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Dimension Formulae for the Polynomial Algebra as a Module over the Steenrod Algebra in Degrees Less than or Equal to 12

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Abstract

Let $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ be the polynomial algebra in n variables x_i, of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well known rules. A major problem in algebraic topology is that of determining $\mathcal{A}^+\mathbf{P}(n)$, the image of the action of the positively graded part of $\mathcal{A}$. We are interested in the related problem of determining a basis for the quotient vector space $\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(\mathbf{n})$. Both $\mathbf{P}(n) = \bigoplus_{d\geq 0}\mathbf{P}^d(n)$ and $\mathbf{Q}(n)$ are graded, where $\mathbf{P}^d(n)$ denotes the set of homogeneous polynomials of degree d. In this paper we give explicit formulae for the dimension of $\mathbf{Q}(n)$ in degrees less than or equal to 12.

Keywords: Steenrod squares, polynomial algebra, hit problem.

1. Introduction

For $n \geq 1$, let $\mathbf{P}(n)$ be the mod-2 cohomology group of the n-fold product of $\mathbb{R}P^\infty$ with itself. Then $\mathbf{P}(n)$ is the polynomial algebra

$$\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$$

in n variables x_i, each of degree 1, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ is the graded associative algebra generated over $\mathbb{F}_2$ by symbols Sq^i for $i \geq 0$, called Steenrod squares subject to the Adem relations (Adem 1957) and $Sq^0 = 1$. Let $\mathbf{P}^d(n)$ denote the homogeneous polynomials of degree d. The action of the Steenrod squares $Sq^i : \mathbf{P}^d(n) \to \mathbf{P}^{d+i}(n)$ is determined by the formula:

$$Sq^i(u) = \begin{cases} u, & i = 0 \\ u^2, & \deg(u) = i \\ 0, & \deg(u) < i, \end{cases}$$

and the Cartan formula

$$Sq^i(uv) = \sum_{r=0}^{i} Sq^r(u)Sq^{i-r}(v).$$

A polynomial $u \in \mathbf{P}^d(n)$ is said to be hit if it is in the image of the action of $\mathcal{A}$ on $\mathbf{P}(n)$, that is, if

$$u = \sum_{i>0} Sq^i(u_i),$$

for some $u_i \in \mathbf{P}(n)$ of degree $d - i$. Let $\mathcal{A}^+\mathbf{P}(n)$ denote the subspace of all hit polynomials. The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors, (Silverman, 1998; Singer, 1991) and (Wood, 1989). We are interested in the related problem of determining a basis for the quotient vector space

$$\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$$

which has also been studied by several authors, (Kameko, 1990; 2003; Peterson, 1987) and (Sum, 2007). Some of the motivation for studying these problems is mentioned in (Nam, 2004). It stems from the Peterson conjecture proved in (Wood, 1989) and various other sources (Peterson, 1989; Singer, 1989).

The following result is useful for determining $\mathcal{A}$-generators for $\mathbf{P}(n)$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m.

In [(Wood, 1989) Theorem 1], R.M.W. Wood proved that:

Theorem 1 (Wood, 1989)**.** *Let $u \in \mathbf{P}(n)$ be a monomial of degree d. If $\alpha(n+d) > n$, then u is hit.*

Thus $\mathbf{Q}^d(n)$ is zero unless $\alpha(n+d) \leq n$ or, equivalently, unless d can be written in the form, $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$ where $\lambda_i \geq 0$. Thus $\mathbf{Q}^d(n) \neq 0$ only if $\mathbf{P}^d(n)$ contains monomials $v = x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$ called spikes. For convenience we shall assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. We, in addition, shall consider a special one when $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0$ and $\lambda_{j-1} = \lambda_j$ only if $j = s$ or $\lambda_{j+1} = 0$. In this case v is called a **minimal spike**.

$\mathbf{Q}(n)$ has been explicitly calculated by Peterson in (Peterson, 1987) for $n = 1, 2$, by (Kameko, 1990) in his thesis, for $n = 3$ and independently by (Kameko, 2003) and (Sum, 2007) for $n = 4$. In this work we shall, unless otherwise stated, be concerned with a basis for $\mathbf{Q}(n)$ consisting of 'admissible monomials', as defined below. Thus when we write $u \in \mathbf{Q}^d(n)$ we mean that u is an admissible monomial of degree d.

We define what it means for a monomial $b = x_1^{e_1} \cdots x_n^{e_n} \in \mathbf{P}(n)$ to be admissible. Write $e_i = \sum_{j\geq 0} \alpha_j(e_i)2^j$ for the binary expansion of each exponent e_i. The expansions are then assembled into a matrix $\beta(b) = (\alpha_j(e_i))$ of digits 0 or 1 with $\alpha_j(e_i)$ in the (i, j)-th position of the matrix.

We then associate with b, two sequences,

$$\begin{aligned} w(b) &= (w_0(b), w_1(b), \ldots, w_j(b), \ldots), \\ e(b) &= (e_1, e_2, \ldots, e_n), \end{aligned}$$

where $w_j(b) = \sum_{i=1}^{n} \alpha_j(e_i)$ for each $j \geq 0$. $w(b)$ is called the **weight vector** of the monomial b and $e(b)$ is called the **exponent vector** of the monomial b.

Given two sequences $p = (u_0, u_1, \ldots, u_l, 0, \ldots)$, $q = (v_0, v_1, \ldots, v_l, 0, \ldots)$, we say $p < q$ if there is a positive integer k such that $u_i = v_i$ for all $i < k$ and $u_k < v_k$. We are now in a position to define an order relation on monomials.

Definition 1. Let a, b be monomials in $\mathbf{P}(n)$. We say that $a < b$ if one of the following holds:

1. $w(a) < w(b)$,
2. $w(a) = w(b)$ and $e(a) < e(b)$.

Note that the order relation on the set of sequences is the lexicographical one.

Following (Kameko, 1990) we define:

Definition 2. A monomial $b \in \mathbf{P}(n)$ is said to be **inadmissible** if there exist monomials $b_1, b_2, \ldots, b_r \in \mathbf{P}(n)$ with $b_j < b$ for each j, $1 \leq j \leq r$, such that

$$b \equiv \left(\sum_{j=1}^{r} b_j\right) \mod \mathcal{A}^+\mathbf{P}(n).$$

b is said to be **admissible** if it is not inadmissible.

Clearly the set of all admissible monomials in $\mathbf{P}(n)$ form a basis for $\mathbf{Q}(n)$.

To put our work in context we require some preliminary observations. For each r, $1 \leq r \leq n$, let

$$\mathbf{X}(r) = \mathrm{Span}\{x_1^{m_1} \cdots x_r^{m_r} \in \mathbf{P}(r) \mid m_1 m_2 \cdots m_r \neq 0\}.$$

Then $\mathbf{X}(r)$ is an $\mathcal{A}$-submodule of $\mathbf{P}(r)$. Let

$$\mathbf{W}(r) = \mathbf{X}(r)/\mathcal{A}^+\mathbf{X}(r).$$

Then for each $n \geq 1$ we have a direct sum decomposition:

$$\mathbf{Q}(n) \cong \bigoplus_{r=1}^{n} \bigoplus_{k=1}^{\binom{n}{r}} \mathbf{W}(r).$$

Thus for any integer $d > 0$ we have the following inexplicit formula for the dimension of $\mathbf{Q}^d(n)$:

$$\dim(\mathbf{Q}^d(n)) = \sum_{r=1}^{n} \binom{n}{r} \dim(\mathbf{W}^d(r)). \quad (1)$$

But $\mathbf{Q}(n)$ is known for , $1 \leq n \leq 4$, and in some cases when $n = 5$. This gives rise to the following explicit formula for $\mathbf{Q}^d(n)$ for $d \leq 5$.

$$\begin{aligned}
\dim(\mathbf{Q}^1(n)) &= \tbinom{n}{1}.\\
\dim(\mathbf{Q}^2(n)) &= \tbinom{n}{2}.\\
\dim(\mathbf{Q}^3(n)) &= \tbinom{n}{1} + \tbinom{n}{2} + \tbinom{n}{3}.\\
\dim(\mathbf{Q}^4(n)) &= \tbinom{n}{2}\cdot 2 + \tbinom{n}{3}\cdot 2 + \tbinom{n}{4}.\\
\dim(\mathbf{Q}^5(n)) &= \tbinom{n}{3}\cdot 3 + \tbinom{n}{4}\cdot 3 + \tbinom{n}{5}.
\end{aligned}$$

We follow the convention that $\binom{n}{i} = 0$ if $i > n$.

Our main result is Theorem 2 below which consists of explicit formulae for $\dim(\mathbf{Q}^d(n)),\ 6 \leq d \leq 12$.

Theorem 2. *For all* $n \geq 1$:

$$\begin{aligned}
\dim(\mathbf{Q}^6(n)) &= \tbinom{n}{2} + \tbinom{n}{3}\cdot 3 + \tbinom{n}{4}\cdot 6 + \tbinom{n}{5}\cdot 4 + \tbinom{n}{6}.\\
\dim(\mathbf{Q}^7(n)) &= \tbinom{n}{1} + \tbinom{n}{2} + \tbinom{n}{3}\cdot 4 + \tbinom{n}{4}\cdot 9 + \tbinom{n}{5}\cdot 10 + \tbinom{n}{6}\cdot 5 + \tbinom{n}{7}.\\
\dim(\mathbf{Q}^8(n)) &= \tbinom{n}{2}\cdot 3 + \tbinom{n}{3}\cdot 6 + \tbinom{n}{4}\cdot 13 + \tbinom{n}{5}\cdot 19 + \tbinom{n}{6}\cdot 15 + \tbinom{n}{7}\cdot 6 + \tbinom{n}{8}.\\
\dim(\mathbf{Q}^9(n)) &= \tbinom{n}{3}\cdot 7 + \tbinom{n}{4}\cdot 18 + \tbinom{n}{5}\cdot 31 + \tbinom{n}{6}\cdot 34 + \tbinom{n}{7}\cdot 21 + \tbinom{n}{8}\cdot 7 + \tbinom{n}{9}.\\
\dim(\mathbf{Q}^{10}(n)) &= \tbinom{n}{2}\cdot 2 + \tbinom{n}{3}\cdot 8 + \tbinom{n}{4}\cdot 26 + \tbinom{n}{5}\cdot 50 + \tbinom{n}{6}\cdot 65 + \tbinom{n}{7}\cdot 55\\
&\quad + \tbinom{n}{8}\cdot 28 + \tbinom{n}{9}\cdot 8 + \tbinom{n}{10}.\\
\dim(\mathbf{Q}^{11}(n)) &= \tbinom{n}{3}\cdot 8 + \tbinom{n}{4}\cdot 32 + \tbinom{n}{5}\cdot 75 + \tbinom{n}{6}\cdot 115 + \tbinom{n}{7}\cdot 120 + \tbinom{n}{8}\cdot 83\\
&\quad + \tbinom{n}{9}\cdot 36 + \tbinom{n}{10}\cdot 9 + \tbinom{n}{11}.\\
\dim(\mathbf{Q}^{12}(n)) &= \tbinom{n}{4}\cdot 21 + \tbinom{n}{5}\cdot 85 + \tbinom{n}{6}\cdot 176 + \tbinom{n}{7}\cdot 231 + \tbinom{n}{8}\cdot 203\\
&\quad + \tbinom{n}{9}\cdot 109 + \tbinom{n}{10}\cdot 45 + \tbinom{n}{11}\cdot 10 + \tbinom{n}{12}.
\end{aligned}$$

Our proof of Theorem 2 is deferred until Section 3.

Ignoring the terms for which $\mathbf{W}^d(r)$ is trivial, the above formulae yield Table 1 below for the dimension of $\mathbf{Q}^d(n)$, $1 \leq d \leq 12$. The numbers on the top row represent n, the number of variables while the serial numbers in the first column represent d, the degree of $\mathbf{Q}(n)$.

Our work is organized as follows. In Section 2, we recall some results on admissible monomials and hit monomials in $\mathbf{P}(n)$. In Section 3 we prove Theorem 2.

2. Preliminaries

In this section we recall some results in (Kameko, 1990; Silverman, 1998; Singer, 1991) and (Mothebe, 2016) on admissible monomials and hit monomials in $\mathbf{P}(n)$.

The following theorem has been used to great effect by Kameko and Sum in computing a basis for $\mathbf{Q}(3)$ and $\mathbf{Q}(4)$ respectively.

Theorem 3 (Kameko, 1990; Sum, 2010)**.** *Let a, b be monomials in $\mathbf{P}(n)$ such that $w_j(a) = 0$ for $j > r > 0$. If b is inadmissible, then ab^{2^r} is also inadmissible.*

Up to permutation of representatives weight order provides a total order relation amongst spikes in a given degree.

Table 1. Results obtained using Theorem 2

d \ n	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	0	1										
3	1	3	7									
4	0	2	8	21								
5	0	0	3	15	46							
6	0	1	6	24	74	190						
7	1	3	10	35	110	301	729					
8	0	3	15	55	174	489	1238	2863				
9	0	0	7	46	191	630	1785	4515	10438			
10	0	2	14	70	280	945	2792	7412	18020	40701		
11	0	0	8	64	315	1205	3900	11151	28917	69234	155035	
12	0	0	0	21	190	1001	3983	13209	38402	100880	243737	550847

It is easy to show that a spike $v = x_1^{2^{\lambda_1}-1}\cdots x_n^{2^{\lambda_n}-1} \in \mathbf{P}^d(n)$ is a minimal spike if its weight order is minimal with respect to other spikes of degree d. In [(Kameko 1990), Theorem 4.2] Masaki Kameko proved that:

Theorem 4 (Kameko, 1990). *Let d be a positive integer and let v be the minimal spike of degree $2d+n$. Define a linear mapping, $f : \mathbf{P}^d(n) \to \mathbf{P}^{2d+n}(n)$, by,*

$$f(x_1^{m_1}\cdots x_n^{m_n}) = x_1^{2m_1+1}\cdots x_n^{2m_n+1}.$$

If $w_0(v) = n$, then f induces an isomorphism $f_* : \mathbf{Q}^d(n) \to \mathbf{Q}^{2d+n}(n)$.

From Wood's theorem and the above result of Kameko the problem of determining $\mathcal{A}$-generators for $\mathbf{P}(n)$ is reduced to the cases for which $w_0(v) \le n-1$ whenever v is a minimal spike of a given degree d.

We recall the following result of Singer on hit polynomials in $\mathbf{P}(n)$. In [(Singer, 1991), Theorem 1.2] W. M. Singer proved that:

Theorem 5 (Singer, 1991). *Let $b \in \mathbf{P}(n)$ be a monomial of degree d, where $\alpha(n+d) \le n$. Let v be the minimal spike of degree d. If $w(b) < w(v)$, then b is hit.*

We note the following stronger version of Theorem 5. Let b be a monomial of degree d. For $l > 0$ define $d_l(b)$ to be the integer $d_l(b) = \sum_{j\ge l} w_j(b)2^{j-l}$.

In [(Silverman, 1998), Theorem 1.2], J. H. Silverman proved that:

Theorem 6 (Silverman, 1998). *Let $b \in \mathbf{P}(n)$ be a monomial of degree d, where $\alpha(n+d) \le n$. Let v be the minimal spike of degree d. If $d_l(b) > d_l(v)$ for some $l \ge 1$, then b is hit.*

We shall require the following result of (Mothebe, 2016):

Theorem 7 (Mothebe, 2016). *If $u = x_1^{m_1}\cdots x_k^{m_k} \in \mathbf{P}^d(k)$ and $v = x_1^{e_1}\cdots x_r^{e_r} \in \mathbf{P}^{d'}(r)$ are admissible monomials, then for each permutation $\sigma \in S_{k+r}$ for which $\sigma(i) < \sigma(j)$, $i < j \le k$ and $\sigma(s) < \sigma(t)$, $k < s < t \le k+r$, the monomial*

$$x_{\sigma(1)}^{m_1}\cdots x_{\sigma(k)}^{m_k} x_{\sigma(k+1)}^{e_1}\cdots x_{\sigma(k+r)}^{e_r} \in \mathbf{P}^{d+d'}(k+r)$$

is admissible.

Theorem 7 is a generalization of the following result of the (Mothebe & Uys, 2015).

Let $u = x_1^{m_1}\cdots x_{n-1}^{m_{n-1}} \in \mathbf{P}(n-1)$ be a monomial of degree d'. Given any pair of integers (j,λ), $1 \le j \le n$, $\lambda \ge 0$, let $h_j^\lambda(u)$ denote the monomial $x_1^{m_1}\cdots x_{j-1}^{m_{j-1}} x_j^{2^\lambda-1} x_{j+1}^{m_j}\cdots x_n^{m_{n-1}} \in \mathbf{P}^{d'+(2^\lambda-1)}(n)$.

Theorem 8 (Mothebe & Uys, 2015). *Let $u \in \mathbf{P}(n-1)$ be a monomial of degree d', where $\alpha(d'+n-1) \le n-1$. If u is admissible, then for each pair of integers (j,λ), $1 \le j \le n$, $\lambda \ge 0$, $h_j^\lambda(u)$ is admissible.*

3. Proof of Theorem 2

The result of Theorem 2 is a consequence of Lemma 9 and Lemma 10 which we prove below.

Lemma 9. *If* $a = x_1^{m_1}...x_n^{m_n} \in \mathbf{X}(n)$ *is an admissible monomial then* $m_1 = 2^{\lambda} - 1$ *for some* $\lambda \geq 1$.

Proof. The lemma is clearly true if $n = 1$. Suppose that $m_1 = 2^{\lambda} - 2$. Let $b = x_1^{e_1}...x_n^{e_n}$ be the monomial obtained from a by replacing m_1 by $2^{\lambda} - 3$. Then $a = Sq^1(b) + x_1^{2^{\lambda}-3}Sq^1(x_2^{m_2}...x_n^{m_n})$ and the fact that all terms in $x_1^{2^{\lambda}-3}Sq^1(x_2^{m_2}...x_n^{m_n})$ are of lower order than a shows that a is inadmissible. But every monomial with $m_1 \neq 2^{\lambda} - 1$ is of the form cd^{2^r} for some monomial $d = x_1^{t_1}...x_n^{t_n}$ with $t_1 = 2^{\lambda} - 2$ so the general result follows from Theorem 3.

Further to Lemma 9, suppose that $a = x_1^{2^{\lambda}-1}...x_n^{m_n}$ with $\lambda \geq 2$. If $m_2 = 2^k - 2$ for some $k \geq 2$, then a is inadmissible.

Recall that for each r, $1 \leq r \leq n$,

$$\mathbf{X}(r) = \text{Span}\{x_1^{m_1} \cdots x_r^{m_r} \in \mathbf{P}(r) \mid m_1 m_2 \cdots m_r \neq 0\}.$$

Then $\mathbf{X}(r)$ is an $\mathcal{A}$-submodule of $\mathbf{P}(r)$. Let

$$\mathbf{W}(r) = \mathbf{X}(r)/\mathcal{A}^+\mathbf{X}(r).$$

Then for each $n \geq 1$ we have a direct sum decomposition:

$$\mathbf{Q}(n) \cong \bigoplus_{r=1}^{n} \bigoplus_{k=1}^{\binom{n}{r}} \mathbf{W}(r).$$

Thus for any integer $d > 0$ we have the following inexplicit formula for the dimension of $\mathbf{Q}^d(n)$

$$\dim(\mathbf{Q}^d(n)) = \sum_{r=1}^{n} \binom{n}{r} \dim(\mathbf{W}^d(r)). \quad (2)$$

The following lemma evaluates Formula (2) explicitly in some cases. It gives explicit formulae for $\dim(\mathbf{W}^d(n))$ for those cases that enables us to obtain $\dim(\mathbf{Q}^d(n))$ for all values of n in the range , $1 \leq n \leq 12$.

Lemma 10.

$$\begin{aligned}
\dim(\mathbf{W}^n(n)) &= 1 && \text{for all } n \geq 1.\\
\dim(\mathbf{W}^n(n-1)) &= n-2 && \text{for all } n \geq 3.\\
\dim(\mathbf{W}^n(n-2)) &= \binom{n-2}{2} && \text{for all } n \geq 6.\\
\dim(\mathbf{W}^n(n-3)) &= \binom{n-4}{3} + (n-3)(n-5) && \text{for all } n \geq 7.\\
\dim(\mathbf{W}^n(n-4)) &= \left(\binom{n-5}{4} - 1\right) + \binom{n-4}{2} + (n-4).\binom{n-6}{2} + \binom{n-5}{2} && \text{for all } n \geq 10.\\
\dim(\mathbf{W}^n(n-5)) &= n - 6 + \frac{(n-5)!}{2\cdot(n-7)(n-9)!} + \binom{n-5}{2} + \binom{n-6}{2} + (n-5)\cdot\binom{n-7}{3} &&\\
&\quad +2\cdot\left(\sum_{i=2}^{n-8}\binom{i}{2}\right) + \binom{n-8}{2} + \left(\binom{n-7}{2} - 1\right) && 10 \leq n \leq 12.\\
\dim(\mathbf{W}^n(n-6)) &= n - 6 + ((n-7)(n-8) - 2) + \binom{n-6}{2} + \binom{n-6}{3} &&\\
&\quad +(n-6)\cdot\binom{n-8}{2} + \frac{(n-7)!}{4\cdot(n-11)!} + (n-8)\cdot\binom{n-9}{2} + \binom{n-9}{2} &&\\
&\quad +\left(\binom{n-6}{3} - 1\right) + 2\cdot\binom{n-7}{3} - (n-9) && 11 \leq n \leq 12.
\end{aligned}$$

Proof. For $n \geq 1$ the basis of $\mathbf{X}^n(n)$ consists of the monomial $x_1x_2\cdots x_{n-1}x_n$. For $n \geq 3$ the basis of $\mathbf{X}^n(n-1)$ consists of the monomial $x_1x_2\cdots x_{n-2}x_{n-1}^2$ and its permutation representatives. For $n \geq 6$ the basis of $\mathbf{X}^n(n-2)$ consists of the monomials:

$$\{a_{n-2} = x_1x_2\cdots x_{n-4}x_{n-3}^2x_{n-2}^2,\ b_{n-2} = x_1x_2\cdots x_{n-4}x_{n-3}x_{n-2}^3\}$$

and their permutation representatives. For convenience we shall ignore those monomials in $\mathbf{X}^n(n-i)$ which, by Theorem 6, are hit. For $n \geq 7$ the basis of $\mathbf{X}^n(n-3)$ consists of the monomials:

$$\{a_{n-3} = x_1x_2\cdots x_{n-6}x_{n-5}^2x_{n-4}^2x_{n-3}^2,\ b_{n-3} = x_1x_2\cdots x_{n-5}x_{n-4}^2x_{n-3}^3,\ c_{n-3} = x_1x_2\cdots x_{n-4}x_{n-3}^4\}$$

and their permutation representatives. If $n \geq 8$, the basis of $\mathbf{X}^n(n-4)$ consists of the monomials:

$$\{\ a_{n-4} = x_1 \cdots x_{n-7}x_{n-6}^2x_{n-5}^2x_{n-4}^3,\ b_{n-4} = x_1 \cdots x_{n-6}x_{n-5}^2x_{n-4}^4,\ c_{n-4} = x_1 \cdots x_{n-6}x_{n-5}^3x_{n-4}^3,\ e_{n-4} = x_1 \cdots x_{n-6}x_{n-5}x_{n-4}^5\}$$

and their permutation representatives, while

$$d_{n-4} = x_1x_2 \cdots x_{n-8}x_{n-7}^2x_{n-6}^2x_{n-5}^2x_{n-4}^2$$

and its permutation representatives have to be included in the list when $n \geq 10$.

If $n \geq 9$ the basis of $\mathbf{X}^n(n-5)$ consists of the monomials:

$$\{a_{n-5} = x_1 \cdots x_{n-8}x_{n-7}^2x_{n-6}^3x_{n-5}^3,\ b_{n-5} = x_1x_2 \cdots x_{n-7}x_{n-6}^3x_{n-5}^4,\ c_{n-5} = x_1x_2 \cdots x_{n-7}x_{n-6}^2x_{n-5}^5,\ d_{n-5} = x_1x_2 \cdots x_{n-6}x_{n-5}^6\}$$

and their permutation representatives, while

$$e_{n-5} = x_1x_2 \cdots x_{n-9}x_{n-8}^2x_{n-7}^2x_{n-6}^2x_{n-5}^3,\ f_{n-5} = x_1x_2 \cdots x_{n-8}x_{n-7}^2x_{n-6}^2x_{n-5}^4$$

and their permutation representatives have to be added to the list when $n \geq 10$ and

$$g_{n-5} = x_1x_2 \cdots x_{n-11}x_{n-10}x_{n-9}^2x_{n-8}^2x_{n-7}^2x_{n-6}^2x_{n-5}^2$$

and its permutation representatives have to be added to the list when $n \geq 13$.

If $n \geq 11$ the basis of $\mathbf{X}^n(n-6)$ consists of the monomials:

$$\{a_{n-6} = x_1 \cdots x_{n-9}x_{n-8}^3x_{n-7}^3x_{n-6}^3,\ b_{n-6} = x_1 \cdots x_{n-7}x_{n-6}^7,\ c_{n-6} = x_1 \cdots x_{n-8}x_{n-7}^3x_{n-6}^5,\ d_{n-6} = x_1 \cdots x_{n-10}x_{n-9}^2x_{n-8}^2x_{n-7}^3x_{n-6}^3,$$
$$e_{n-6} = x_1x_2 \cdots x_{n-9}x_{n-8}^2x_{n-7}^3x_{n-6}^4,\ f_{n-6} = x_1x_2 \cdots x_{n-8}x_{n-7}^2x_{n-6}^6,\ g_{n-6} = x_1x_2 \cdots x_{n-9}x_{n-8}^2x_{n-7}^2x_{n-6}^5\}$$

and their permutation representatives, while

$$h_{n-6} = x_1x_2 \cdots x_{n-11}x_{n-10}^2x_{n-9}^2x_{n-8}^2x_{n-7}^2x_{n-6}^3,\ k_{n-6} = x_1x_2 \cdots x_{n-10}x_{n-9}^2x_{n-8}^2x_{n-7}^2x_{n-6}^4$$

and their permutation representatives have to be added to the list when $n \geq 13$ and

$$l_{n-6} = x_1x_2 \cdots x_{n-12}x_{n-11}^2x_{n-10}^2x_{n-9}^2x_{n-8}^2x_{n-7}^2x_{n-6}^2$$

and its permutation representatives has to be added to the list when $n \geq 14$.

We claim that for all pairs (n, i), $1 \leq i \leq 4$, and all n as specified in Lemma 10, $\mathbf{W}^{n+1}(n-i+1)$ is the vector space sum

$$\mathbf{W}^{n+1}(n-i+1) = \sum_{j=1}^{n-i+1} h_j^1(\mathbf{W}^n(n-i)), \tag{3}$$

while this is true for $n = 10, 11$ when $i = 5$ and for $n = 11$ when $i = 6$. This shall suffice for a proof of Lemma 10 since the formulae in the lemma give the number of elements $\mathbf{W}^{n+1}(n-i+1)$ obtained from $\mathbf{W}^n(n-i)$ in this inductive manner.

For each i, $0 \leq i \leq 6$, we identify, for the initial value n_0 of n, all elements in $\mathbf{X}^{n_0}(n_0 - i)$ which are admissible, that is, we determine the admissible monomial basis for $\mathbf{W}^{n_0}(n_0 - i)$. We then show, for a given integer i, and all $n \geq n_0$, that each element of $\mathbf{X}^{n+1}(n-i+1)$ which does not belong to the set

$$\{h_j^1(x_1^{m_1} \cdots x_{n-i}^{m_{n-i}}) \mid x_1^{m_1} \cdots x_{n-i}^{m_{n-i}} \in \mathbf{W}^n(n-i),\ 1 \leq j \leq n-i+1\},$$

is inadmissible.

Clearly $\dim(\mathbf{W}^n(n)) = 1$ for all $n \geq 1$ since $x_1x_2 \cdots x_{n-1}x_n$ is the only admissible element in $\mathbf{X}^n(n)$, while $\dim(\mathbf{W}^n(n-1)) = n-2$ for all $n \geq 3$ since there are $n-2$ permutation representatives of $x_1x_2 \cdots x_{n-2}x_{n-1}^2$ which are admissible.

If $i = 2$, then $n_0 = 6$ and it is known that $\mathbf{W}^6(4)$ is generated by $x_1x_2x_3x_4^3$ and all its permutation representatives together with the monomials $x_1x_2^2x_3x_4^2$ and $x_1x_2x_3^2x_4^2$. If $n > 6$, then all permutation representatives of b_{n-2} are admissible and the only permutation representatives of a_{n-2} in $\mathbf{X}^n(n-2)$ that may not be obtained from the basis of $\mathbf{W}^6(4)$ by inductively applying Equation (3) are those of the form $x_1^2 \cdots x_{n-2}^{m_{n-2}}$ as well as the monomial $x_1x_2^2x_3^2x_4 \cdots x_{n-2}$, all of which are clearly inadmissible. Thus $\dim(\mathbf{W}^n(n-2)) = \binom{n-2}{2}$, for all $n \geq 6$, since a_{n-2}, b_{n-2} have, respectively, $\binom{n-3}{2}-1$, $n-2$ permutation representatives which are admissible and $\binom{n-3}{2} - 1 + n - 2 = \binom{n-2}{2}$.

If $i = 3$, then $n_0 = 7$ and it is known that $\mathbf{W}^7(4)$ is generated by the monomials

$$x_1x_2^2x_3^2x_4^2 \text{ and } x_1x_2x_3^2x_4^3,\ x_1x_2x_3^3x_4^2,\ x_1x_2^3x_3x_4^2,\ x_1^3x_2x_3x_4^2,\ x_1x_2^2x_3x_4^3\ x_1x_2^2x_3^3x_4,\ x_1x_2^3x_3^2x_4,\ x_1^3x_2x_3^2x_4.$$

It is easy to show that the monomial c_{n-3} and all its permutation representatives are inadmissible for all $n \geq 7$.

If $n > 7$, then the only permutation representatives of a_{n-3} and b_{n-3} in $\mathbf{X}^n(n-3)$ that may not be obtained from the basis of $\mathbf{W}^7(4)$ by inductively applying Equation (3) are those of the form $x_1^2 \cdots x_{n-3}^{m_{n-3}}$ as well as the monomial $x_1^3x_2^2x_3x_4\cdots x_{n-3}$, all of which are clearly inadmissible. Thus $\dim(\mathbf{W}^n(n-3)) = \binom{n-4}{3} + (n-3)(n-5)$, for all $n \geq 7$, since a_{n-3}, b_{n-3} have, respectively, $\binom{n-4}{3}$, $(n-3)(n-5)$ permutation representatives which are admissible.

If $i = 4$, then $n_0 = 10$. We first note that $\mathbf{W}^8(4)$ is generated by $x_1x_2x_3^3x_4^3$ and all its permutation representatives, as well as the monomials in the following sets:

$A = \{x_1x_2x_3^2x_4^4,\ x_1x_2^2x_3x_4^4,\ x_1x_2^2x_3^4x_4\}$ and $B = \{x_1^3x_2x_3^2x_4^2,\ x_1x_2^3x_3^2x_4^2,\ x_1x_2^2x_3^3x_4^2,\ x_1x_2^2x_3^2x_4^3\}$.

It is easy to show that the monomial e_{n-4} and all its permutation representatives are inadmissible for all $n \geq 8$.

If $n > 8$, then we know that all the permutation representatives of c_{n-4} are admissible. The only permutation representatives of a_{n-4} and b_{n-4} in $\mathbf{X}^n(n-4)$ that may not be obtained from the basis of $\mathbf{W}^8(4)$ by inductively applying Equation (3) are those of the form $x_1^2\cdots x_{n-4}^{m_{n-4}}$ and of the form $x_1^3x_2^2x_3^{m_3}x_4^{m_4}\cdots x_{n-4}^{m_{n-4}}$ as well as those with a factor of the form $x_i^4x_j^2$, $i < j$, all of which are clearly inadmissible. If $n = 10$, then by Theorem 7, the following permutation representatives of d_6 in $\mathbf{X}^{10}(6)$ are admissible, namely those in the set:

$C = \{x_1x_2x_3^2x_4^2x_5^2x_6^2, x_1x_2^2x_3x_4^2x_5^2x_6^2, x_1x_2^2x_3^2x_4x_5^2x_6^2, x_1x_2^2x_3^2x_4^2x_5x_6^2\}$,

The monomial $x_1x_2^2x_3^2x_4^2x_5^2x_6$ is inadmissible since

$$x_1x_2^2x_3^2x_4^2x_5^2x_6 = Sq^4(x_1x_2x_3x_4x_5x_6) + Sq^3(x_1^2x_2x_3x_4x_5x_6) + Sq^1(x_1^4x_2x_3x_4x_5x_6) + x_1x_2^2x_3^2x_4^2x_5x_6^2 + x_1x_2^2x_3^2x_4x_5^2x_6^2$$
$$+\, x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_1x_2x_3^2x_4^2x_5^2x_6^2.$$

The other permutation representatives of d_6 are inadmissible since they are of the form $x_1^2\cdots x_6^{m_6}$.

If $n > 10$, then the only permutation representatives of d_{n-4} in $\mathbf{X}^n(n-4)$ that may not be obtained from d_6 by inductively applying Equation (3) are those of the form $x_1^2\cdots x_{n-4}^{m_{n-4}}$ as well the monomial $x_1x_2^2x_3^2x_4^2x_5^2x_6\cdots x_{n-4}$, all of which are clearly is inadmissible. Thus

$$\dim(\mathbf{W}^n(n-4)) \quad = \quad \left(\binom{n-5}{4}-1\right) + \binom{n-4}{2} + (n-4).\binom{n-6}{2} + \binom{n-5}{2}, \quad \text{for all } n \geq 10,$$

since a_{n-4}, b_{n-4}, c_{n-4}, d_{n-4}, have, respectively,

$$(n-4).\binom{n-6}{2},\ \binom{n-5}{2},\ \binom{n-4}{2},\ \left(\binom{n-5}{4}-1\right)$$

permutation representatives which are admissible. If , $8 \leq n < 10$, then we omit the term $\left(\binom{n-5}{4}-1\right)$ in the expression for the value of $\dim(\mathbf{W}^n(n-4))$.

If $i = 5$, then $n_0 = 10$. We first note that $\mathbf{W}^9(4)$ is generated by monomials in the following sets:

$A = \{x_1x_2x_3x_4^6,\ x_1x_2x_3^6x_4,\ x_1x_2^6x_3x_4\}$, $\quad B = \{x_1x_2x_3^2x_4^5,\ x_1x_2^2x_3x_4^5,\ x_1x_2^2x_3^5x_4\}$,

$C = \{x_1x_2^2x_3^3x_4^3,\ x_1x_2^3x_3^2x_4^3,\ x_1x_2^3x_3^3x_4^2,\ x_1^3x_2x_3^2x_4^3, x_1^3x_2x_3^3x_4^2,\ x_1^3x_2^3x_3x_4^2\}$, and

$D = \{x_1x_2x_3^3x_4^4,\ x_1x_2^3x_3x_4^4,\ x_1x_2^3x_3^4x_4,\ x_1^3x_2x_3x_4^4, x_1^3x_2x_3^4x_4,\ x_1^3x_2^4x_3x_4\}$.

If $n > 9$, then the only permutation representative of a_{n-5} in $\mathbf{X}^n(n-5)$ that may not be obtained from the basis of $\mathbf{W}^9(4)$ by inductively applying Equation (3) is that of the form $x_1^6x_2\cdots x_{n-5}$ which is clearly inadmissible while the only permutation representatives of b_{n-5}, c_{n-5} and d_{n-5} in $\mathbf{X}^n(n-5)$ that may not be obtained from the basis of $\mathbf{W}^9(4)$ by inductively applying Equation (3) are, respectively, those with a factor of the form $x_i^5x_j^2$, $i < j$, $x_i^2x_j$, $i < j$, and $x_i^4x_j^3$, $i < j$, all of which are clearly inadmissible.

If $i = 5$ and $n = 10$, then by Theorem 7, the following permutation representatives of e_5 and f_5 are admissible in $\mathbf{X}^{10}(5)$, namely those in the sets:

$E = \{x_1^3x_2x_3^2x_4^2x_5^2,\ x_1x_2^3x_3^2x_4^2x_5^2,\ x_1x_2^2x_3^3x_4^2x_5^2,\ x_1x_2^2x_3^2x_4^3x_5^2,\ x_1x_2^2x_3^2x_4^2x_5^3\}$,

$F = \{x_1x_2x_3^2x_4^2x_5^4,\ x_1x_2x_3^2x_4^4x_5^2,\ x_1x_2^2x_3x_4^2x_5^4,\ x_1x_2^2x_3^4x_4x_5^2,\ x_1x_2^2x_3x_4^4x_5^2\}$.

Permutation representatives of e_5 which are not in E are those of the form $x_1^2\cdots x_5^{m_5}$ as well as those of the form $x_1^3x_2^2x_3^{m_3}\cdots x_5^{m_5}$, all of which are clearly inadmissible. Permutation representatives of f_5 which are not in F are those of the form $x_1^2\cdots x_5^{m_5}$ as well as those with a factor of the form $x_i^4x_j^2x_k^2,\ i < j,k$, and the three monomials in the set $G = \{x_1x_2^2x_3^4x_4^2x_5,\ x_1x_2^2x_3^2x_4^4x_5,\ x_1x_2^2x_3^2x_4x_5^4\}$, all of which are clearly inadmissible. For example

$$x_1x_2^2x_3^4x_4^2x_5 = Sq^2(x_1x_2x_3^4x_4x_5) + Sq^1(x_1^2x_2x_3^4x_4x_5) + x_1x_2^2x_3^4x_4x_5^2 + x_1x_2x_3^4x_4^2x_5^2.$$

If $n > 10$, then the only permutation representatives of e_{n-5} in $\mathbf{X}^n(n-5)$ that may not be obtained from the basis of $\mathbf{W}^{10}(5)$ by inductively applying Equation (3) are those of the form $x_1^2\cdots x_{n-5}^{m_{n-5}}$ as well as those of the form $x_1^3x_2^2x_3^{m_3}\cdots x_{n-5}^{m_{n-5}}$, all of which are clearly inadmissible. On the other hand the permutation representatives of f_{n-5} that may not be obtained from the basis of $\mathbf{W}^{10}(5)$ by inductively applying Equation (3) are those of the form $x_1^2\cdots x_{n-5}^{m_{n-5}}$ as well as those with a factor of the form $x_i^4x_j^2x_k^2,\ i < j,k$, and the three monomials in the set:

$$G = \{x_1x_2^2x_3^4x_4^2x_5\cdots x_{n-5},\ x_1x_2^2x_3^2x_4^4x_5\cdots x_{n-5},\ x_1x_2^2x_3^2x_4x_5^4\cdots x_{n-5}\},$$

all of which are clearly inadmissible. Thus

$$\dim(\mathbf{W}^n(n-5)) \quad\geq\quad n-6+\tfrac{(n-5)!}{2\cdot(n-7)(n-9)!}+\tbinom{n-5}{2}+\tbinom{n-6}{2}+(n-5)\cdot\tbinom{n-7}{3}+2\cdot\left(\textstyle\sum_{i=2}^{n-8}\tbinom{i}{2}\right)+\tbinom{n-8}{2}+\left(\tbinom{n-7}{2}-1\right),$$

since $a_{n-5}, b_{n-5}, c_{n-5}, d_{n-5}, e_{n-5}, f_{n-5}$, have, respectively,

$$\frac{(n-5)!}{2\cdot(n-7)(n-9)!},\ \binom{n-5}{2},\ \binom{n-6}{2},\ n-6,\ (n-5)\cdot\binom{n-7}{3},\ 2\cdot\left(\sum_{i=2}^{n-8}\binom{i}{2}\right)+\binom{n-8}{2}+\left(\binom{n-7}{2}-1\right),$$

permutation representatives which are admissible. Equality holds when, $10 \leq n \leq 12$.

If $i = 6$ then $n_0 = 11$. We know that all permutation representatives of a_{n-6} and b_{n-6} are admissible for all $n \geq 11$. By Theorem 7, the following permutation representatives of $c_5,\ d_5,\ e_5,\ f_5$ and g_5 are admissible in $\mathbf{X}^{11}(5)$, namely those in the sets:

$A = \{x_1x_2^2x_3^2x_4^3x_5^3,\ x_1x_2^2x_3^3x_4^2x_5^3,\ x_1x_2^3x_3^2x_4^2x_5^3,\ x_1^3x_2x_3^2x_4^2x_5^3,\ x_1x_2^2x_3^3x_4^3x_5^2,\ x_1x_2^3x_3^2x_4^3x_5^2,\ x_1^3x_2x_3^2x_4^3x_5^2,\ x_1x_2^3x_3^3x_4^2x_5^2,\ x_1^3x_2x_3^3x_4^2x_5^2,$
$x_1^3x_2^3x_3x_4^2x_5^2\}$,

$B = \{x_1x_2^2x_3x_4^6x_5,\ x_1x_2^2x_3x_4x_5^6,\ x_1x_2x_3^2x_4x_5^6,\ x_1x_2x_3x_4^2x_5^6,\ x_1x_2x_3x_4^6x_5^2,\ x_1x_2^6x_3x_4^2x_5,\ x_1x_2^6x_3x_4x_5^2,\ x_1x_2x_3^6x_4^2x_5,\ x_1x_2x_3^6x_4x_5^2,$
$x_1x_2x_3^2x_4^6x_5\}$,

$C = \{x_1x_2^3x_3^4x_4x_5^2,\ x_1x_2^3x_3^4x_4^2x_5,\ x_1x_2^3x_3x_4^4x_5^2,\ x_1x_2x_3^3x_4^4x_5^2,\ x_1^3x_2x_3^4x_4x_5^2,\ x_1^3x_2x_3x_4^4x_5^2,\ x_1^3x_2^4x_3x_4^2x_5,\ x_1^3x_2^4x_3x_4x_5^2,\ x_1^3x_2x_3^4x_4^2x_5,$
$x_1^3x_2x_3x_4^2x_5^4,\ x_1x_2^3x_3x_4^2x_5^4,\ x_1x_2x_3^3x_4^2x_5^4,\ x_1x_2x_3^2x_4^3x_5^4,\ x_1x_2x_3^2x_4^4x_5^3,\ x_1^3x_2x_3^2x_4x_5^4,\ x_1x_2^3x_3^2x_4x_5^4,\ x_1x_2^2x_3^3x_4x_5^4,\ x_1x_2^2x_3x_4^3x_5^4,$
$x_1x_2^2x_3x_4^4x_5^3,\ x_1^3x_2x_3^2x_4^4x_5,\ x_1x_2^3x_3^2x_4^4x_5,\ x_1x_2^2x_3^3x_4^4x_5,\ x_1x_2^2x_3^4x_4^3x_5,\ x_1x_2^2x_3^4x_4x_5^3\}$,

$D = \{x_1x_2x_3x_4^3x_5^5,\ x_1x_2x_3^3x_4x_5^5,\ x_1x_2^3x_3x_4x_5^5,\ x_1^3x_2x_3x_4x_5^5,\ x_1x_2x_3^3x_4^5x_5,\ x_1x_2^3x_3x_4^5x_5,\ x_1^3x_2x_3x_4^5x_5,\ x_1x_2^3x_3^5x_4x_5,\ x_1^3x_2x_3^5x_4x_5,$
$x_1^3x_2^5x_3x_4x_5\}$,

$E = \{x_1x_2x_3^2x_4^5x_5^2,\ x_1x_2^2x_3x_4^5x_5^2,\ x_1x_2^2x_3^5x_4x_5^2,\ x_1x_2^2x_3^5x_4^2x_5,\ x_1x_2^2x_3x_4^2x_5^5,\ x_1x_2x_3^2x_4^2x_5^5\}$.

It is known, (Mothebe 2009), that $\dim(\mathbf{Q}^{11}(5)) = 315$, so the above permutation representatives of the monomials of $a_5,\ b_5\ c_5,\ d_5,\ e_5,\ f_5$ and g_5 form a complete list of admissible monomials in $\mathbf{X}^{11}(5)$. We claim that

$$\mathbf{W}^{12}(6) = \sum_{j=1}^{6} h_j^1(\mathbf{W}^{11}(5)). \tag{4}$$

If $n > 11$, then the only permutation representatives of $c_{n-6}, d_{n-6}, e_{n-6}, f_{n-6}$ and g_{n-6} in $\mathbf{X}^n(n-6)$ that may not be obtained from the basis of $\mathbf{W}^{11}(5)$ by inductively applying Equation (3) are those of the form $x_1^2\cdots x_{n-6}^{m_{n-6}},\ x_1^3x_2^2\cdots x_{n-6}^{m_{n-6}}$ or, respectively, those of the form $x_1^3x_2^3x_3^2\cdots x_{n-6}^{m_{n-6}},\ x_1^6\cdots x_{n-6}^{m_{n-6}}$ or $x_1x_2^2x_3^6x_4\cdots x_{n-6}$ or $x_1x_2^6x_3^2x_4\cdots x_{n-6},\ x_1^4\cdots x_{n-6}^{m_{n-6}}$ or $x_1^3x_2^4x_3^2\cdots x_{n-6}$ or $x_1x_2^4\cdots x_{n-6}^{m_{n-6}}$ or $x_1x_2x_3^4\cdots x_{n-6}^{m_{n-6}}$, those with a factors of the form $x_i^5x_j^3,\ i < j,\ x_1x_2^2x_3^2\cdots x_{n-6}^{m_{n-6}}$ or those with a factors of the form $x_i^5x_j^2x_k^2,\ i < j,k$, all of which are clearly inadmissible. Thus

$$\begin{aligned}\dim(\mathbf{W}^n(n-6)) \quad\geq\quad & n-6+((n-7)(n-8)-2)+\tbinom{n-6}{2}+\tbinom{n-6}{3}+2\cdot\tbinom{n-7}{3}-(n-9)\\ & +(n-6).\tbinom{n-8}{2}+\tbinom{n-6}{3}-1+\tfrac{(n-7)!}{4\cdot(n-11)!}+(n-8).\tbinom{n-9}{2}+\tbinom{n-9}{2},\end{aligned}$$

since $a_{n-6}, b_{n-6}, c_{n-6}, d_{n-6}, e_{n-6}, f_{n-6}, g_{n-6}$, have, respectively,

$$\binom{n-6}{3},\ n{-}6,\ \binom{n-6}{2},\ \frac{(n-7)!}{4\cdot(n-11)!}+(n{-}8)\cdot\binom{n-9}{2}+\binom{n-9}{2},\ (n{-}6)\cdot\binom{n-8}{2}+\binom{n-6}{3}-1,\ (n{-}7)(n{-}8)-2,\ 2\cdot\binom{n-7}{3}-(n{-}9)$$

permutation representatives which are admissible. Equality holds when $n = 11, 12$.

It is known, (Sum & Phuc, 2013), that $\dim(\mathbf{Q}^{12}(5)) = 190$. Since $\dim(\mathbf{Q}^{12}(4)) = 21$, we must have $\dim(\mathbf{W}^{12}(5)) = 85$. This completes the proof of the lemma hence that of Theorem 2.

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