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Hopf-bifurcation Limit Cycles of an Extended Rosenzweig-MacArthur Model

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Abstract

In this paper, we formulated a new topologically equivalence dynamics of an Extended Rosenzweig-MacArthur Model. Also, we investigated the local stability criteria, and determine the existence of co-dimension-1 Hopf-bifurcation limit cycles as the bifurcation-parameter changes. We discussed the dynamical complexities of this model using numerical responses, solution curves and phase-space diagrams.

Keywords: Hopf-bifurcation, stability criteria, limit cycles

1. Introduction

In general, any nonlinear dynamical system contains certain parameters called bifurcation parameters or controlled-free parameters, and thus it's an imperative to study the qualitative behaviors of such robust systems as the parameters are varied. This study of bifurcation analysis includes the post-critical behaviors of the nonlinear system in the neighborhood of the critical points called Hopf-bifurcation limit cycles or periodic solutions (Liao &Yu, 2007). The complexities of such nonlinear dynamics include heteroclinic orbits, homoclinic orbits and chaos which envisaged essential global behaviors (Kutnzetsov, 1995; Sun & Luo 2005; Wang & Zhao, 2011).

Mathematical models of multiple interacting species that exhibit such rich local and global dynamical behaviors (i.e., stability of equilibra, local and global bifurcations, limit cycles, peak-to-peak dynamics) includes Rosenzweig-MacArthur tri-trophic food chain models; it predicts and depicts real ecological system (Rosenweig-MacArthur, 1963; Keshet-Edelstien, 2005; Brauer, & Chavez-Castillo, 2012; Kar, Ghorai, Batabyal, 2012; Sanjaya, Sunaryo, Salleh, & Mamat 2013; Haque, Ali, & Chakravarty, 2013).

In this paper, we obtained a topologically equivalence dynamics of an Extended Rosenweig-MarArthur Model, and studied codimension-1 Hopf-bifurcation limit cycles (periodic behaviors). These limit cycles occur naturally in the ecosystem, and using CASS (maple) we obtained their solution space and phase space diagrams, see (Naji, Upadhyay, & Rai, 2010; Candalen, & Rinaldi, 1999; Wiggin, 2010; Seydel, 2010, & Shavin, 2015).

2. Model Formulation and Boundedness

The Extended Rosenzweig-MacArthur (ERM) model formulated and studied by Feng, Freeze, Lu, and Rocco (2014) is given as:

$$\frac{dx_1}{dt} = rx_1\left\{1-\frac{x_1}{K}\right\} - a_2\frac{x_1}{b_1+x_1}x_2 - a_3\frac{x_1}{b_1+x_1}x_3$$

$$\frac{dx_2}{dt} = c_2a_2\frac{x_2}{b_1+x_1}x_1 - d_2x_2 - a_3\frac{x_2}{b_2+x_2}x_3$$

$$\frac{dx_3}{dt} = c_3a_3\frac{x_2}{b_2+x_2}x_3 - d_3x_3 + c_3a_3\frac{x_1}{b_1+x_1}x_3 \qquad 1.1$$

where, $x_1(t),\ x_2(t), and\ x_3(t)$ are population biomass densities of prey, predator and super-predator respectively, and constant parameters with significance ecological implications.

We obtained a topologically equivalence dynamical model through the non-dimensionalization of the state variables as follows:

$$\frac{dx}{d\tau}=\alpha x\left(1-\frac{x}{\kappa}\right)-\eta\frac{x}{1+x}y-\frac{x}{1+x}z$$

$$\frac{dy}{d\tau}=\epsilon\frac{x}{1+x}y-\xi y-\sigma\frac{y}{1+y}z$$

$$\frac{dz}{d\tau}=\beta\frac{y}{1+y}z-\mu z+\beta\frac{x}{1+x}z \tag{1.2}$$

where, $x(\tau)=\frac{x_2(t)}{b_1}, y(\tau)=\frac{x_2(t)}{b_2}, z(\tau)=\frac{x_3(t)}{b_1}, \alpha=\frac{r}{a_3}, \kappa=\frac{K}{b_1}, \eta=\frac{a_2b_2}{a_3b_1}, \epsilon=\frac{c_2\,a_2}{a_3}\quad \xi=\frac{d_2}{a_3},\ \sigma=\frac{b_1}{b_2}, \mu=\frac{d_3}{a_3}, \tau=$

$a_3t, and\ c_3=\beta$

For ecological significance, the parameters are assumed to be positive. State variables are invariant on the positive octant defined as $\Re_+^3=(x(t),y(t),z(t)|x(t)\geq 0, y(t)\geq 0, z(t)\geq 0)$, $\forall\ t\geq 0$, satisfying initial conditions; $x(0)=x_0$, $y(0)=y_0, z(0)=z_0$. The population functions of the interacting species satisfies the boundedness conditions;

$$(0,0,0)\leq\big(x(t),y(t),z(t)\big)\leq\left(K, y_0e^{\left(\frac{\epsilon\kappa}{1+\kappa}-\xi\right)\tau}, z_0e^{\left(\frac{2\beta\kappa+\beta}{1+\kappa}-\mu\right)\tau}\right)$$

3. Existence and Positivity of Trivial and Semi-trivial Equilibria of the Model

We obtain the critical point of model (1.2), by solving the system during it steady-state; independent of time, and deduce the positivity conditions of each critical point. The model exhibited the following trivial, and semi-trivial equilibria points:

$$E_0(x^*=0,y^*=0,z^*=0),\ E_1(x^*=K,y^*=0,z^*=0),$$

$$E_2\left(x^*=\frac{\xi}{\epsilon-\xi}, y^*=\frac{\alpha\epsilon(\kappa\epsilon-\kappa\xi-\xi)}{\eta\kappa(\epsilon-\xi)^2}, z^*=0\right); if\ \epsilon>\xi, \kappa>\frac{\xi}{\epsilon-\xi}$$

$$E_3\left(x^*=\frac{\mu}{\beta-\mu}, y^*=0, z^*=\frac{\alpha\beta(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^2}\right); if\ \beta>\mu,\qquad \kappa>\frac{\mu}{\beta-\mu}$$

Assume a critical point say, $\boldsymbol{X}^*=(x^*,y^*,z^*)$ then, the linearization of system (1.2) yields the community matrix as follows;

$$J(x^*,y^*,z^*)=\begin{bmatrix}J_{11} & J_{12} & J_{13}\\ J_{21} & J_{22} & J_{23}\\ J_{31} & J_{32} & J_{33}\end{bmatrix}$$

$$=\begin{bmatrix}\alpha\left(\frac{\kappa-2x^*}{\kappa}\right)-\frac{\eta y^*+z^*}{(1+x^*)^2} & -\frac{\eta x^*}{1+x^*} & -\frac{x^*}{1+x^*}\\ \frac{\epsilon y^*}{(1+x^*)^2} & \frac{\epsilon x^*}{1+x^*}-\xi-\frac{\sigma z^*}{(1+y^*)^2} & -\frac{\sigma y^*}{1+y^*}\\ \frac{\beta z^*}{(1+x^*)^2} & \frac{\beta z^*}{(1+y^*)^2} & \frac{\beta y^*}{1+y^*}-\mu-\frac{\beta x^*}{1+x^*}\end{bmatrix} \tag{1.3}$$

Descartes' rule of sign variation guarantees the existence of coexistence equilibrium of the model as stated in the following lemma:

3.10 Lemma I: Existence of Coexistence Equilibrium Point.

The model (1.2) has a positive coexistence equilibrium point, say; $E_4(X=x^*,Y=y^*,Z=z^*)$ if;

$$(i)\ \ y^*=\frac{\mu(1+x^*)-\beta x^*}{(\beta-\mu)(1+x^*)+\beta x^*},\ z^*=\frac{\beta(\epsilon x^*-\xi(1+x^*))}{\sigma((\beta-\mu)(1+x^*)+\beta x^*)},$$

$$P(X) = X^3 + P_0X^2 + P_1X + P_2 = 0, P_0 < 0, P_1 < 0, P_2 < 0, \frac{\xi}{\epsilon - \xi} < X < \frac{\mu}{\beta - \mu}$$

$$P_0 = \frac{2\alpha\sigma\kappa\beta - \alpha\sigma\kappa\mu - 3\alpha\sigma\beta + 2\alpha\sigma\mu}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

$$(iii)\ P_1 = \frac{3\alpha\sigma\kappa\beta - 2\alpha\sigma\kappa\mu + \kappa\sigma\eta\beta - \kappa\sigma\eta\mu - \alpha\sigma\beta + \alpha\sigma\mu + \kappa\beta\xi - \beta\kappa\epsilon}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

$$P_2 = \frac{\alpha\beta\sigma\kappa - \alpha\sigma\kappa\mu - \eta\kappa\mu\sigma + \kappa\beta\xi}{\alpha\sigma\mu - 2\alpha\sigma\beta}$$

4. Stability Analysis of Equilibria of the Model

4.1 Lemma II: Stability of Prey Equilibrium point. $E_1(x^* = K, y^* = 0, z^* = 0)$ is unstable if $\frac{\varepsilon\kappa}{1+\kappa} > \xi \ or \ \frac{\beta\kappa}{1+\kappa} > \mu$, locally asymptotically stable if $\frac{\varepsilon\kappa}{1+\kappa} < \xi \ and \ \frac{\beta\kappa}{1+\kappa} < \mu$, globally asymptotically stable if locally asymptotically stable, and satisfies; $\frac{\varepsilon\kappa}{1+\kappa} < \xi \ and \ \beta\frac{1+2\kappa}{1+\kappa} < \mu$.

4.2 Lemma III: Stability of Prey-Predator Equilibrium Point.

The equilibrium point

$$E_2\left(x^* = \frac{\xi}{\epsilon - \xi}, y^* = \frac{\alpha\epsilon(\kappa\epsilon - \kappa\xi - \xi)}{\eta\kappa(\epsilon - \xi)^2}, z^* = 0\right); if\ \epsilon > \xi, \kappa > \frac{\xi}{\epsilon - \xi}$$

is unstable if

$$\frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]} > 0$$

Locally asymptotically stable if

$$\frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]} < 0\ and$$

$$\frac{\xi}{\varepsilon - \xi} < \kappa < \frac{\varepsilon + \xi}{\varepsilon - \xi}$$

4.3 Lemma IV: Stability of Coexistence Equilibrium Point.

Consider the community matric of the coexistence equilibrium point, $E_4(X = x^*, Y = y^*, Z = z^*)$;

$$\begin{bmatrix} J_{11} & -\frac{\eta x^*}{1 + x^*} & \frac{-x^*}{1 + x^*} \\ -\frac{\varepsilon(x^*\xi - x^*\varepsilon + \xi)}{\sigma(x^{*2} + 2x^* + 1)(2\beta x^* - \mu x^* - \mu)} & J_{22} & \frac{\sigma(\beta x^* - \mu x^* - \mu)}{\beta(1 + x^*)} \\ -\frac{\beta^2(\xi x^* - \varepsilon x^* + \xi)}{\sigma(2\beta x^* - \mu x^* + \beta - \mu)(x^{*2} + 2x^* + 1)} & J_{32} & 0 \end{bmatrix}$$

where,

$$J_{11} = \left(\frac{1}{(2\beta x^* - \mu x^* + \beta - \mu)(x^{*2} + 2x^* + 1}\right) *$$

$$(2\alpha\beta\kappa\sigma x^{*3} - 4\alpha\beta x^{*4} - \alpha\kappa\mu\sigma x^{*4} + 5\alpha\beta\kappa\sigma x^{*2} - 10\alpha\beta\sigma x^{*3} - 3\alpha\kappa\mu\sigma x^{*2} + 6\alpha\mu\sigma x^{*3} + 4\alpha\beta\sigma\kappa x^* - 8\alpha\beta\sigma x^{*2}$$
$$- 3\alpha\sigma\kappa\mu x^* + 6\alpha\mu\sigma x^{*2} + \beta\eta\kappa\sigma x^* - \eta\kappa\mu\sigma x^* + \alpha\beta\sigma\kappa - 2\alpha\beta\sigma x^* - \alpha\kappa\mu\sigma + 2\alpha\sigma\mu x^* + \beta\kappa\xi x^*$$
$$- \beta\kappa\varepsilon x^* - \eta\kappa\mu\sigma + \beta\kappa\xi)$$

$$J_{22} = \frac{\beta\xi x^{*2} - \beta\varepsilon x^{*2} - \mu\xi x^{*2} + \mu\epsilon x^{*2} + \beta\xi x^* - 2\mu\xi x^* + \mu\varepsilon x^* - \mu\xi}{\beta(x^{*2} + 2x^* + 1)}$$

$$J_{32} = -\frac{(2\beta x^* - \mu x^* + \beta - \mu)(\xi x^* - \varepsilon x^* + \xi)}{\sigma(x^{*2} + 2x^* + 1)}$$

The characteristic polynomial of the community matrix yields

$$P(\lambda) = |J(x^*, y^*, z^*) - \lambda I| \equiv \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$$
$$= \lambda^3 - trac(J(x^*, y^*, z^*)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det((J(x^*, y^*, z^*)) = 0$$

where $A_{11}, A_{22}, and A_{33}$ are the cofactors of the community matrix of coexistence equilibrium point. Here,

$$A_1 = -(J_{11} + J_{22} + J_{33}),\ A_2 = J_{11}J_{33} + J_{22}J_{33} + J_{11}J_{22} - J_{12}J_{21} - J_{23}J_{32} - J_{13}J_{31}$$
$$A_3 = J_{13}J_{31}J_{22} + J_{23}J_{32}J_{11} + J_{21}J_{33}J_{12} - J_{12}J_{23}J_{31} - J_{13}J_{21}J_{32} - J_{11}J_{22}J_{33}$$

Then, applying Routh-Hurwitz conditions on the characteristic polynomial yields,

$$A_1 > 0, A_3 > 0 \ and \ A_1A_2 - A_3 > 0 \qquad 1.4$$

where $A_1, A_2, and\ A_3$ are coefficient of characteristic polynomial of the community matrix at the coexistence equilibrium point.

Thus, the positive coexistence equilibrium point, $E_4(X = x^*, Y = y^*, Z = z^*)$ is locally and asymptotically stable.

5. Existence of Hopf-bifurcation at Semi-trivial Equilibria

5.1 Proposition I: Hopf-bifurcation at Prey-predator Equilibrium Point.

Suppose the prey-predator equilibrium exist, say; $E_2(x^*, y^*, z^*)$ and satisfies $> \xi,\ \kappa > \frac{\xi}{\varepsilon-\xi},\ and\ \frac{\xi}{\varepsilon-\xi} < \kappa < \frac{\varepsilon+\xi}{\varepsilon-\xi}$.

Then, as the bifurcation parameter, κ varies there exist a critical bifurcating threshold, $\kappa = \kappa^*$ such that the equilibrium point, $E_3(x^*, y^*, z^*)$ is asymptotically stable for $\kappa \in (0, \kappa^*)$, and degenerates to a stable limit cycle for $\kappa > \kappa^*$. Then, model (1.2) exhibit a supercritical Hopf-bifurcation from this equilibrium if

$$\kappa^* = \frac{\varepsilon + \xi}{\epsilon - \xi};\ \ \left(\frac{d\psi_1(\kappa)}{d\kappa}\right)_{\kappa=\kappa^*} = \frac{\alpha\xi(\varepsilon - \xi)}{2\varepsilon(\epsilon + \xi)} \neq 0.$$

Proof:

Evaluate the community matrix (4.0) at the equilibrium point,

$$E_2\left(x^* = \frac{\xi}{\epsilon - \xi}, y^* = \frac{\alpha\epsilon(\kappa\epsilon - \kappa\xi - \xi)}{\eta\kappa(\epsilon - \xi)^2}, z^* = 0\right)$$

yields the characteristic polynomial, say;

$$P(\lambda) = \left|J\left(\frac{\xi}{\epsilon - \xi}, \frac{\alpha\epsilon(\kappa\epsilon - \kappa\xi - \xi)}{\eta\kappa(\epsilon - \xi)^2}, 0\right) - I\lambda\right| = 0$$

$$= \begin{vmatrix} \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{\varepsilon\kappa(\xi - \varepsilon)} & -\frac{\xi\eta}{\varepsilon} - \lambda & -\frac{\xi}{\varepsilon} \\ -\frac{\alpha(\kappa\xi - \varepsilon\kappa + \xi)}{\eta\kappa} & 0 - \lambda & \frac{\alpha\sigma\varepsilon(\kappa\xi - \varepsilon\kappa + \xi)}{\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)} \\ 0 & 0 & J_{33} - \lambda \end{vmatrix} = 0$$

$$J_{33} = \frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]}$$

$$P(\lambda) = (J_{33} - \lambda)\left(\lambda^2 - \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{\varepsilon\kappa(\xi - \varepsilon)}\lambda - \frac{\alpha\xi(\kappa\xi - \varepsilon\kappa + \xi)}{\varepsilon\kappa}\right) = 0 \quad 1.5$$

Consider the characteristic polynomial of prey-predator community matrix, say;

$$P(\lambda) =$$

$$\left(\frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]} - \lambda\right) *$$

$$\left(\lambda^2 - \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{\varepsilon\kappa(\xi - \varepsilon)}\lambda - \frac{\alpha\xi(\kappa\xi - \varepsilon\kappa + \xi)}{\varepsilon\kappa}\right) = 0$$

solving quadratic expression yields the roots as;

$$\lambda_{1,2} = \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{2\varepsilon\kappa(\xi - \varepsilon)} \pm \sqrt{\left(\frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{2\varepsilon\kappa(\xi - \varepsilon)}\right)^2 + \frac{\alpha\xi(\kappa\xi - \varepsilon\kappa + \xi)}{\varepsilon\kappa}} \equiv \psi_1(\kappa) \pm i\psi_2(\kappa)$$

$$\lambda_3 = \frac{(\beta\xi - \mu\varepsilon)[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)] - \alpha\beta\varepsilon^2(\kappa\xi - \kappa\varepsilon + \xi)}{\varepsilon[\eta\kappa(\xi - \varepsilon)^2 - \alpha\varepsilon(\kappa\xi - \kappa\varepsilon + \xi)]}$$

Hopf-bifurcation occurs when, $\lambda_{1,2} = \pm i\psi_2(\kappa)$, solving $\psi_1(\kappa) = 0$, yields the critical bifurcating threshold say;

$$\kappa^* = \frac{\varepsilon + \xi}{\epsilon - \xi} \Rightarrow \psi_1(\kappa) = \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{2\varepsilon\kappa(\xi - \varepsilon)}; \lambda_{1,2} = \pm i\psi_2(\kappa^*) = \pm i\sqrt{\frac{\alpha\xi(\epsilon - \xi)}{(\varepsilon + \xi)}}$$

$$\lambda_3 = \frac{\alpha\beta\xi\varepsilon^2 + \alpha\beta\varepsilon^3 - \alpha\mu\varepsilon^3 - \beta\eta\xi^3 + \beta\eta\xi\varepsilon^2 + \eta\mu\xi^2\varepsilon - \mu\eta\varepsilon^3}{\varepsilon(\alpha\varepsilon^2 - \eta\xi^2 + \eta\varepsilon^2)}$$

Suppose the positive invariant parameter space $\Re_+^4(\alpha, \varepsilon, \xi, \kappa)$ is a smooth analytic function of bifurcation parameter, and differentiate $\psi_1(\kappa) = \frac{\xi\alpha(\kappa\xi - \varepsilon\kappa + \xi + \varepsilon)}{2\varepsilon\kappa(\xi - \varepsilon)}$ w.r.t κ , the transversality condition follows as;

$$\left(\frac{d\psi_1(\kappa)}{d\kappa}\right)_{\kappa=\kappa^*} = \frac{\alpha\xi(\varepsilon - \xi)}{2\varepsilon(\epsilon + \xi)} \neq 0$$

Hence, the prove is complete.

5.2 Proposition II: Hopf-bifurcation at Prey Super-predator Equilibrium Point. Suppose the prey super-predator equilibrium exist, say; $E_3(x^*, y^*, z^*)$ and satisfies $\kappa > \mu,\ \kappa > \frac{\mu}{\beta - \mu},\ and\ \frac{\mu}{\beta - \mu} < \kappa < \frac{\beta + \mu}{\beta - \mu}$. Then, as the bifurcation parameter, β varies there exist a critical bifurcating threshold, $\beta = \beta^*$ such that the equilibrium point, $E_3(x^*, y^*, z^*)$ is asymptotically stable for $\beta \in (0, \beta^*)$, and degenerates to a stable limit cycle for $\beta > \beta^*$. Then, model (3.0) exhibit a supercritical Hopf-bifurcation at the equilibrium if,

$$\beta^{*}=\frac{\kappa\mu+\mu}{\kappa-1};\quad \left(\frac{d\psi_{1}(\beta)}{d\beta}\right)_{\beta=\beta^{*}}=\frac{\alpha(\kappa-1)^{3}}{4\mu\kappa(\kappa+1)}\neq 0$$

Proof:

Evaluating the community matrix (1.2) at the equilibrium point,

$$E_{3}\left(x^{*}=\frac{\mu}{\beta-\mu},y^{*}=0,z^{*}=\frac{\alpha\beta(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}}\right)$$

yields the characteristic polynomial,

$$P(\lambda)=\left|J\left(\frac{\mu}{\beta-\mu},0,\frac{\alpha\beta(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}}\right)-I\lambda\right|=0$$

$$\begin{vmatrix}\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{\beta\kappa(\beta-\mu)}-\lambda & -\frac{\mu\eta}{\beta} & -\frac{\mu}{\beta}\\ 0 & \frac{\kappa(\varepsilon\mu-\xi\beta)(\beta-\mu)^{2}-\alpha\beta^{2}\sigma(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}}-\lambda & 0\\ \frac{\alpha(\kappa\beta-\kappa\mu-\mu)}{\kappa} & \frac{\alpha\beta^{2}(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}} & 0-\lambda\end{vmatrix}$$

$$=0$$

$$P(\lambda)=\left(\frac{\kappa(\varepsilon\mu-\xi\beta)(\beta-\mu)^{2}-\alpha\beta^{2}\sigma(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}}-\lambda\right)*\left(\lambda^{2}-\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{\beta\kappa(\beta-\mu)}\lambda+\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu)}{\kappa\beta}\right)=0 \qquad 1.6$$

solving quadratically equation (7.0) yields the roots as;

$$\lambda_{1,2}=$$

$$\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{2\beta\kappa(\beta-\mu)}\pm\sqrt{\left(\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{2\beta\kappa(\beta-\mu)}\right)^{2}-\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu)}{\kappa\beta}}$$

$$\equiv\psi_{1}(\beta)\pm i\psi_{2}(\beta),$$

$$\lambda_{3}=\frac{\kappa(\varepsilon\mu-\xi\beta)(\beta-\mu)^{2}-\alpha\beta^{2}\sigma(\kappa\beta-\kappa\mu-\mu)}{\kappa(\beta-\mu)^{2}}$$

Hopf-bifurcation occurs when, $\lambda_{1,2}=\pm i\psi_{2}(\beta)$, solving $\psi_{1}(\beta^{*})=0$, yields the critical bifurcating threshold say;

$$\beta^{*}=\frac{\kappa\mu+\mu}{\kappa-1};\Rightarrow\psi_{1}(\beta)=\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{2\beta\kappa(\beta-\mu)};\ \lambda_{1,2}=\pm i\psi_{2}(\beta)=\pm i\sqrt{\frac{\alpha\mu}{\kappa}}$$

$$\lambda_{3}=-\frac{1}{4}\left(\frac{\alpha\sigma\kappa^{3}+3\alpha\sigma\kappa^{2}+3\alpha\sigma\kappa+4\kappa^{2}\xi-4\varepsilon\kappa^{2}+\alpha\sigma+4\kappa\xi+4\varepsilon\kappa}{\kappa(\kappa+1)}\right)$$

Suppose the positive invariant parameter space $\Re_{+}^{4}(\alpha,\mu,\beta,\kappa)$ is a smooth analytic function of bifurcation parameter, and differentiate $\psi_{1}(\beta)=\frac{\alpha\mu(\kappa\beta-\kappa\mu-\mu-\beta)}{2\beta\kappa(\beta-\mu)}$ w.r.t β the transversality condition follows as;

$$\left(\frac{d\psi_{1}(\beta)}{d\beta}\right)_{\beta=\beta^{*}}=\frac{\alpha(\kappa-1)^{3}}{4\mu\kappa(\kappa+1)}\neq 0$$

Hence, the prove is complete.

5.2 Proposition III: Existence of Hopf-Bifurcation at Coexistence Equilibrium Point.

Suppose the positive invariant parameter space $\Re_{+}^{8}(\alpha,\beta,\sigma,\varepsilon,\kappa,\mu,\xi,\eta)$ is a smooth analytic function of the bifurcation parameter β, necessary and sufficient conditions for local asymptotic stability of $E_4(X = x^*, Y = y^*, Z = z^*)$ holds. Then the model (1.2) degenerates to a family of periodic cycles, bifurcating the equilibrium point, $E_4(X = x^*, Y = y^*, Z = z^*)$ in the neighborhood of the critical bifurcating threshold β^* such that $\beta \in (\beta^* - \delta, \beta^* + \delta); \delta > 0$ provided the following conditions are satisfied,

$$(i)\ A_1(\beta^*) > 0,\ A_3(\beta^*) > 0;\ \phi(\beta^*) = A_1(\beta^*)A_2(\beta^*) - A_3(\beta^*) = 0$$

$$(ii)\ \left(\frac{\partial \psi_1}{\partial \beta}\right)_{\beta=\beta^*} = -\frac{\left(A_2 \frac{\partial A_1}{\partial \beta} - \frac{\partial A_3}{\partial \beta} + A_1 \frac{\partial A_2}{\partial \beta}\right)}{2\left(A_1{}^2 + A_2\right)} \neq 0$$

Proof:

Consider the characteristic polynomial of the coexistence equilibrium point and assume the roots are non-zero, as $A_1 \neq 0$. Set the eigenvalues as smooth analytic function of the bifurcation parameter, β as;

$$\lambda_{1,2} = \psi_1 \pm i\psi_1 \equiv \psi_1(\beta) \pm i\psi_2(\beta)\ ;\ \lambda_3 = C(\beta) \in \Re \tag{1.7}$$

Then, at critical bifurcating threshold, $\beta = \beta^*$ we have that the pair of complex conjugate degenerates, and satisfies

$$\lambda_{1,2} = \pm i\psi_2(\beta^*)\ ;\ \psi_2(\beta^*) > 0, \quad \psi_1(\beta^*) = 0\ and\ \lambda_3 = C(\beta^*) \in \Re \tag{1.8}$$

On substituting eqn. (1.7) into the characteristic polynomial of coexistence equilibrium point, equating real and imaginary parts to zero yields;

$$P(\lambda) = \left(A_3 - A_1\,\psi_2{}^2\right) + i\left(A_2\psi_2 - \psi_2{}^3\right) = 0$$

$$A_3 = A_1\,\psi_2{}^2;\ A_2\psi_2 = \psi_2{}^3, \Rightarrow A_1A_2 - A_3 = 0$$

Hence the critical bifurcating threshold occurs on a smooth analytic function, say;

$$\phi(\beta^*) = A_1(\beta^*)A_2(\beta^*) - A_3(\beta^*);\ \lambda_{1,2} = \pm i\sqrt{A_2}, \lambda_3 = -A_1,\ A_1, A_3 > 0$$

Next, it suffices to verify the transversality condition, substitute $\lambda_1 = \psi_1 + i\psi_1$ into the characteristic polynomial, and equate real and imaginary parts yields;

$$-3\psi_1\,\psi_2{}^2 - A_1\,\psi_2{}^2 + \psi_1{}^3 + A_1\,\psi_1{}^2 + A_2\psi_1 + A_3 = 0 \tag{1.9}$$

$$-\psi_2{}^3 + 3\psi_2\,\psi_1{}^2 + 2\psi_1\psi_2A_1 + \psi_2A_2 = 0 \tag{2.0}$$

Diffentiating partially equations (1.9 & 2.0), w.r.t. β and applying condition of eqns. (1.7 & 1.8) yields;

$$\frac{\partial \psi_1}{\partial \beta}\left(-3\,\psi_2{}^2 + A_2\right) - 2A_1\psi_2\frac{\partial \psi_2}{\partial \beta} = \psi_2{}^2\frac{\partial A_1}{\partial \beta} - \frac{\partial A_3}{\partial \beta} \tag{2.1}$$

$$\frac{\partial \psi_2}{\partial \beta}\left(-3\,\psi_2{}^2 + A_2\right) + 2A_1\psi_2\frac{\partial \psi_1}{\partial \beta} = -\psi_2\frac{\partial A_2}{\partial \beta} \tag{2.2}$$

Solving equations (2.1 & 2.2) simultaneously for $\frac{\partial \psi_1}{\partial \beta}$ and evaluating at critical bifurcating parameter yields;

$$\left(\frac{\partial \psi_1}{\partial \beta}\right)_{\beta=\beta^*} = -\frac{\left(A_2 \frac{\partial A_1}{\partial \beta} - \frac{\partial A_3}{\partial \beta} + A_1 \frac{\partial A_2}{\partial \beta}\right)}{2\left(A_1{}^2 + A_2\right)}$$

Hence, there exist a family of periodic solutions bifurcating from positive coexistence equilibrium point $E_4(X = x^*, Y = y^*, Z = z^*)$ in the neighborhood of the bifurcating threshold, $\beta = \beta^*$. That is Hopf-bifurcation occurs when the bifurcation parameter is open for $\beta \in (\beta^* - \delta,\ \beta^* + \delta)$.

6. Numerical Responses of Hopf-bifurcation Limit Cycles of the Semi-trivial Equilibrium Points

6.1 Hopf-bifurcation Limit Cycle of Prey-predator Equilibrium Point

Given the bifurcation parameter κ, observe that at critical bifurcating threshold $\kappa^* = 4.3333$, a stable limit cycle degenerates for $\kappa = 4.4433 > 4.3333$ with eigenvalues $\lambda_{1,2} = \pm i0.5925 \; and \; \lambda_3 = -1.41025$. Additionally, the transversality condition is satisfied for $\left(\frac{d\psi_1(\kappa)}{d\kappa}\right)_{\kappa=\kappa^*} = 0.18 \neq 0$.

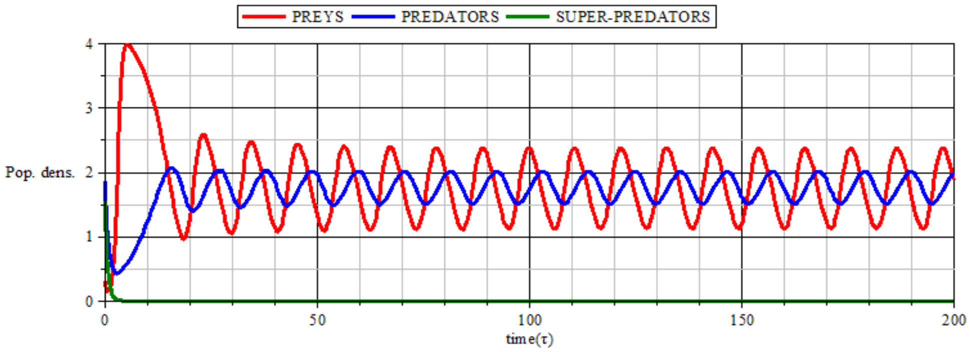

Fig. 1.0 Periodic solutions of prey-predator species

Fig. 1.0 illustrate cyclic periodic behaviors of prey-predator species; a super critical Hopf-bifurcation with degeneracy from the stable prey-predator equilibrium point. Its satisfies ecological parameters; $\alpha = 2.60, \kappa = 2.9231, \eta = 2.40, \varepsilon = 0.96, \xi = 0.60, \sigma = 1.0, \mu = 1.60, \beta = 0.15$, and subject to initial conditions $x(0) = 0.3077, y(0) = 1.8462, z(0) = 1.5385$.

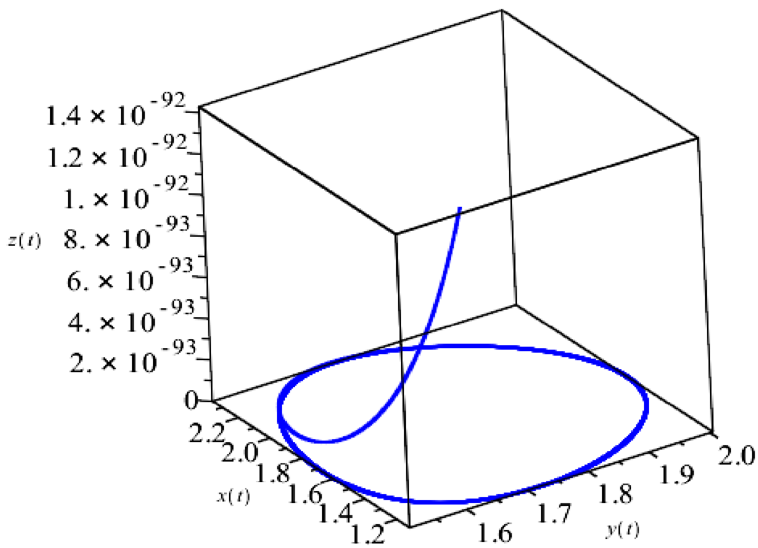

Fig. 2.0: Limit cycle of prey-predator oscillations.

Fig. 2.0 illustrate the phase-space diagram of prey-predator limit cycle of the model.

6.2 Hopf-bifurcation Limit Cycle of Prey-super Predator Equilibrium Point

Given the bifurcation parameter κ, observe that at critical bifurcating threshold $\beta^* = 0.40905$, a stable limit cycle degenerates for $\beta = 0.4512 > 0.40905$ with eigenvalues $\lambda_{1,2} = \pm i0.2284 \; and \; \lambda_3 = -1.6787$. Additionally, the transversality condition is satisfied for $\left(\frac{d\psi_1(\kappa)}{d\kappa}\right)_{\beta=\beta^*} = 1.7780 \neq 0$.

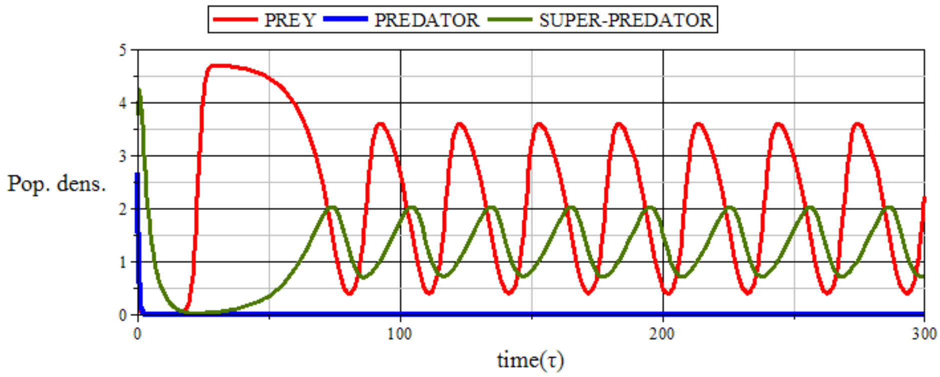

Fig. 3.0 Periodic solution of prey-superpredator oscillation

Fig. 3.0 illustrate cyclic periodic behaviors of prey super-predator interactions; a super critical Hopf-bifurcation with degeneracy from the stable prey super-predator equilibrium point. It satisfies ecological parameters; $\alpha = 0.9091, \kappa = 4.75, \eta = 0.4545, \varepsilon = 0.1136, \xi = 0.9091, \sigma = 0.5333, \mu = 0.2727, \beta = 0.4110$, and subject to initial conditions $x(0) = 1.25, y(0) = 2.6667, z(0) = 3.75$. Fig. 4.0 illustrate the phase-space diagram of prey and super-predator limit cycle of the model.

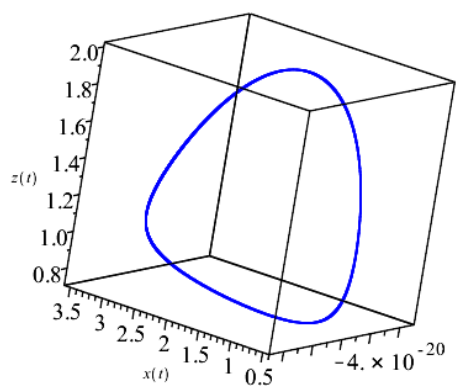

Fig. 4.0 Limit cycle of prey super-predator oscillations

6.3 Hopf-bifurcation Limit cycles of Coexistence Equilibrium Point.

We observe that as bifurcation parameter β, varies the system exhibit strange (chaotic) attractors for parameter value $\beta = 0.2245 < \beta^*$; quasi-periodic behaviors for, $\beta = 0.3655 < \beta^*$(Fig. 5.0); periodic behaviors for $\beta = 0.4913 < \beta^*$(Fig. 7.0); and asymptotic behavours for $\beta = 0.543 > \beta^*$. Fig. 6.0 and fig. 8.0 illustrate the phase-space diagram of quasi-periodic behavior and limit cycles of the interacting species at the coexistence equilibrium points, respectively. Hence, using proposition III, it suffices to claim that there exist a critical bifurcating threshold β^*, for $\beta = \{\beta^*: 0.4913 < \beta^* < 0.543\}$.

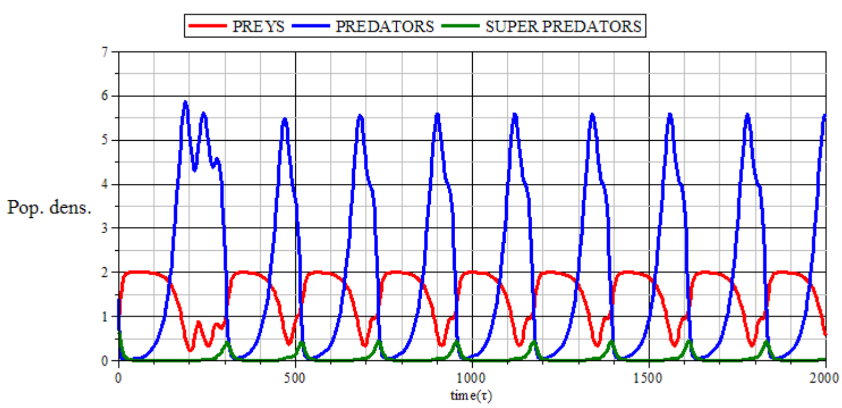

Fig. 5.0 Quasi-periodic solutions of coexistence equilibrium point (β = 0.4655)

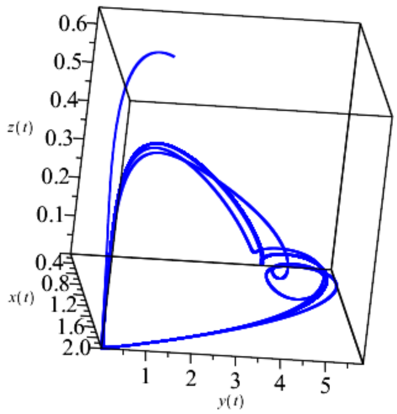

Fig. 6.0 Quasi-periodic behavior

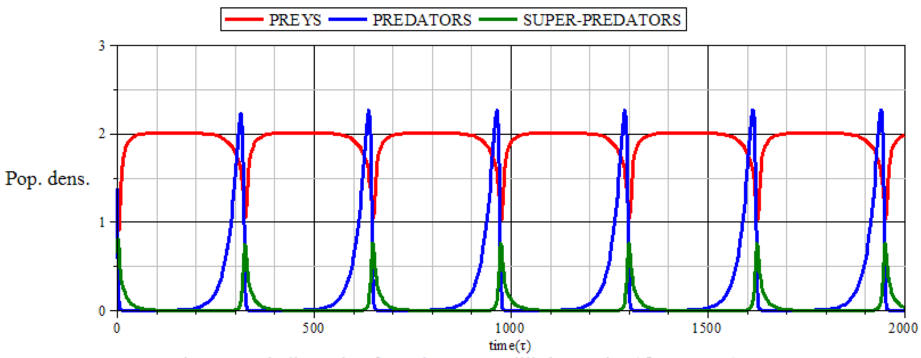

Fig. 7.0 Periodic cycle of coexistence equilibrium point (β = 0.4913)

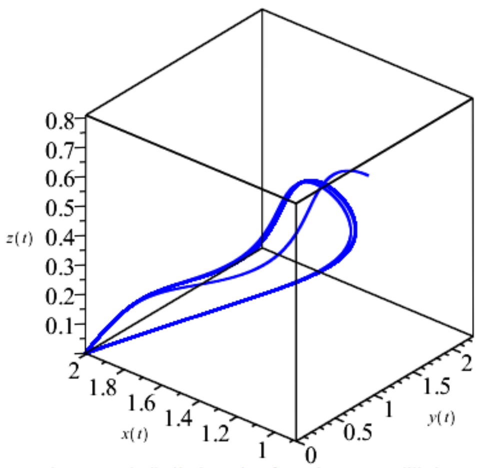

Fig. 8.0 Periodic limit cycle of coexistence equilibrium point

7. Conclusion

In this paper, we study a new dimensionless model as an extension of the Rosenzweig-MacArthur model; a tri-trophic food chain model that depicts and predicts the populations of interacting species in the natural ecological system. We established ultimate boundedness conditions and exponential convergence of populations of the interacting species. Using local stability conditions, we determine the existence of codimension-1 Hopf-bifurcation limit cycles of the model as the free-parameter changes. Ecologically, as the super-predator biomass efficiency increases through its ecological interactions with other species, the system oscillates periodically, and the species coexisted based on qualitative theory of dynamical system.

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