Existence of Solutions to the Boundary Value Problems for a Class of P-Laplacian Equations at Resonance

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Abstract

By the generalizing the extension of the continuous theorem of Ge and Ren and constructing suitable Banach spaces and operators, we investigate the existence- solutions to the boundary value problems for a class of p-Laplacian equations. Finally an example is given to illustrate our results.

Keywords: continuous theorem, resonance, p-Laplacian equations, boundary value problem

1. Introduction

In this paper,we will study the boundary value problem

$$\begin{cases} (\varphi_p(u''))'(t) = f(t, v, v', v'') \\ (\varphi_p(v''))'(t) = g(t, u, u', u'') \\ u(0) = u''(0) = 0, v(0) = v''(0) = 0 \\ u'(1) = \int_0^1 k_1(t)u'(t)dt, v'(1) = \int_0^1 k_2(t)v'(t)dt \end{cases} \tag{1.1}$$

where$\varphi_p(s) = |s|^{p-2}s, p > 1, \int_0^1 k_i(t)dt = 1, i = 1, 2$

In this paper,we will always suppose that

(H_1) $k_i(t) \in L^1[0, 1]$ are nonnegative and$\| k_i \|_1= 1$, where$\| k_i \|_1= \int_0^1 | k_i(t) | dt, i = 1, 2.$

$(H_2) f(t, u, v, w), g(t, u, v, w)$is continuous in $[0, 1] \times R^3$

2. Preliminaries

Definition 2.1 *Let X and Y be a two Banach spaces with norms* $\| \cdot \|_X, \| \cdot \|_Y$*,respectively.A continuous operator* $M : X \cap domM \longrightarrow Y$*is said to be quasi-linear if*

(i)$ImM := M(X \cap domM)$ *is a closed subset of* Y*,*

(ii)$KerM := \{x \in X \cap domM : Mx = 0\}$ *is linearly homeomorphic to* $R^n, n < \infty$.

where domM denote the domain of operator M.

Let $X_1 = \text{Ker}M$ and X_2 be complement space of X_1 in X,then $X = X_1 \bigoplus X_2$.Let $P : X \longrightarrow X_1$ be a projector and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$

Definition2.2 *Suppose* $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0, 1]$ *is a continuous and bounded operator.Denote* N_1 *by* N*.Let* $\Sigma_\lambda = \{x \in \overline{\Omega} \cap domM : Mx = N_\lambda x\}$*.*$N_\lambda$ *is said M-quasi-compact in* $\overline{\Omega}$ *if there exists a vector subspace* Y_1 *of* Y *satisfying* $dimY_1 = dimX_1$ *and two operators* Q, R *with* $Q : Y \to Y_1, QY = Y_1$*,being continuous ,bounded,and satisfying* $Q(I - Q) = 0$*,*$R : \overline{\Omega} \times [0, 1] \to X_2 \cap domM$ *continuous and compact such that for* $\lambda \in [0, 1]$*,*
(a) $(I - Q)N_\lambda(\overline{\Omega}) \subset ImM \subset (I - Q)Y$*,*
(b)$QN_\lambda x = \theta, \lambda \in (0, 1) \Leftrightarrow QNx = \theta$*,*
(c)$R(\cdot, 0)$ *is zero operator and* $R(\cdot, 0)|_\Sigma = (I - P)|_\Sigma$ *,*
(d)$M[P + R(\cdot, 0)] = (I - Q)N_\lambda$

Theorem 2.1 *Let X and Y be two Banach spaces with norms* $\| \cdot \|_X, \| \cdot \|_Y$*,respectively,and* $\Omega \subset X$ *be an open and bounded nonempty set.Suppose*

$$M : X \cap domM \to Y$$

is a quasi-linear operator and that $N_\lambda : \overline{\Omega} \to Y, \lambda \in [0,1]$ is M-quasi-compact.In addition,if the following conditions hold:
$(C_1) Mx \neq N_\lambda x, \forall x \in \partial\Omega \cap domM, \lambda \in (0,1)$,
$(C_2) deg\{JQN, \Omega \cap KerM, 0\} \neq 0$
then the abstract equation $Mx = Nx$ has at least one solution in $domM \cap \overline{\Omega}$,where $N = N_1$,$J : ImQ \to KerM$ is a homeomorphism with $J(\theta) = \theta$.

Proof. The proof is similar to the one of lemma 2.1 and Theorem 2.1 in [Ge et al., 2004].

We can easily get the following inequalities.
Lemma2.1 *For any $u, v \geq 0$,we have*
(1)$\varphi_p(u, v) \leq \varphi_p(u) + \varphi_p(v), 1 < p \leq 2$.
(2)$\varphi_p(u, v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v)), p \geq 2$.

In the following,we will always suppose that q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main Results

Let $X = C^2[0,1] \times C^2[0,1]$ with norm $\|(u,v)\| = \|u\| + \|v\|$,where$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$,
$Y = C[0,1] \times C[0,1] \times C[0,1] \times C[0,1]$with norm $\|(y_1, y_2, y_3, y_4)\| = \max\{\|y_1\|_\infty, \|y_2\|_\infty, \|y_3\|_\infty, \|y_4\|_\infty\}$ with $\|y\|_\infty = \max_{t\in[0,1]}|y(t)|$.We know that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces.

Define operators $M : X \cap domM \longrightarrow Y$,$N_\lambda : X \to Y$ as follows:

$$M(u,v) = \begin{pmatrix} (\varphi_p(u''))'(t) \\ (\varphi_p(v''))'(t) \\ T_1(\varphi_p(u''))'(t) \\ T_2(\varphi_p(v''))'(t) \end{pmatrix}, N_\lambda(u,v) = \begin{pmatrix} \lambda f(t, v(t)v'(t), v''(t)) \\ \lambda g(t, u(t), u'(t), u''(t)) \\ 0 \\ 0) \end{pmatrix}$$

where $T_i y = c_i, i = 1,2.y \in C[0,1], c_1, c_2$ satisfy

$$\int_0^1 k_i(t) \int_t^1 \varphi_q(\int_0^s (y(r) - c_i)dr)dsdt = 0 \tag{0.1}$$

$$\text{dom}M = \{(u,v) \in X | \varphi_p(u''), \varphi_p(v'') \in C^1[0,1], u(0) = u''(0) = v(0) = v''(0) = 0\}$$

Lemma3.1 *For $y \in C[0,1]$,there is only one constant $c_i \in R$ such that$T_i y = c_i$,with $|c_i| \leq \|y\|_\infty$,and $T_i : C[0,1] \to R$, $i = 1,2$. are continuous .*

The proof is similar to Lemma 3.1 in [Weihua, 2014].

It is clear that $(u,v) \in \text{dom}M$ is a solution if and only if it satisfies $M(u,v) = N(u,v)$ where $N = N_1$.For convenience, let$(a,b,c,d)^L = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

Lemma3.2 *M is a quasi-linear operator.*

Proof. It is easy to get $\text{Ker}M = \{(b_1 t, b_2 t) \mid b_1, b_2 \in R\} := X_1$.

For $(u,v) \in X \cap \text{dom}M$,if $M(u,v) = (y_1, y_2, c_1, c_2)^L$,then c_1, c_2 satisfy (3.1).On the other hand ,if $y_i \in C[0,1], T_i y_i = c_i$, $i = 1,2$,take

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \int_0^t (t-s)\varphi_q(\int_0^s y_1(r)dr)ds \\ \int_0^t (t-s)\varphi_q(\int_0^s y_2(r)dr)ds \end{pmatrix}.$$

We can get $(u,v) \in X \cap \text{dom}M$ and $M(u,v) = (y_1, y_2, c_1, c_2)^L$,then c_1, c_2.Thus

$$\text{Im}M = \{(y_1, y_2, c_1, c_2)^L \mid y \in C[0,1], c_1, c_2 \text{satisfy (3.1)}\}.$$

By the continuity of $T_i, i = 1,2$,we see that $\text{Im}M \subset Y$ is closed.So,M is quasi-linear.The proof is completed.

Lemma3.3 $T_i(c) = c, T_i(y + c) = T_i(y) + c, T_i(cy) = cT_i(y), i = 1,2, c \in R, y \in C[0,1]$

Proof. The proof is simple.Therfore we omit it.

Take a projector $P : X \to X_1$ and an operator $Q : Y \to Y_1$ as follows:

$$(P(u,v))(t) = (u'(0)t, v'(0)t), Q(y_u, y_v, y_1, y_2)^L = (0, 0, T_1y_1 - T_1y_u, T_2y_2 - T_2y_v)^L$$

where $Y_1 = \{(0, 0, c_1, c_2)^L \mid c_i \in R, i = 1.2\}$.Obviously $QY = Y_1$and $\dim Y_1 = \dim X_1$.

By the continuity and boundedness of $T_i, i = 1, 2$,we can easily see that Q is continuous and bounded in$Y$.It follows Lemma3.3 that $Q(I - Q)(y_u, y_v, y_1, y_2)^L = (0, 0, 0, 0)^L, y_u, y_v, y_1, y_2 \in C[0, 1]$

Define a operator $R : X \times [0, 1] \to X_2$

$$R(u,v,\lambda)(t) = \begin{pmatrix} \int_0^t (t-s)\varphi_q(\int_0^s \lambda f(r, v(r), v'(r), v''(r))dr)ds \\ \int_0^t (t-s)\varphi_q(\int_0^s \lambda g(r, u(r), u'(r), u''(r))dr)ds \end{pmatrix},$$

where $\mathrm{Ker}M \bigoplus X_2 = X$.By (H_2) and the Arzela-Asscoli theorem,we can get $R : \overline{\Omega} \times [0, 1] \to X_2$ is continuous and compact,where $\Omega \subset X$ is a bounded set.

Lemma3.4 *Assume that $\Omega \in X$ is an open bounded set.Then N_λ is M-quasi-compact in $\overline{\Omega}$.*

Proof. It is clear that $\mathrm{Im}P = \mathrm{Ker}M$, $QN_\lambda(u,v) = \theta \Leftrightarrow QN(u,v) = \theta$ and $R(\cdot,\cdot,0) = 0$. for $(u,v) \in \overline{\Omega}$,

$$(I-Q)N_\lambda(u,v) = \begin{pmatrix} \lambda f(t, v(t), v'(t), v''(t)) \\ \lambda g(t, u(t), u'(t), u''(t)) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ T_1(\lambda f(t, v(t), v'(t), v''(t))) \\ T_2(\lambda g(t, u(t), u'(t), u''(t)) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda f(t, v(t), v'(t), v''(t)) \\ \lambda g(t, u(t), u'(t), u''(t)) \\ T_1(\lambda f(t, v(t), v'(t), v''(t))) \\ T_2(\lambda g(t, u(t), u'(t), u''(t)) \end{pmatrix} \in \mathrm{Im}M.$$

Since $\mathrm{Im}M \subset \mathrm{Ker}Q$, and $y = Qy + (I - Q)y$,we obtain $\mathrm{Im}M \subset (I - Q)Y$.Thus $(I - Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im}M \subset (I - Q)Y$.
For $(u, v) \in \Sigma_\lambda = \{(u, v) \in \overline{\Omega} \cap \mathrm{dom}M \mid M(u, v) = N_\lambda(u, v)\}$

$$R(u,v,\lambda) = \begin{pmatrix} \int_0^t (t-s)\varphi_q(\int_0^s \lambda f(r, v(r), v'(r), v''(r))dr)ds \\ \int_0^t (t-s)\varphi_q(\int_0^s \lambda g(t, u(t), u'(t), u''(t))dr)ds \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^t (t-s)\varphi_q(\int_0^s \varphi_p(u''))'(r)dr)ds \\ \int_0^t (t-s)\varphi_q(\int_0^s \varphi_p(v''))'(r)dr)ds \end{pmatrix}$$

$$= \begin{pmatrix} u(t) - u'(0)t \\ v(t) - v'(0)t \end{pmatrix} = (I-P)(u,v).$$

i.e. Definition 2.2(c)holds.For $u \in \overline{\Omega}$,we have

$$M[P(u,v) + R(u,v,\lambda)] = \begin{bmatrix} \lambda f(r, v(r), v'(r), v''(r)) \\ \lambda g(t, u(t), u'(t), u''(t)) \\ T_1(\lambda f(t, v(t), v'(t), v''(t))) \\ T_2(\lambda g(t, u(t), u'(t), u''(t)) \end{bmatrix} = (I-Q)N_\lambda(u,v)$$

Thus,Definition 2.2(d)holds.Therefore,N_λ is M-quasi-compact in $\overline{\Omega}$.The proof is completed.

Theorem 3.1 *Assume that the following conditions hold:*
(H_3)There exist nonnegative constants K_1, K_2 such that one of (1)and (2) holds:
(1)

$$B_1 f(t, A_1, B_1, C_1) > 0, \quad t \in [0, 1], |B_1| > K_1, A_1, C_1 \in R$$

and

$$B_2 g(t, A_2, B_2, C_2) > 0, \quad t \in [0, 1], |B_2| > K_2, A_2, C_2 \in R$$

(2)

$$B_1 f(t, A_1, B_1, C_1) < 0, \quad t \in [0, 1], |B_1| > K_1, A_1, C_1 \in R$$

and

$$B_2 g(t, A_2, B_2, C_2) < 0, \quad t \in [0, 1], |B_2| > K_2, A_2, C_2 \in R$$

(H_4) *There exist nonnegative functions* $a_i(t), b_i(t), c_i(t), e_i(t) \in L^1[0, 1], i = 1, 2$ *such that*

$$| f(t, x, y, z) | \le a_1(t)\varphi_p(| x |) + b_1(t)\varphi_p(| y |) + c_1(t)\varphi_p(| z |) + e_1(t)$$

and

$$| g(t, x, y, z) | \le a_2(t)\varphi_p(| x |) + b_2(t)\varphi_p(| y |) + c_2(t)\varphi_p(| z |) + e_2(t)$$

where $\varphi_q(\|a_i(t)\|_1 + \|b_i(t)\|_1 + \|c_i(t)\|_1,) < 2^{2-q}$ *if* $1 < p \le 2$;$\varphi_q(2^{p-2}\|a_i(t)\|_1 + 2^{p-2}\|b_i(t)\|_1 + \|c_i(t)\|_1,) < 1$ *if* $p \ge 2$
The boundary value problem (1.1) has at least one solution.

Lemma3.5 *Suppose* (H_3) *and* (H_4) *hold.Then*

$$\Omega_1 = \{(u, v) \in domM \mid M(u, v) = N_\lambda(u, v)\}$$

is bounded in X.

Proof. For $(u, v) \in \Omega_1$,we have $QN_\lambda(u, v) = 0, i.e.$

$$T_1(\lambda f(t, v(t), v'(t), v''(t))) = 0, T_2(\lambda g(t, u(t), u'(t), u''(t)) = 0$$

ByH_3,there exist constants $t_0, t_1 \in [0, 1]$ such that $| u'(t_0) | \le K_2$ and $| v'(t_1) | \le K_1$.Since $u(t) = \int_0^t u'(s)ds$, $u'(t) = u'(t_0) + \int_{t_0}^t u''(s)ds$, $v(t) = \int_0^t v'(s)ds$, $v'(t) = v'(t_1) + \int_{t_1}^t v''(s)ds$,,then that

$$\begin{array}{c} | u(t) | \le \| u' \|_\infty, | u'(t) | \le K_2 + \| u'' \|_\infty, \quad t \in [0, 1]. \\ | v(t) | \le \| v' \|_\infty, | v'(t) | \le K_1 + \| v'' \|_\infty, \quad t \in [0, 1]. \end{array} \tag{0.2}$$

It follow from $M(u, v) = N_\lambda(u, v)$,$(H_4)$ and (3.2)that

$$\begin{aligned} | u''(t) | &= | \varphi_q(\int_0^t \lambda f(s, v(s), v'(s), v''(s))ds) | \\ &\le \varphi_q(\int_0^1 a_1(t)\varphi_p(| v |) + b_1(t)\varphi_p(| v' |) + c_1(t)\varphi_p(| v'' |) + e_1(t)dt) \\ &\le \varphi_q[(\|a_1\|_1 + \|b_1\|_1)\varphi_p(K_1 + \| v'' \|_\infty) + \|c_1\|_1\varphi_p(\| v'' \|_\infty) + \|e_1\|_1] \end{aligned}$$

If $1 < p \le 2$,by Lemma 2.1,we get

$$\begin{aligned} | u''(t) | &\le \varphi_q(D_1 + C_1\varphi_p \| v'' \|_\infty) \\ &\le 2^{q-2}[\varphi_q(D_1) + \varphi_q(C_1) \| v'' \|_\infty] \\ &\le 2^{q-2}[\varphi_q(D_1) + \varphi_q(C_1) \| v \|] \end{aligned}$$

thus

$$\| u'' \|_\infty \le 2^{q-2}[\varphi_q(D_1) + \varphi_q(C_1) \| v \|]$$

where $C_1 = \|a_1\|_1 + \|b_1\|_1 + \|c_1\|_1, D_1 = (\|a_1\|_1 + \|b_1\|_1)\varphi_p(K_1) + \|e_1\|_1$. On the other hand, since $| u'(t) | \le K_2 + \| u'' \|_\infty, t \in [0, 1]$,we get $\| u' \|_\infty \le K_2 + \| u'' \|_\infty$ since $| u(t) | \le \| u' \|_\infty, t \in [0, 1]$,we get$\| u \|_\infty \le K_2 + \| u'' \|_\infty$.Thus

$$\| u \| \le K_2 + 2^{q-2}[\varphi_q(D_1) + \varphi_q(C_1) \| v \|]. \tag{0.3}$$

Similarly,

$$\| v \| \le K_1 + 2^{q-2}[\varphi_q(D_2) + \varphi_q(C_2) \| u \|] \tag{0.4}$$

where $C_2 = \|a_2\|_1 + \|b_2\|_1 + \|c_2\|_1, D_2 = (\|a_2\|_1 + \|b_2\|_1)\varphi_p(K_2) + \|e_2\|_1$ Take$M_1 = \max\{2^{q-2}\varphi_q(C_1), 2^{q-2}\varphi_q(C_2)\}$,we get

$$\| u \| \le K_2 + 2^{q-2}\varphi_q(D_1) + M_1 \| v \|,$$

$$\| v \| \le K_1 + 2^{q-2}\varphi_q(D_2) + M_1 \| u \| .$$

Since $\| (u, v) \| = \| u \| + \| v \|$, we get

$$\| (u, v) \| \leq \frac{K_1 + K_2 + 2^{q-2}\varphi_q(D_1) + 2^{q-2}\varphi_q(D_2)}{1 - M_1}$$

If $p > 2$,by Lemma 2.1,we get

$$| u''(t) | \leq \varphi_q(D_3 + C_3\varphi_p \| v'' \|_\infty) \leq \varphi_q(D_3) + \varphi_q(C_3) \| v'' \|_\infty$$

$$\leq \varphi_q(D_3) + \varphi_q(C_3) \| v \|,$$

thus

$$\| u'' \|_\infty \leq \varphi_q(D_3) + \varphi_q(C_3) \| v \| .$$

By (3.2),we get

$$\| u \| \leq K_2 + \varphi_q(D_3) + \varphi_q(C_3) \| v \| .$$

Similarly

$$\| v \| \leq K_1 + \varphi_q(D_4) + \varphi_q(C_4) \| v \| .$$

where $C_3 = 2^{p-2}(\|a_1\|_1 + \|b_1\|_1) + \|c_1\|_1, D_3 = 2^{p-2}(\|a_1\|_1 + \|b_1\|_1)\varphi_p(K_1) + \|e_1\|_1$, $C_4 = 2^{p-2}(\|a_2\|_1 + \|b_2\|_1) + \|c_2\|_1, D_4 = 2^{p-2}(\|a_2\|_1 + \|b_2\|_1)\varphi_p(K_2) + \|e_2\|_1$,Take $M_2 = \max\{\varphi_q(C_3), \varphi_q(C_4)\}$,we get

$$\| (u, v) \| \leq \frac{K_1 + K_2 + \varphi_q(D_3) + \varphi_q(D_4)}{1 - M_2}$$

So we obtain Lemma 3.5.

Remark1 If we take $\|(u, v)\| = \max\{\|u\|, \|v\|\}$,Lemmma 3.5 still holds.We only need (3.3) into (3.4),we can obtain the Lemma.

Lemma3.6 *Assume (H_3) holds,then*

$$\Omega_2 = \{(u, v) \in KerM \mid QN(u, v) = 0\}$$

is bounded in X,where$N = N_1$.

Proof. For $(u, v) \in \Omega_2$,we have $(u, v) = (b_1t, b_2t)$,then $T_1 f(t, b_2t, b_2, 0) = 0, T_2 f(t, b_1t, b_1, 0) = 0$.By (H_3),we get$| b_1 | \leq K_2, | b_2 | \leq K_1$.So,$\Omega_2$ is bounded.

Proof of theorem3.1 Let $\Omega = \{(u, v) \in X \mid \| (u, v) \| < r\}$,where r is large enough such that $K_1 + K_2 < r < +\infty$ and $\overline{\Omega_1} \cup \overline{\Omega_2} \subset \Omega$.

By lemma3.5 and Lemma3.6,we can get if $(u, v) \in \mathrm{dom}M \cap \partial\Omega$, then $M(u, v) \neq N_\lambda(u, v)$, if $(u, v) \in \mathrm{Ker}M \cap \partial\Omega$, then $QN(u, v) \neq 0$.

Let

$$H(u, v, \delta) = \rho\delta(u, v)^L + (1 - \delta)JQN(u, v)^L.\delta \in [0, 1], (u, v) \in \mathrm{Ker}M \cap \overline{\Omega}.$$

where $J : \mathrm{ImQ} \to \mathrm{KerM}$ is a homeomorphism with $J(0, 0, b_1, b_2)^L = (b_2t, b_1t)^L$,

$$\rho = \begin{cases} -1 & , \ \text{if}(H_3)(1)\text{holds} \\ 1 & , \ \text{if}(H_3)(2)\text{holds} \end{cases}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & , \ x > 0 \\ -1 & , \ x < 0 \end{cases}$$

For $(u, v) \in \mathrm{KerM} \cap \partial\Omega$,we have $(u, v) = (b_1t, b_2t) \neq (0, 0)$

$$H(u, v, \delta) = \rho\delta \begin{pmatrix} b_1t \\ b_2t \end{pmatrix} + (1 - \delta) \begin{pmatrix} -T_2g(t, b_1t, b_1, 0)t \\ -T_1 f(t, b_2t, b_2, 0)t \end{pmatrix}$$

If $\delta = 1, h(u, v, 1) = \rho(b_1t, b_2t)^L \neq (0, 0)^L$.If $\delta = 0, h(u, v, 0) = JQN(b_1t, b_2t)^L \neq (0, 0)^L$. For $0 < \delta < 1$,we now prove that $H(u, v, \delta) \neq (0, 0)^L$.Otherwise,If $H(u, v, \delta) = (0, 0)^L$,then

$$\begin{pmatrix} T_2g(t, b_1t, b_1, 0) \\ T_1 f(t, b_2t, b_2, 0) \end{pmatrix} = \begin{pmatrix} \frac{\rho\delta}{1-\delta} b_1 \\ \frac{\rho\delta}{1-\delta} b_2 \end{pmatrix}$$

Since $\|(u, v)\| = r > K_1 + K_2$,we have $|b_1| > K_2$ or $|b_2| > K_1$,If $|b_2| > K_1$we have

$$T_1(b_2 f(t, b_2 t, b_2, 0)) = b_2 T_1(f(t, b_2 t, b_2, 0)) = \frac{\rho\delta}{1-\delta} b_2^2$$

$$\text{sgn}(T_1(b_2 f(t, b_2 t, b_2, 0))) = \text{sgn}[b_2 f(t, b_2 t, b_2, 0)] = \text{sgn}(\frac{\rho\delta}{1-\delta} b_2^2) = \text{sgn}(\rho)$$

if $|b_1| > K_2$,we have

$$T_2(b_1 g(t, b_1 t, b_1, 0)) = b_1 T_2(g(t, b_1 t, b_1, 0)) = \frac{\rho\delta}{1-\delta} b_1^2$$

$$\text{sgn}(T_2(b_1 g(t, b_1 t, b_1, 0))) = \text{sgn}[b_1 g(t, b_1 t, b_1, 0)] = \text{sgn}(\frac{\rho\delta}{1-\delta} b_1^2) = \text{sgn}(\rho)$$

This is a contradiction with the definition of ρ. So,$H(u, v, \delta) \neq 0, (u, v) \in \text{KerM} \cap \partial\Omega, \delta \in [0, 1]$.

By the homotopy of degree ,we get $\deg(JQN, \Omega \cap \text{KerM}, 0) = \deg(H(\cdot, \cdot, 0), \Omega \cap \text{Ker}M, 0) = \deg(H(\cdot, \cdot, 1), \Omega \cap \text{Ker}M, 0) = \deg(\rho I, \Omega \cap \ \text{Ker}M, 0) \neq 0$ By Theorem 2.1,we find that (1.1) has at least one solution in$\overline{\Omega}$ .The proof is completed.

Remark2 If $\|(u, v)\| = \max\{\|u\|, \|v\|\}$,Lemma 3.6 still holds.We can take $\max\{K_1, K_2\} < r < \infty$,we still get the Theorem 3.1.

4. Example

Let us consider the following boundary value problem

$$\begin{cases} (\varphi_p(u''))'(t) = \frac{t^3}{4}\sin x^5 + \frac{1}{8}y^5 + \frac{t^2}{4}\sin z^5 + \cos t \\ (\varphi_p(v''))'(t) = \frac{t^4}{8}\cos x^5 + \frac{1}{16}y^5 + \frac{t^3}{12}\cos z^5 + \sin t \\ u(0) = u''(0) = 0, v(0) = v''(0) = 0 \\ u'(1) = \int_0^1 2tu'(t)dt, v'(1) = \int_0^1 3t^2 v'(t)dt \end{cases} \tag{4.1}$$

where $p = 6$.

Corresponding to problem(1.1),we have $q = \frac{6}{5}, a_1(t) = \frac{t^3}{4}, b_1(t) = \frac{1}{8}, c_1(t) = \frac{t^2}{4}, e_1(t) = cost, k_1(t) = 2t$, $a_2(t) = \frac{t^4}{8}$, $b_2(t) = \frac{1}{16}, c_2(t) = \frac{t^3}{12}, e_1(t) = sint, k_2(t) = 3t^2$. Take $K_1 = 2, K_2 = 3$,we can get $(H_1) - (H_4)$ hold.By Theorem3.1,we have the problem(4.1)has at least one solution.

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