^*(u_iu_{i+1}) = 4i; \ 1 \leq i \leq n - 1$
- $f^*(u_1u_n) = 4n - 3$
- $f^*(u_iv_i) = 4i - 2; \ 1 \leq i \leq n - 1$
- $f^*(u_nv_n) = 4n - 1$

Thus vertices and edges together get distinct labels from $\{1, 2, 3, \ldots, p+q\}$

Hence $C_n \oplus K_1$ is a Super Lehmer-3 mean graph

**Example: 2.10**

The labeling pattern of $C_6 \oplus K_1$ is

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Figure 4.

Figure 5.
Theorem: 2.11

\(nP_m\) is a Super Lehmer-3 mean graph

*Proof:*

Let \(v_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\) be the vertices of \(nP_m\). Then the edge set is \(E=\{v_{ij}v_{ije} i \leq n, 1 \leq j \leq m-1\}\).

Define a function \(f:V(nP_m)\to\{1, 2, \ldots, p+q\}\) by

\[f^*(v_{ij}) = (2m-1)(i-1)+(2j-1); 1 \leq i \leq n, 1 \leq j \leq m\]

Thus the induced edge labels are

\[f^*(v_{ij}v_{ije}) = (2m-1)(i-1)+(2j); 1 \leq i \leq n, 1 \leq j \leq m-1\]

Thus \(f\) provides a Super Lehmer-3 mean labeling of \(nP_m\).

*Example: 2.12*

A Super Lehmer-3 mean labeling of \(5P_6\) is given below

![Figure 6](http://jmr.ccsenet.org)

Theorem: 2.13

\((P_n\cup K_1)\cup P_m\) is a Super Lehmer-3 mean graph.

*Proof:*

Let \(G\) be a graph obtained by the union of \((P_n\cup K_1)\) and \(P_m\). Let \((P_n\cup K_1)\) be a graph with \(n\) vertices \(u_1, u_2, \ldots, u_n\) and \(v_1, v_2, \ldots, v_n\) respectively. Let the vertices of \(P_m\) be \(w_1, w_2, \ldots, w_m\).

Define a function \(f:V(G)\to\{1, 2, \ldots, p+q\}\) by

\[f(u_i) = 4i-3; 1 \leq i \leq n\]
\[f(v_i) = 4n-1; 1 \leq i \leq n\]
\[f(w_j) = 2j+1; 1 \leq j \leq m\]

Then the induced edge labels are

\[f^*(u_iu_{i+1}) = 4i-2; 1 \leq i \leq n\]
\[f^*(u_i v_i) = (2j+1); 1 \leq j \leq m-1\]

Thus the vertices and edges together get distinct labels from \(\{1, 2, \ldots, p+q\}\).

This provides a Super Lehmer-3 mean labeling for \((P_n\cup K_1)\cup P_m\).

*Example: 2.14*

The Super Lehmer-3 mean labeling of \((P_n\cup K_1)\cup P_3\) is
Theorem: 2.15

(Kite) \( P_m \) is a Super Lehmer-3 mean graph.

Proof:

Let \( G \) be a graph obtained from the union of kite and path

The vertices of kite be \( u_1, u_2, \ldots, u_n \) and \( uvw \). Identify \( u \) with \( u_n \), \( uvw \) be a cycle

Let \( P_m \) be a graph with \( m \) vertices. Then the resultant graph is \( G \)

Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
  f(u_i) &= 2i - 1; \quad 1 \leq i \leq n \\
  f(v) &= 2n + 1 \\
  f(w) &= 2n + 4 \\
  f(x_j) &= f(w) + (2j - 1); \quad 1 \leq j \leq m
\end{align*}
\]

Thus the edges are labeled with

\[
\begin{align*}
  f^*(u_iu_{i+1}) &= 2i; \quad 1 \leq i \leq n - 1 \\
  f^*(u_nv) &= 2n \\
  f^*(u_hw) &= 2n + 2 \\
  f^*(vw) &= 2n + 3 \\
  f^*(x_jx_{j+1}) &= f(wv) + (2j - 1); \quad 1 \leq j \leq m
\end{align*}
\]

By the above labeling pattern \( \{f(V(G)) \cup f(e)/e \in E(G)\} = \{1, 2, 3, \ldots, p+q\} \)

Hence \( G \) admits a Super Lehmer-3 mean labeling.

Example: 2.16

The Super Lehmer-3 mean labeling pattern is given below

Theorem: 2.17

\((C_n \circ K_1) \cup P_m\) is a Super Lehmer-3 mean graph

Proof:

Let \( u_1, u_2, u_3, \ldots, u_n, u_1 \) be the vertices of a cycle \( C_n \). Add a new vertices \( v_i \) such that \( v_i \) is adjacent to \( u_i \), \( 1 \leq i \leq n \).

Let \( P_m \) be a path with \( m \) vertices

Define a function \( f: V((C_n \circ K_1) \cup P_m) \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
  f(u_1) &= 3 \\
  f(u_i) &= 4i - 3; \quad 2 \leq i \leq n - 1
\end{align*}
\]
Thus vertices and edges together get distinct labels from \{1, 2, 3 \ldots p+q\}.

Hence \((C_n \cup K_1) \cup P_m\) is a Super Lehmer-3 mean graph

**Example: 2.18**

A Super Lehmer-3 mean labeling of \((C_6 \cup K_1) \cup P_5\) is

![Figure 9.](image_url)

**Theorem: 2.19**

\((C_n \cup K_1) \cup (P_m \cup K_1)\) is a Super Lehmer-3 mean graph

**Proof:**

Let \(u_1, u_2, u_3, \ldots, u_n, u_1\) be a cycle \(C_n\). Add a new vertices \(v_i\) such that \(v_i\) is adjacent to \(u_i\), \(1 \leq i \leq n\).

Let \((P_m \cup K_1)\) is a comb of \(m\) vertices \(w_1, w_2, \ldots, w_m\); \(x_1, x_2, \ldots x_m\) respectively.

The graph \(G\) is defined by a function \(f: V(G) \rightarrow \{1, 2, \ldots, p+q\}\) by

\[
\begin{align*}
f(u_1) &= 3 \\
f(u_i) &= 4i-3; \ 2 \leq i \leq n-1 \\
f(u_n) &= 4n-2 \\
f(v_1) &= 1 \\
f(v_i) &= 4i-1; \ 2 \leq i \leq n-1 \\
f(v_n) &= 4n \\
f(w_j) &= f(v_n) + (4j-3); \ 1 \leq j \leq m \\
f(x_j) &= f(v_n) + (4j-1); \ 1 \leq j \leq m
\end{align*}
\]

Then the vertices and edges together get distinct labels from \{1, 2, 3 \ldots p+q\}.

Hence \((C_n \cup K_1) \cup (P_m \cup K_1)\) forms a Super Lehmer-3 mean graph

**Example: 2.20**

A Super Lehmer-3 mean labeling of \((C_6 \cup K_1) \cup P_5\) is
3. Conclusion

Hence the union of two Super Lehmer-3 mean graph is again a Super Lehmer-3 mean graph.

References


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