Commutativity of $\Gamma$-Generalized Boolean Semirings with Derivations

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Abstract

In this paper the notion of derivations on $\Gamma$-generalized Boolean semiring are established, namely $\Gamma$-(f, g) derivation and $\Gamma$-(f, g) generalized derivation. We also investigate the commutativity of prime $\Gamma$-generalized Boolean semiring admitting $\Gamma$-(f, g) derivation and $\Gamma$-(f, g) generalized derivation satisfying some conditions.

Keywords: $\Gamma$-generalized Boolean semiring, semiring, commutativity, derivation

1. Introduction

There has been a great deal of work concerning commutativity of prime rings and prime near rings with derivations or generalized derivations satisfying certain differential identity (Ali, 2012; Asci, 2007; Bell, 2012; Rehman, 2011; Quadri, 2003). The notion of semiring was first introduced by H.S. Vandiver (Vandiver, 1934) in 1934 and a generalization of semiring, $\Gamma$-semiring was first studied by M.K. Rao (Rao, 1995).

In 1987, H.E. Bell and G. Mason (Bell & Mason, 1987) introduced derivations on $\Gamma$-near rings and studied some basic properties. The concept of $\Gamma$-derivations in $\Gamma$-near ring was introduced by Jun, Kim and Cho (Jun, 2003). Then Asci(Asci, 2007) investigated some commutativity conditions for $\Gamma$-near rings with derivations. Kazaz and Alkan (Kazaz & Alkan, 2008) introduced the notion of two-side $\Gamma$-$\alpha$ derivation of $\Gamma$-near rings and investigated some commutativity of prime and semiprime $\Gamma$-near rings. In 2011, the notion of derivations in prime $\Gamma$-semiring was introduced by M.A. Javed et al (Javed et al, 2013). In 2013, K.K. Dey and A.C. Paul (Dey & Paul, 2013) studied on generalized derivations of prime gamma ring. Later in 2014, M.R. Khan and M.M. Hasnain (Khan & Hasnain, 2014) introduced the notion of generalized $\Gamma$-derivation in $\Gamma$-near rings and investigated some basic properties.

In this paper, we introduce the notion of $\Gamma$-(f, g) derivations and $\Gamma$-(f, g) generalized derivations on $\Gamma$-generalized Boolean semirings, and investigate some related properties. We also investigate some commutativity results for $\Gamma$-generalized Boolean semiring involving $\Gamma$-(f, g) derivation and $\Gamma$-(f, g) generalized derivation.

2. Preliminaries

We first recall some definitions and prove lemmas use in proving our main results.

A $\Gamma$-generalized Boolean semiring (or simply $\Gamma$-GB-semiring) is a triple $(R, +, \Gamma)$, where

$(1)$ $(R, +)$ is an abelian group.

$(2)$ $\Gamma$ is a nonempty finite set of binary operations satisfying the following properties

(i) $a\alpha b \in R$ for all $a, b \in R$ and $\alpha \in \Gamma$,

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$ for all $a, b, c \in R$ and $\alpha \in \Gamma$,

(iii) $a\alpha(b\beta c) = (a\alpha b)\beta c = (b\alpha a)\beta c$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$,

(iv) $a\alpha(b\beta c) = a\beta(b\alpha c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

The following are some basic properties on $\Gamma$-GB-semiring then the proof is straightforward and hence omitted. For any $a, b, c \in R$ and $\alpha \in \Gamma$, we have

(i) $-(-a) = a$,

(ii) $a\alpha 0 = 0$,

(iii) $a\alpha(-b) = -(a\alpha b)$,

(iv) $a\alpha(b - c) = (a\alpha b) - (a\alpha c)$,

(v) $-(a + b) = -a - b$, 


Lemma 2.1, we have $\Delta$ follows that $0$ or $I$.

To show that $\Delta$ is commutative, let $f$ be a prime $\Gamma$-GB-semiring if $0$ or $I$.

Next, we start with following lemmas which will be used extensively.

The center of $R$, written $Z(R)$, is defined to be the set

$$Z(R) = \{a \in R | aab = baa \text{ for all } b \in R \text{ and } a \in \Gamma\}$$

Next, we start with following lemmas which will be used extensively.

**Lemma 2.1.** Let $R$ be a $\Gamma$-GB-semiring. If $x \in Z(R)$ then $yax \in Z(R)$ and $xay \in Z(R)$ for all $y \in R$ and $a \in \Gamma$.

**Proof.** Let $x \in Z(R), y, z \in R$, and $a \in \Gamma$. Then

$$(yax)z = x(ayz) = (ybz)x = z(ayx) = y(aza) = yz(a) = yz$$

This completes the proof.

**Lemma 2.2.** Let $R$ be a prime $\Gamma$-GB-semiring such that $0aa = a$ for all $a \in R$ and $a \in \Gamma$ and let $I \neq \{0\}$ be a $\Gamma$-ideal of $R$. Then for any $x, y \in R$

(i) If $xI = I = \{0\}$, then $x = 0$.

(ii) If $IIx = I = \{0\}$, then $x = 0$.

(iii) If $xIvy = I = \{0\}$, then $x = 0$ or $y = 0$.

**Proof.** (i) Let $x \in R$ be such that $xI = \{0\}$. Since $I \neq \{0\}$, there exists nonzero $z$ in $I$. We have $x\Gamma Gz \subseteq xI = \{0\}$ and so $x\Gamma Gz = \{0\}$. Since $R$ is prime and $z \neq 0$, it follows that $x = 0$.

(ii) Let $x \in R$ be such that $IIx = \{0\}$. Since $I \neq \{0\}$, there exists nonzero $z$ in $I$ and since $zbr = (0 + z)b - 0b \in I$ for all $r \in R$ and $\beta \in \Gamma, z\Gamma R \subseteq I$. We have $z\Gamma Rx \subseteq IIx = \{0\}$ and so $z\Gamma Rx = \{0\}$. Since $R$ is prime and $z \neq 0$, it follows that $x = 0$.

(iii) Let $x, y \in R$ be such that $xIvy = \{0\}$. Then $x\Gamma R\Gamma y \subseteq xIvy = \{0\}$ and so $x\Gamma R\Gamma y = \{0\}$. Since $R$ is prime, it follows that $x = 0$ or $Iy = \{0\}$. By (ii) we get $y = 0$.

**Lemma 2.3.** Let $R$ be a prime $\Gamma$-GB-semiring and $\Delta$ be a nonzero function from $R$ into $R$. Then $\Delta(x) \in Z(R)$ for all $x \in R$ if and only if $R$ is commutative.

**Proof.** If $R$ is commutative, then it is obvious that $\Delta(x) \in Z(R)$ for all $x \in R$. Suppose that $\Delta(x) \in Z(R)$ for all $x \in R$. By Lemma 2.1, we have $\Delta(x)y \in Z(R)$ for all $y \in R$ and $a \in \Gamma$. It follows that $[\Delta(x)y, t] = 0$ for all $t, x, y \in R$ and $a, \beta \in \Gamma$.

To show that $R$ is commutative, let $x, y \in R$ and $a \in R$. Since $\Delta$ is a nonzero function on $R$, there exists $z \in R$ such that
Δ(z) ≠ 0. For any t ∈ R and β, γ ∈ Γ we have
Δ(z)βγ[x, y]0 = [Δ(z)]β(γxy), y]0 = 0. So, Δ(z)ΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓΓGamma
An additive mapping \( D : R \to R \) is called a left (resp. right) \( \Gamma \)-\((f, g)\) generalized derivation if there exists nonzero \( \Gamma \)-\((f, g)\) derivation \( d \) on \( R \) satisfying

\[
D(xay) = f(x)ad(y) + D(x)ag(y) \quad (\text{resp. } D(xay) = f(x)\alpha D(y) + d(x)\alpha g(y))
\]

for all \( x, y \in R \) and \( \alpha \in \Gamma \).

**Lemma 3.2.** Let \( R \) be a \( \Gamma \)-GB-semiring and \( D \) be a left \( \Gamma \)-\((f, g)\) generalized derivation on \( R \). Then

\[
[f(x)ad(y) + D(x)ag(y)] \beta g(z) = f(x)ad(y)\beta g(z) + D(x)ag(y)\beta g(z).
\]

**Proof.** Let \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \), we have

\[
D((x\alpha y)\beta z) = f(x\alpha y)\beta d(z) + D((x\alpha z)\beta y) = f(x)af(y)\beta d(z) + (f(x)ad(y) + D(x)ag(y)) \beta g(z) \quad \text{and}
\]

\[
D(x\alpha y(\beta z)) = f(x)ad(y)\beta d(z) + D(x)ag(y)\beta z = f(x)af(y)\beta d(z) + (f(x)ad(y) + D(x)ag(y)) \beta g(z)
\]

Since \( D((x\alpha y)\beta z) = D(x\alpha (y\beta z)) \),

\[
(f(x)ad(y) + D(x)ag(y)) \beta g(z) = f(x)ad(y)\beta g(z) + D(x)ag(y)\beta g(z).
\]

This completes the proof.

**Corollary 3.3.** Let \( R \) be a \( \Gamma \)-GB-semiring. Let \( d \) be a \( \Gamma \)-\((f, g)\) derivation on \( R \) and \( f, g \) be automorphisms on \( R \). Then

\[
[f(x)ad(y) + D(x)ag(y)] \beta g(z) = f(x)ad(y)\beta g(z) + D(x)ag(y)\beta g(z).
\]

**Lemma 3.4.** Let \( R \) be a prime \( \Gamma \)-GB-semiring. Let \( D \) be a nonzero \( \Gamma \)-\((f, g)\) generalized derivation on \( R \) and \( f, g \) be automorphisms on \( R \). If \( f(x)ad(y) + D(x)ag(y) \in Z(R) \) for all \( x, y \in R \) and \( \alpha \in \Gamma \) then \( R \) is commutative.

4. Commutativity of \( \Gamma \)-generalized Boolean Semirings

In this section, we show that \( \Gamma \)-generalized Boolean semiring with derivations satisfying certain conditions are commutative.

**Theorem 4.1.** Let \( R \) be a prime \( \Gamma \)-GB-semiring and let \( f, g \) be automorphisms on \( R \). If \( d \) is a nonzero \( \Gamma \)-\((f, g)\) derivation on \( R \) satisfying any one of the following

(i) \( [d(x), g(y)]_\alpha = [f(x), g(y)]_\alpha \),

(ii) \( d(x, y)_\alpha = [f(x), g(y)]_\alpha \),

(iii) \( (d(x) \circ g(y))_\alpha = (f(x) \circ g(y))_\alpha \),

(iv) \( d(x \circ y)_\alpha = (f(x) \circ g(y))_\alpha \),

(v) \( d(x \circ y)_\alpha = [f(x), g(y)]_\alpha \),

(vi) \( d(x, y)_\alpha = (f(x) \circ g(y))_\alpha \),

for all \( x, y, z \in R \) and \( \alpha \in \Gamma \). Then \( R \) is commutative.

**Proof.** (i) Assume that \( [d(x), g(y)]_\alpha = [f(x), g(y)]_\alpha \) for all \( x, y \in R \) and \( \alpha \in \Gamma \). Replacing \( x \) by \( z\beta x \), we obtain \( [d(z\beta x), g(y)]_\alpha = [f(z\beta x), g(y)]_\alpha \) for all \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \). Then

\[
d(z\beta x)ag(y) - g(y)ad(z\beta x) = [f(z\beta x), g(y)]_\alpha
\]

\[
f(z)\beta d(x)ag(y) + d(z)\beta g(x)ag(y) - g(y)af(z)\beta d(x) - g(y)ad(z)\beta g(x) = [f(z)\beta f(x), g(y)]_\alpha
\]

\[
f(z)\beta d(x, y)_\alpha + d(z)\beta g(x, y)_\alpha = [f(z)\beta f(x, y)]_\alpha
\]

Since \( d \neq 0 \), there exists \( z \in R \) such that \( d(z) \neq 0 \). By Lemma 2.5, it follows that \( R \) is commutative.

The proof of (ii) - (vi) are obtained similarly to that of (i).

**Theorem 4.2.** Let \( R \) be a prime \( \Gamma \)-GB-semiring and \( f, g \) be automorphisms on \( R \). If \( d \) is a nonzero \( \Gamma \)-\((f, g)\) derivation on \( R \) such that

(i) \( [d(x), y]_\alpha \in Z(R) \), or

(ii) \( (d(x) \circ y)_\alpha \in Z(R) \),

for all \( x, y \in R \) and \( \alpha \in \Gamma \). Then \( R \) is commutative.

**Proof.** This follows directly from Lemma 2.4.
Theorem 4.3. Let $R$ be a prime $\Gamma$-GB-semiring such that $0aa = 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let $d$ be a nonzero $\Gamma$-(f, f) derivation on $R$ where $f$ is a nonzero automorphism on $R$. If

(i) $d[x, y]_\alpha = [d(x), f(y)]_\alpha$, or

(ii) $d(x \circ y)_\alpha = (d(x) \circ f(y))_\alpha,$

for all $x, y \in R$ and $\alpha \in \Gamma$. Then $R$ is commutative.

Proof. (i) Assume that $d[x, y]_\alpha = [d(x), f(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$. Replacing $x$ by $\beta x$, we obtain $d[z\beta x, y]_\alpha = [dz\beta x, f(y)]_\alpha$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then


d([z\beta x]y − y[z\beta x]) = d(z\beta x)f(y) − f(y)ad(z\beta x)

d(z\beta x)ay + d(z\beta x)af(y) − f(y)ad(z\beta x) − d(z\beta x)f(y) − f(y)ad(z\beta x) = 0.

d(z\beta x)f(x) = 0.

Hence $f(z)\beta[f(x), d(y)]_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show that $R$ is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$. We have $f(z)\beta[tf(x), d(y)]_\alpha = f(z)\beta[f(tyx), d(y)]_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$.

Since $f$ is surjective, $f(z)\Gamma R[f(x), d(y)]_\alpha = [0].$

Since $R$ is prime and $f(z) \neq 0$, $[f(x), d(y)]_\alpha = 0.$

And since $f$ is surjective on $R$, $d(y) \in Z(R).$ By Lemma 2.3, it follows that $R$ is commutative.

(ii) Using similar techniques as above, we obtain $f(z)\beta(f(x) \circ d(y))_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show $R$ is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$. We have $f(z)\beta[f(t)\gamma(f(x) \circ d(y))_\alpha = f(z)\beta[f(tyx) \circ d(y))_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$.

Since $f$ is surjective, $f(z)\Gamma R[f(x) \circ d(y)]_\alpha = [0].$

Since $R$ is prime and $f(z) \neq 0$, $(f(x) \circ d(y))_\alpha = 0 \in Z(R).$

By Theorem 4.2(ii), it follows that $R$ is commutative. This completes the proof.

Theorem 4.4. Let $R$ be a nonzero prime $\Gamma$-GB-semiring such that $0aa = 0$ for all $a \in R$ and $\alpha \in \Gamma$ and $f, g$ be automorphism on $R$. Let $D$ be a left $\Gamma$-(f, g) generalized derivation on $R$ satisfying

(i) $[D(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$, or

(ii) $(D(x) \circ g(y))_\alpha = (f(x) \circ g(y))_\alpha,$

for all $x, y \in R$ and $\alpha \in \Gamma$. If there exists $0 \neq z \in R$ such that $D(z) = 0$, then $R$ is commutative.

Proof. (i) Assume that $[D(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$. Replacing $x$ by $\beta x$, we obtain $[D(z\beta x), g(y)]_\alpha = [f(z\beta x), g(y)]_\alpha$. For each $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ we have

$$D(z\beta x)g(y) − g(y)\alpha D(z\beta x) = [f(z)\beta f(x), g(y)]_\alpha$$

$$f(z)\beta \alpha \beta d(x) + D(x)\beta \beta g(x)\alpha g(y) − g(y)\alpha f(z)\beta \beta d(x) + D(x)\beta \beta g(x)\alpha g(y) − g(y)\alpha = [f(z)\beta \beta f(x), g(y)]_\alpha$$

$$f(z)\beta \beta d(x)g(y) − g(y)\alpha f(z)\beta \beta d(x)g(y) − g(y)\alpha D(z\beta x)g(x) = [f(z)\beta \beta f(x), g(y)]_\alpha$$

$$f(z)\beta \beta \beta d(x, g(y))_\alpha + D(z\beta \beta g(x, g(y))_\alpha = [f(z)\beta \beta f(x), g(y)]_\alpha$$

Hence $f(z)\beta[\beta d(x, g(y))_\alpha − [f(x), g(y)]_\alpha + D(z\beta \beta g(x, g(y))_\alpha = 0.$

To show $R$ is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since there exists $0 \neq z \in R$ such that $D(z) = 0$, we have

$$f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha) = f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha)$$

$$f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha)$$

Thus $f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha) = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$.

Since $g$ is surjective, $f(z)\Gamma R([d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha) = [0].$

Since $f$ is injective, $f(z) \neq 0$. And $R$ is prime, we have $[d(x), g(y)]_\alpha − [f(x), g(y)]_\alpha = 0.$
Thus \([d(x), g(y)]_a = [f(x), g(y)]_a\) for all \(x, y \in R\) and \(\alpha \in \Gamma\).

By Theorem 4.1(ii), it follows that \(R\) is commutative.

(ii) Using similar techniques as above, we obtain

\[
f(z)\beta((d(x) \circ g(y))_a) - (f(x) \circ g(y))_a + D(z)\beta(g(x) \circ g(y))_a = 0\quad \text{for all } x, y, z \in R \text{ and } \alpha, \beta \in \Gamma.
\]

To show \(R\) is commutative, let \(x, y \in R\) and \(\alpha \in \Gamma\). Since there exists \(0 \neq z \in R\) such that \(D(z) = 0\), we have

\[
f(z)\beta((d(x) \circ g(y))_a) - (f(x) \circ g(y))_a = f(z)\beta(g(t)y((d(x) \circ g(y))_a - g(t)\gamma(f(x) \circ g(y))_a)
\]

\[
= f(z)\beta((d(x) \circ g(tyy))_a - (f(x) \circ g(tyy))_a)
\]

\[
= f(x)\beta((d(x) \circ g(tyy))_a - (f(x) \circ g(tyy))_a)
\]

\[
+ D(z)\beta(g(x) \circ g(tyy))_a = 0
\]

Thus \(f(z)\beta(g(t)y((d(x) \circ g(y))_a - (f(x) \circ g(y))_a) = 0\) for all \(t, x, y, z \in R\) and \(\alpha, \beta, \gamma \in \Gamma\).

The same argument in the proof of (i) and by Theorem 4.1(iii) we conclude that \(R\) is commutative. This completes the proof.

**Theorem 4.5.** Let \(R\) be a prime \(\Gamma\)-GB-semiring and \(f, g\) be automorphisms on \(R\). Let \(D\) be a right \(\Gamma\)-(\(f, g\)) generalized derivation on \(R\) satisfying any one of the following

(i) \(D(x, y)_a = [f(x), g(y)]_a\),

(ii) \(D(x \circ y)_a = (f(x) \circ g(y))_a\),

(iii) \(D(x) \circ y)_a = [f(x), g(y)]_a\),

(iv) \(D(x)_a = (f(x) \circ g(y))_a\),

for all \(x, y \in R\) and \(\alpha \in \Gamma\). Then \(R\) is commutative.

**Proof.** (i) Assume that \(D(x, y)_a = [f(x), g(y)]_a\) for all \(x, y \in R\) and \(\alpha \in \Gamma\). Replacing \(x\) by \(z\beta x\), we obtain \(D[z\beta x, y)_a = [f(z\beta x), g(y)]_a\). For each \(x, y, z \in R\) and \(\alpha, \beta \in \Gamma\) we have

\[
D(z\beta [x, y]_a) = f(z)\beta D[x, y]_a + d(z)\beta [g, x, y]_a = f(z)\beta [f(x), g(y)]_a
\]

\[
f(z)\beta D[x, y]_a - [f(x), g(y)]_a + d(z)\beta [g, x, y]_a = 0
\]

\[
d(z)\beta [g, x, y]_a = 0.
\]

To show that \(R\) is commutative, let \(x, y \in R\) and \(\alpha \in \Gamma\). Since \(d \neq 0\), there exists \(z \in R\) such that \(d(z) \neq 0\), we have \(d(z)\beta [g, x, y]_a = 0\). By Lemma 2.5, it follows that \(R\) is commutative.

The proof of (ii) - (iv) are obtained similarly to that of (i).

**Theorem 4.6** Let \(R\) be a prime \(\Gamma\)-GB-semiring and \(f, g\) be automorphisms on \(R\). Let \(D\) be a nonzero left (resp. right) \(\Gamma\)-(\(f, g\)) generalized derivation on \(R\) such that

(i) \([D(x), y]_a \in Z(R)\), or

(ii) \((D(x) \circ y)_a \in Z(R)\),

for all \(x, y \in R\) and \(\alpha \in \Gamma\). Then \(R\) is commutative.

**Proof.** This follows directly from Lemma 2.4.

**Theorem 4.7.** Let \(R\) be a prime \(\Gamma\)-GB-semiring such that \(0aa = 0\) for all \(a \in R\) and \(\alpha \in \Gamma\). Let \(f\) be a nonzero automorphism on \(R\). If \(D\) is a left \(\Gamma\)-(\(f, f\)) generalized derivation on \(R\) such that

(i) \([D(x), y]_a = [D(x), f(y)]_a\), or

(ii) \((D(x) \circ y)_a = (D(x) \circ f(y))_a\),

for all \(x, y \in R\) and \(\alpha \in \Gamma\). Then \(R\) is commutative.

**Proof.** (i) Assume that \([D(x, y)]_a = [D(x, f(y))_a\) for all \(x, y \in R\) and \(\alpha \in \Gamma\)

Replacing \(x\) by \(z\beta x\), we obtain \([D[z\beta x, y]_a = [D(z\beta x), f(y)]_a\).

For each \(x, y, z \in R\) and \(\alpha, \beta \in \Gamma\), we have
\[
D(z\beta[x,y]_\alpha) = D(z\beta x)\alpha f(y) - f(y)\alpha D(z\beta x)
\]
\[
f(z)\beta d[x,y]_\alpha + D(z\beta)f[x,y]_\alpha = d(z\beta x)\alpha f(y) - f(y)\alpha D(z\beta x)
\]
\[
f(z)\beta f(x)\alpha d(y) + f(z)\beta d(x)\alpha f(y) - f(z)\beta f(y)\alpha d(x) - f(z)\beta d(y)\alpha f(x) + D(z)\beta f(x)\alpha f(y) - f(y)\alpha f(z)\beta d(x) - f(y)\alpha D(z)\beta f(x)
\]
so, \( f(z)\beta [f(x), d(y)]_\alpha = 0 \) for all \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \).

To show that \( R \) is commutative, let \( x, y \in R \) and \( \alpha \in \Gamma \). Since \( f \neq 0 \), there exists \( z \in R \) such that \( f(z) \neq 0 \), we have
\[
f(z)\beta f(t)\gamma [f(x), d(y)]_\alpha = f(z)\beta f(ty)\alpha f(x) = f(z)\beta [f(x), d(y)]_\alpha = 0
\]
Since \( f \) is surjective, \( f(z)\Gamma RT [f(x), d(y)]_\alpha = 0 \).

Since \( R \) is prime and \( f(z) \neq 0 \), \( [f(x), d(y)]_\alpha = 0 \).

Since \( f \) is surjective, \( d(y) \in Z(R) \). By Lemma 2.3, it follows that \( R \) is commutative.

(ii) Using similar techniques as above, we have, \( f(z)\beta (f(x) \circ d(y))_\alpha = 0 \) for all \( x, y, z \in R \) and \( \alpha, \beta \in \Gamma \).

To show \( R \) is commutative, let \( x, y \in R \) and \( \alpha \in \Gamma \). Since \( f \neq 0 \), there exists \( z \in R \) such that \( f(z) \neq 0 \), we have
\[
f(z)\beta f(t)\gamma (f(x) \circ d(y))_\alpha = f(z)\beta (f(ty) \circ d(y))_\alpha = 0
\]
Since \( f \) is surjective, \( f(z)\Gamma R T (f(x) \circ d(y))_\alpha = 0 \).

Since \( R \) is prime and \( f(z) \neq 0 \), \( (f(x) \circ d(y))_\alpha = 0 \in Z(R) \).

By Theorem 4.2(ii), it follows that \( R \) is commutative. This completes the proof.

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References


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