Strong Masting Conjecture for Multiple Size Hexagonal Tessellation in GSM Network Design

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Received: February 18, 2016 Accepted: March 7, 2016 Online Published: May 18, 2016

doi:10.5539/jmr.v8n3p44 URL: http://dx.doi.org/10.5539/jmr.v8n3p44

Abstract

One way to improve cellular network performance is to use efficient handover method and design pattern among other factors. The efficient design pattern has been proven geometrically to be hexagonal (Hales, 2001, pp. 1- 22) due to its maximum tessellable area coverage. But uneven geographical distribution of subscribers requires tessellable hexagons of different radii due to variation of costs of GSM masts. This will call for an overlap difference. The constraint of minimum overlap difference for multiple cell range is a new area that is untapped in cell planning. This paper addresses such multiple size hexagonal tessellation problem using a conjecture. Data from MTN River State-Nigeria, was collected. Multiple Size Hexagonal Tessellation Model (MSHTM) conjecture for masting three (3) different size MTN GSM masts in River State, accounted for least overlap difference with area of 148.3km^2 using 36 GSM masts instead of the original 21.48 km^2 for 50 GSM masts. Our conjecture generally holds for k-different ($k \geq 2$) cell range.

Keywords: disks, frequency, GSM masts, hexagon, overlap difference, tessellation.

1. Introduction

Cell planning is the most significant operations in GSM design network. It includes the choice of design pattern (triangular, square or hexagon), geographic, environmental and network parameters such as terrain and artificial structures, base station location and transmission power among others. But the hexagonal design has least overlap and hence has a strength higher than both a square and an equi-triangular polygon. The hexagon motivated circular shaped cells are produced when two or more sector signal radiated antenna are used. These circular cells overlap significantly and is crucial for subscriber movement-handover. This overlap removes signal loss due to no coverage or ensures soft handover due to better overlapping regions. How much overlap difference to be permitted in the uniform design network has been studied extensively by Donkoh E. K. et.al (2015a). Uneven geographical distribution of subscribers require varied cell ranges. This will result in variation in overlap difference. Both the design pattern and geometry of overlap is complex but the result is more economical as fewer GSM masts will be used. Realistically, how much overlap has been a thriving challenge in recent times due to variation in subscriber geographical distribution, affordability and significance of mobile telephone in this age of technology.

2. Related Literature

Antenna signal radiation in GSM network design takes many form. But the most profitable radiation pattern is known to be the sector motivated circular shape (Azad, 2012.). Due to it convenience and usefulness manufacturers of GSM antenna like Mobile Mark, Multiband Technologies, Global Source, Asian Creation among others have designed GSM antenna's with variety of sector angles including 30^0, 60^0, 90^0, 120^0 as the common ones. Nonetheless, designing GSM network due to uneven geographical distribution of subscribers requires geometry of 2-D hexagonal tessellation . Unfortunately, multiple size hexagonal design has since not been generally practiced due to the complexity in the position of the base station that offers minimum overlap difference. This paper uses the least tessellable polygonal overlap difference to conjecture a multiple size design formula for hexagons of several dimensions.

Donkoh & Opoku (2016, pp.33) emphasize with geometric proof that the hexagon has the least overlap difference of 13.4% and hence has the strongest tessellable area coverage.

Donkoh E. K et.al (2015a, pp. 5) investigated the hexagonal overlap difference for a uniform 0.6km cell range of 50 GSM masts design and obtain an overlap difference of 5.788km using 35 GSM masts instead of the original 26.884km.

This however confirms the economies of scale for design pattern minimizing overlap difference with smooth handover.

Hamad-Ameen J.J (2008, pp. 393), studied frequency and cell planning with clusters (K=7,9,13) and without clusters for uniform cell range and concluded that it is very efficient in GSM masting.

Donkoh E. K et. al (2015c), the authors considered GSM network design with two different cell range namely 1km and 3km obtaining the least overlap difference of 15.27km as compared to the original 30.95km.

In application to earth and mineral science, Raposo (2011), uses vertex-clustering of uniform hexagonal tessellation to simplify scale-specific automated map line.

Beyond applications of Christaller (1933) classic theory, hexagonal tessellation has been advocated for thematic cartography by Carr et.al (1992), and has been used to study cluster perception in animated maps (Griffin, 2006), as well as color perception (Brewer, 1996).

However, geometry of multiple size hexagonal tessellation in covering bounded areas, optimizing the overlap difference for k-different $(k \geq 2)$ cell ranges have not been studied. We therefore proposed the strong multiple size masting conjecture for solving the bounded area coverage problem in GSM network design.

3. Computational Experience

We consider intersecting circular cells superimposed on non-hexagonal polygon with cell range R_1 and R_2 and [1]apothems r_1 and r_2 respectively. In Figure 1, XZ is one side of the non-hexagonal polygon and the cells overlap for covering purposes with an overlap difference(d) of $R_2 - r_2 + R_1 - r_1$.

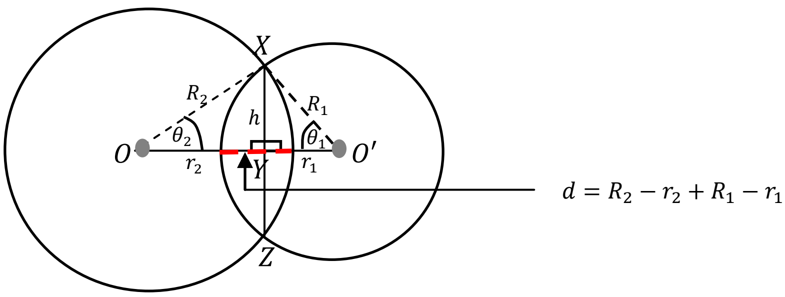


Figure 1. Overlap difference for Non-Uniform cell Range R_1, R_2.

From triangle XOY in Figure 1, $R_2^2 = h^2 + r_2^2$ where XY is a side of the right triangle.

$$h = \sqrt{R_2^2 - r_2^2} \tag{1}$$

$$\sin\theta_2 = \frac{h}{R_2}$$

$$h = R_2 \sin\theta_2 \tag{2}$$

Equating (1) and (2),

$$R_2 \sin\theta_2 = \sqrt{R_2^2 - r_2^2}$$

$$\sin\theta_2 = \sqrt{1 - \left(\frac{r_2^2}{R_2^2}\right)} \tag{3}$$

Also, $r_2 = R_2 cos\theta_2$

Similarly in triangle $\mathrm{XO'Y}, \mathrm{R}_1^2 = \mathrm{h}^2 + \mathrm{r}_1^2$ where h is a side of the right triangle.

$$h = \sqrt{R_1^2 - r_1^2} \tag{4}$$

$$\sin\theta_1 = \frac{h}{R_1}$$

$$h = R_1 \sin\theta_1 \tag{5}$$

Equating (4) and (5),

[1] Apothem is a line drawn from the center of a regular polygon to an edge and perpendicular to that edge. It is the perpendicular bisector of that edge and also the radius of the inscribed circle to that polygon.

$$R_1 \sin\theta_1 = \sqrt{R_1^2 - r_1^2}$$

$$\sin\theta_1 = \sqrt{1 - \left(\frac{r_1^2}{R_1^2}\right)} \tag{6}$$

Similarly in triangle $XO'Y, R_1^2 = h^2 + r_1^2$ where h is a side of the right triangle.
Also, $r_1 = R_1 cos\theta_1$
Equating (2) and (5) and extending it to radius R_2

$$R_1 \sin\theta_1 = R_2 \sin\theta_2$$

$$R_1 = R_2 \frac{\sin\theta_2}{\sin\theta_1}$$

$$R_1 = \frac{R_1 R_2 \sin\theta_2}{\sqrt{R_1^2 - r_1^2}}$$

$$\sin\theta_2 = \frac{\sqrt{R_1^2 - r_1^2}}{R_2} \tag{7}$$

Similarly, in triangle $XO'Y$,

$$\sin\theta_1 = \frac{\sqrt{R_2^2 - r_2^2}}{R_1} \tag{8}$$

A single overlap difference for Figure 2 is

$$d_1 = R_2 - r_2 + R_1 - r_1 \tag{9}$$

$$d_1 = \sum_{i=1}^{2} R_i - r_i$$

$$d_1 = \sum_{i=1}^{2} R_i(1 - cos\theta_i) \tag{10}$$

Generally n non-uniform overlaps,

$$d_n = \sum_{i=1}^{2n} R_i - r_i$$

$$d_n = \sum_{i=1}^{2n} R_i(1 - cos\theta_i) \tag{11}$$

Equation (9), (10) or (11) is used to calculate the overlap difference for non-uniform disks as shown in Figure 1. We established a formula for calculating the area of a pair of overlap for non-hexagonal polygon inscribed disks. From Figure 1 we have:

Area of single overlap difference = $(\text{Area of sector } XO'Z - \text{Area of triangle } XO'Z\,) + (\text{Area of sector } XOZ - \text{Area of triangle } XOZ\,)$

$$A_d = \frac{1}{2} R_1^2(\theta_1 - sin\theta_1) + \frac{1}{2} R_2^2(\theta_2 - sin\theta_2) \tag{12}$$

Substituting (7) and (8) into (12)

$$A_d = \frac{1}{2} R_1^2 \left[sin^{-1}\left(\frac{\sqrt{R_2^2 - r_2^2}}{R_1}\right) - \frac{\sqrt{R_2^2 - r_2^2}}{R_1} \right] + \frac{1}{2} R_2^2 \left[sin^{-1}\left(\frac{\sqrt{R_1^2 - r_1^2}}{R_2}\right) - \frac{\sqrt{R_1^2 - r_1^2}}{R_2} \right] \tag{13}$$

Generally for n different overlaps,

$$A_n = \frac{1}{2} \sum_{i=1}^{2n} R_i^2(\theta_i - sin\theta_i)$$

$$A_d = \frac{1}{2} \sum_{i=1}^{2n} R_i^2 \left[sin^{-1}\left(\frac{\sqrt{R_{i+1}^2 - r_{i+1}^2}}{R_i}\right) - \frac{\sqrt{R_{i+1}^2 - r_{i+1}^2}}{R_i} \right] \tag{14}$$

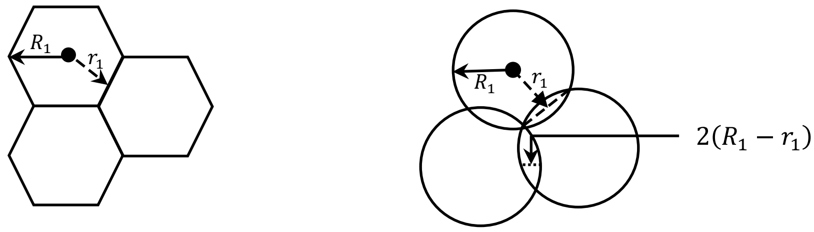
Equation (13) is the area of each non-uniform cell range in terms of overlap difference that is not created by hexagon. The value A_d can be calculated from Figure 1 in Autocad environment or using equation (13) as shown in column 4 and 8 in Table 2 where R_1 and R_2 are the cell ranges of the GSM masts. The equation $d = d_m - d_n$ is the overlap difference between disks with centres m and n.

3.1 Overlap for Optimal Disks Covering in Hexagonal Tessellation.

Donkoh E. K et.al (2015b, pp.26) in their study of overlap dimensions in cyclic tessellable regular polygon emphasize that hexagon has better overlap difference of 13.7% as compared to 29.3% for square and 50.0% for equi-triangular polygon.

$$\text{Overlap difference} = \begin{cases} Hexagon, & 0 < Strong \leq 13.7\% \\ Square, & 29.3\% \leq Moderate < 50\% \\ Equi-triangular, & Weak \geq 50\% \end{cases}$$

Figure 2(a) illustrates the hexagonal cell layout with inradius and circumradius of the hexagonal cell as r_1 and R_1, respectively. In Figure 2(b), cells partially overlapped because R_1 equals to the hexagon's circumradius.



(a) Hexagonal cell layout (b) Idealized circular layout

Figure 2. Cell Layout Model's For GSM Networks.

3.2 Generalized Overlap Difference in Hexagon-Inscribed Disks.

Overlap in cell planning ensures smooth handover in GSM network with a differential effect such as increasing the number of GSM masts, environmental destruction etc and therefore must be kept as minimal as possible. A typical overlap may arise as a result of uniform or non-uniform cell range. Ample research work has been done on the uniform cell range (Donkoh E. K, et al 2015a). We minimize the overlap difference for non-uniform cell range of two different GSM antenna masts with radii R_1 and R_2 ($R_2 > R_1$) and corresponding apothem r_1 and r_2 to be $R_1 - r_1 + 2R_2 - r_2$. A mixture of non-uniform cell radius with wrong frequency assignment results in large overlap difference and in effect increases interference (such as cross talk, background noise, error in digital signaling-missed calls, blocked calls, dropped calls, etc). It also has the differential advantage of reducing cost of coverage area as multiple cell range are factored in the design. Figure 4 shows 1: k side length, for k∈ $\mathbb{N} \geq 2$ overlap difference for two different radii R_1 and R_2.

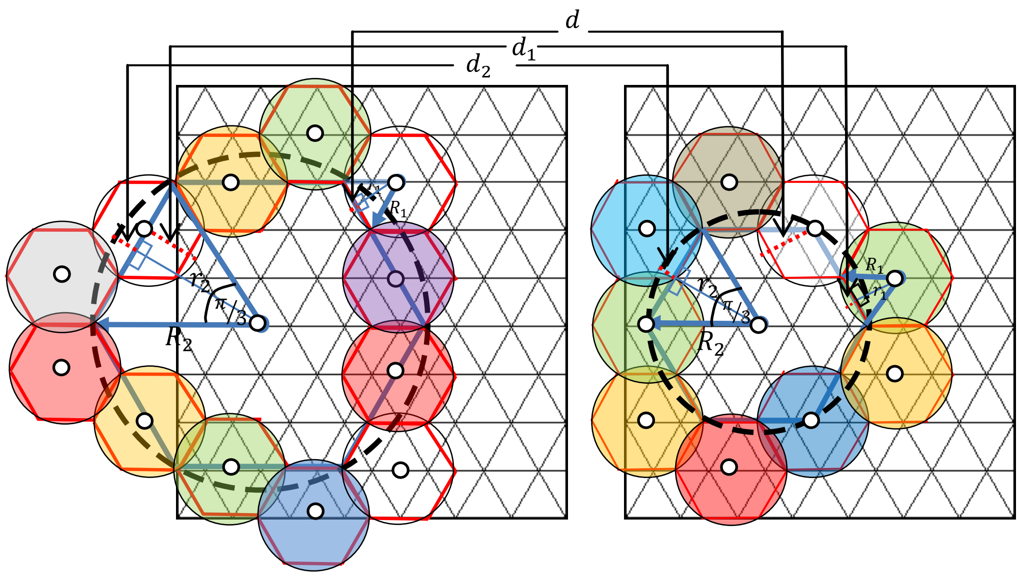


Figure 4. Overlap difference for non-uniform disks, represented by ····, and GSM cell by O.

Let $d_{R_2,R_1} = d + d_1 + d_2$ represents the global least overlap difference for two different size disks superimposed on tessellable hexagon in the ratio 1:k, where

$$d_{R_2,R_1} = R_1 - r_1 + R_1 + R_2 - r_2$$

$$d_{R_2,R_1} = R_2 - r_2 + 2R_1 - r_1 \tag{18}$$

Theorem: The apothem r_n created by n sided tessellable regular polygon inscribed in a disk of radius R_1 is

$$r_n = R_1 Cos\left(\frac{\pi}{n}\right).$$

Proof.

Donkoh et al (2016, pp.33-34) gave a formal proof of this theorem. Equation (18) then becomes

$$d_{R_2,R_1} = R_2 - R_2 cos\left(\frac{\pi}{n}\right) + 2R_1 - R_1 cos\left(\frac{\pi}{n}\right)$$

$$d_{R_2,R_1} = R_2\left[1 - cos\left(\frac{\pi}{n}\right)\right] + R_1\left[2 - cos\left(\frac{\pi}{n}\right)\right] \tag{19}$$

Since the polygon is a hexagon $n = 6$ sided but with different radii. Thus

$$d_{R_2,R_1} = R_2\left[1 - cos\left(\frac{\pi}{6}\right)\right] + R_1\left[2 - cos\left(\frac{\pi}{6}\right)\right]$$

$$d_{R_2,R_1} = \frac{1}{2}\left[(2-\sqrt{3})R_2 + (4-\sqrt{3})R_1\right] \tag{20}$$

Equation (20) is the least overlap difference for GSM network design using two different radii since the $1{:}n$ size hexagon tile completely.

3.3 Masting Conjecture

Generally for k different tessellable regular polygons $n_1, n_2, \dots n_k$ inscribed in disks with respective radii $R_k, R_{k-1}, \dots, R_1$ $(where\ R_k > R_{k-1})$ the least overlap difference is

$$d_{n_k,n_{k-1},\dots n_1} = R_k\left[1 - cos\left(\frac{\pi}{n_1}\right)\right] + R_{k-1}\left[2 - cos\left(\frac{\pi}{n_2}\right)\right] + \cdots + R_1\left[k - cos\left(\frac{\pi}{n_k}\right)\right].$$

$$d_{n_k,n_{k-1},\dots n_1} = \sum_{l=1}^{k} R_{k-l+1}\left[l - cos\left(\frac{\pi}{n_k}\right)\right] \tag{21}$$

Since we are tilling with hexagon $n_k = 6, \forall\ k \in \mathbb{R}$

$$d_{n_k,n_{k-1},\dots n_1} = \sum_{l=1}^{k} R_{k-l+1}\left[l - cos\left(\frac{\pi}{6}\right)\right]$$

$$d_{n_k,n_{k-1},\dots n_1} = \sum_{l=1}^{k} R_{k-l+1}\left[l - \frac{\sqrt{3}}{2}\right] \tag{22}$$

For three different tessellable regular hexagon the least overlap difference can be obtain from equation (22) to be

$$d_{R_3,R_2,R_1} = \sum_{l=1}^{k=3} R_{k-l+1}\left[l - \frac{\sqrt{3}}{2}\right]$$

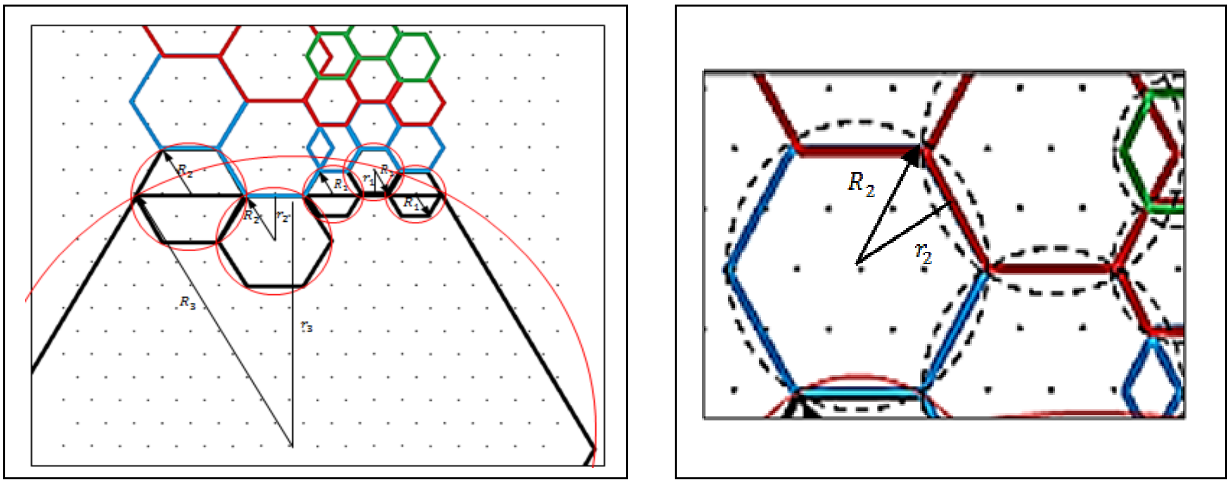
$$= R_3\left(1 - \frac{\sqrt{3}}{2}\right) + R_2\left(2 - \frac{\sqrt{3}}{2}\right) + R_1\left(3 - \frac{\sqrt{3}}{2}\right)$$

$$= R_3 - R_3\frac{\sqrt{3}}{2} + 2R_2 - R_2\frac{\sqrt{3}}{2} + 3R_1 - R_1\frac{\sqrt{3}}{2}$$

$$= R_3 - R_3 cos\left(\frac{\pi}{6}\right) + 2R_2 - R_2 cos\left(\frac{\pi}{6}\right) + 3R_1 - R_1 cos\left(\frac{\pi}{6}\right)$$

$$d_{n_3,n_2,n_1} = R_3 - r_3 + 2R_2 - r_2 + 3R_1 - r_1 \tag{23}$$

This is the one sided least overlap difference of triple non-uniform hexagonal tessellation for masting in GSM network. Equation (23) significantly informs cell planners how to design the GSM network for least overlap difference. Figure 6 illustrate this concept.



6(a): Triple different size hexagonal tessellation 6(b): Section of hexagonal tiling with radius R_2

Figure 6. Triple Size Hexagonal Tessellation

Continuous upward tiling will lead us to the following overlap differences .

Case I: Hexagon with Radius R_2 as in 6(b)

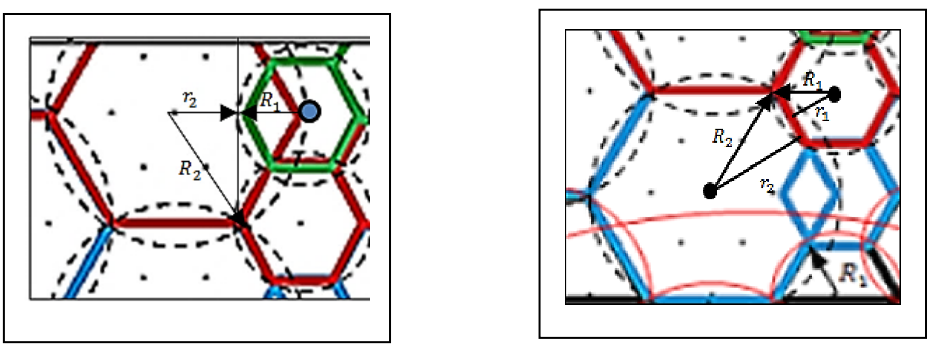
Overlap difference will be

$$d = 2(R_2 - r_2)$$
$$d = (2 - \sqrt{3})R_2$$

For k overlaps, the difference will be

$$d_k = (2 - \sqrt{3})kR_2 \tag{24}$$

Case II: Hexagons with radii R_2 and R_1



6(c): Section of hexagon tilling with radii R_2, R_1 6(d): Section of hexagon tiling with radii R_2, R_1

Overlap difference will be

$$d = R_2 - r_2 + R_1 - r_1$$
$$d = R_2\left(1 - cos\frac{\pi}{6}\right) + R_1\left(1 - cos\frac{\pi}{6}\right)$$
$$d = \frac{1}{2}(2 - \sqrt{3})(R_1 + R_2)$$

For m such overlaps, the difference will be

$$d_m = \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)m \tag{25}$$

Case III: Hexagons with Radii R_2 and R_1

Overlap difference is

$$d = R_2 - r_2 + R_1 - r_1$$

$$d = R_2\left[1-cos\left(\frac{\pi}{6}\right)\right] + R_1\left[1-cos\left(\frac{\pi}{6}\right)\right]$$

$$d = \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)$$

For p such overlaps, the difference will be

$$d_p = \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)p \tag{26}$$

Case IV: Hexagons with Radius R_1

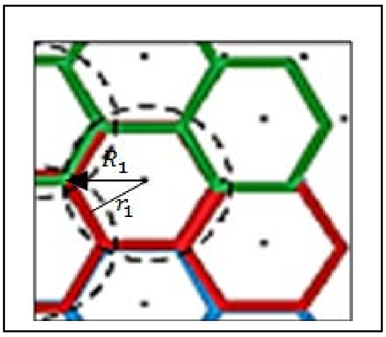


Figure 6(e). Section of hexagonal tilling with radius R_1

Overlap difference will be

$$d = 2(R_1 - r_1)$$

$$d = 2\left(R_1 - R_1 cos\frac{\pi}{6}\right)$$

For n such overlaps, the difference will be

$$d_n = \left(2-\sqrt{3}\right)nR_1 \tag{27}$$

Generally, the various cases put together gives us total overlap difference

$$d = 1(R_3 - r_3 + 2R_2 - r_2 + 3R_1 - r_1) + \left(2-\sqrt{3}\right)kR_2 + \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)p + \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)m + \left(2-\sqrt{3}\right)nR_1$$

$$d = \left[R_3\left(1-\frac{\sqrt{3}}{2}\right) + R_2\left(2-\frac{\sqrt{3}}{2}\right) + R_1\left(3-\frac{\sqrt{3}}{2}\right)\right] + \left(2-\sqrt{3}\right)(kR_2 + nR_1) + \frac{1}{2}\left(2-\sqrt{3}\right)(R_1+R_2)(m+p) \tag{28}$$

Table 1. WGS-84 coordinates of 50 MTN GSM masts in River State, Nigeria.

Location	Geographical Coordinates		Grid Coordinates	
	Latitude	Longitudes	Easterns (Xm)	Northings(Ym
1. AGUDAMA STR., BY GARRISON	4^0 48′ 14.28446″ N	7^0 00′ 24.89204″ E	278949.137	531317.354
2. 11B ELECHI BEACH, DIOBU	4^0 46′ 59.28462″ N	6^0 59′ 41.57230″E	277607.260	529016.990
3. 16 NHEDIOHANMA, DIOBU, PHC	4^0 47′ 41.24449″ N	6^0 59′ 18.15230 E	276889.199	530308.265
4. APEX MILL LTD. TRANS AMADI INDUS. AREA	4^0 4′ 33.60453″ N	7^0 01′ 36.11181″ E	281145.856	531904.560
5. BY 3 KING'S AVENUE, ABULOMA, OZUBOKO	4^0 47′ 01.14485″ N	7^0 02′ 18.95184″ E	282458.036	529060.136
6. EJUAN COMMUNITY, ABULOMA	4^0 47′ 33.90477″ N	7^0 02′ 30.59174″ E	282819.658	530065.587

7. NZIMIRO STREET SHELL RA, OLD GRA	$4^0\,47'\,20.16467''$ N	$7^0\,00'\,53.33206''$ E	279820.847	529652.068
8. OCO MILLER IND.. SERV. LTD., TRANS AMADI	$4^0\,48'\,30.12458''$ N	$7^0\,02'\,12.23173''$ E	282258.753	531794.438
9. OFF JOHN OGBODA STREET, NGWOR STR.	$4^0\,48'\,18.00450''$ N	$7^0\,00'\,48.47199''$ E	279676.210	531429.531
10. OKIS AWO CLOSE, RAINBOW	$4^0\,47'\,50.34468''$ N	$7^0\,01'\,50.39185''$ E	281582.136	530574.219
11. OLD GRA FORCE AVENU	$4^0\,46'\,45.00473''$ N	$7^0\,00'\,32.87214''$ E	279187.121	528573.665
12. OPP GIGGLES CYBER CAFE, BOROKIRI	$4^0\,44'\,54.54525''$ N	$7^0\,02'\,30.05200''$ E	282789.100	525169.673
13. OPP NANA'S HOTELS, MOORE HOUSE STR.	$4^0\,45'\,26.40512''$ N	$7^0\,02'\,14.45200''$ E	282311.053	526149.861
14. ORUTA COMPOUND, OZUBOKO-AM	$4^0\,46'\,26.64496''$ N	$7^0\,02'\,35.21184''$ E	282956.163	527998.775
15. PLOT 305 BOROKIRI, FILLAREA	$4^0\,45'\,10.62516''$ N	$7^0\,02'\,02.39203''$ E	281937.962	525666.112
16. NKPOGU BYE-PASS, TOKI HOTEL ROAD	$4^0\,48'\,12.94455''$ N	$7^0\,01'\,12.45193''$ E	280414.825	531271.930
17. ST MARY'S CATHOLIC CH. LAGOS BUS STOP	$4^0\,45'\,43.56496''$ N	$7^0\,01'\,08.57216''$ E	280282.014	526682.863
18. CHINDAH ESTATE, UST, PORT HARCOURT	$4^0\,48'\,11.54438''$ N	$6^0\,59'\,11.65228''$ E	276691.599	531239.780
19. 23 DICK TIGER, STREET, DIOBU	$4^0\,47'\,21.42455''$ N	$6^0\,59'\,22.43231''$ E	277019.328	529698.935
20. MILE ONE POLICE STATION	$4^0\,47'\,27.60457''$ N	$6^0\,59'\,51.17222''$ E	277905.683	529886.217
21. OPP 100 ABEL JUMBO ST, MILE 2 DIOBU PHC	$4^0\,47'\,35.44447''$ N	$6^0\,59'\,05.95232''$ E	276512.658	530131.171
22. OMEGA BEACH EASTERN BY PASS	$4^0\,46'\,47.94479''$ N	$7^0\,01'\,19.55203''$ E	280626.117	528659.839
23. NNOKAM., RUMUOKWOKUNU VILL.	$4^0\,48'\,25.64431''$ N	$6^0\,59'\,00.35228''$ E	276344.602	531674.010
24. OPP 3 DICK NWOKE STR, OGBUNABALI	$4^0\,47'\,37.26461''$ N	$7^0\,00'\,44.75204''$ E	279557.926	530178.201
25. 4 NZIMIRO STREET, PORT HARCOURT	$4^0\,47'\,45.30456''$ N	$7^0\,00'\,20.21210''$ E	278802.305	530427.412
26. IMMACULATE CATH. HT PARISH,MILE 3, DIOBU	$4^0\,47'\,59.04442''$ N	$6^0\,59'\,13.75227''$ E	276755.193	530855.544
27. 9 EZEBUNWO CLOSE, OROWURUKWO	$4^0\,48'\,41.76436''$ N	$7^0\,00'\,12.53205''$ E	278570.664	532162.743
28. BY ADARI-OBU LANE, ABULOMA	$4^0\,46'\,48.72494''$ N	$7^0\,02'\,56.45173''$ E	283612.721	528675.271
29. ROAD E, UST CAMPUS	$4^0\,47'\,21.14448''$ N	$6^0\,58'\,44.25243''$ E	275842.544	529693.789
30. CHIEF ODUM CLOSE, EASTERN BYE-PASS	$4^0\,47'\,52.56457''$ N	$7^0\,00'\,48.89200''$ E	279686.886	530647.896
31. ROAD 3, AGIP ESTATE	$4^0\,48'\,34.14426''$ N	$6^0\,58'\,36.15233''$ E	275599.515	531937.367
32. 11 WONODI STREET, GRA PHASE III	$4^0\,48'\,34.14439''$ N	$6^0\,59'\,56.35210''$ E	278071.310	531930.090
33. 31B FORCES AVENUE, OLD GRA	$4^0\,47'\,03.72470''$ N	$7^0\,00'\,32.99212''$ E	279192.486	529148.794
34. CHRISTIAN COUNCIL COLL., ELEKAHIA	$4^0\,48'\,39.18446''$ N	$7^0\,00'\,57.89194''$ E	279968.423	532079.405
35. OUR LADY FATIMAH'S COL., NEW MKT LAYOUT	$4^0\,45'\,27.06508''$ N	$7^0\,01'\,53.93203''$ E	281678.645	526171.937
36. BY RESURRECTION MINISTRIES, BOROKIRI	$4^0\,44'\,39.12525''$ N	$7^0\,02'\,18.53204''$ E	282432.687	524696.934
37. 5 ABUJA BYPASS, MILE 3, DIOBU	$4^0\,48'\,10.04441''$ N	$6^0\,59'\,39.35220''$ E	277545.195	531191.187
38. 6 ABOBIRI STREET, OFF INDUSTRY ROAD,	$4^0\,45'\,48.66494''$ N	$7^0\,00'\,49.97219''$ E	279709.176	526841.199
39. ABONNEMA WHARF RD, RCCG KIDNEY PARISH	$4^0\,46'\,32.40475''$ N	$7^0\,00'\,11.99225''$ E	278542.448	528188.416
40. BY MARINE BASE COMM. BANK PREMISES	$4^0\,46'\,08.22491''$ N	$7^0\,01'\,18.65207''$ E	280594.871	527439.601
41. 8RECLAMATION LAYOUT OFF HARBOUR ROAD	$4^0\,45'\,30.48496''$ N	$7^0\,00'\,42.65224''$ E	279481.951	526283.300
42. 3 ENWENABURU AVENUE, ELIOGBOLU	$4^0\,51'\,46.38391''$ N	$7^0\,00'\,55.25165''$ E	279903.855	537831.021
43. AGGREY RD HOUSING ESTATE, PORT HARC.	$4^0\,45'\,45.24502''$ N	$7^0\,01'\,57.05202''$ E	281776.402	526730.203
44. BEHIND CHINDAH BAR,IHUNWO OROGBUM RD	$4^0\,48'\,29.44433''$ N	$6^0\,59'\,24.75220''$ E	277096.965	531788.546
45. FACULTY OF LAW BUILD., UST CAMPUS	$4^0\,47'\,53.14438''$ N	$6^0\,58'\,41.25237''$ E	275752.978	530677.224
46. UST CAMPUS	$4^0\,47'\,52.54438''$ N	$6^0\,58'\,37.35240''$ E	275632.722	534326.027
47. INSIDE EL-SHADDAI INT'L INC. PREMISES, AMADI	$4^0\,49'\,52.30000''$ N	$7^0\,01'01.68000''$ E	280091.597	530775.220
48. OPP 6A WOKE LANE OGBUNABALI	$4^0\,47'\,56.88454''$ N	$7^0\,00'\,32.21205''$ E	279173.186	530782.113

49.	CHIEF AKAROLO ESTATE, ELEKAHIA	$4^0\ 49'\ 04.98441''$ N	$7^0\ 01'\ 29.27181''$ E	280937.839	532869.255
50.	DANGOTE PREM., TRANS AMADI, OGINIGBA	$4^0\ 49'\ 16.44446''$ N	$7^0\ 02'\ 15.95166''$ E	282377.492	533217.187

A plot of the 50 MTN GSM masts coordinates is shown in Figure 7(a). We write a matlab code for the covering as shown in Figure 7(b).

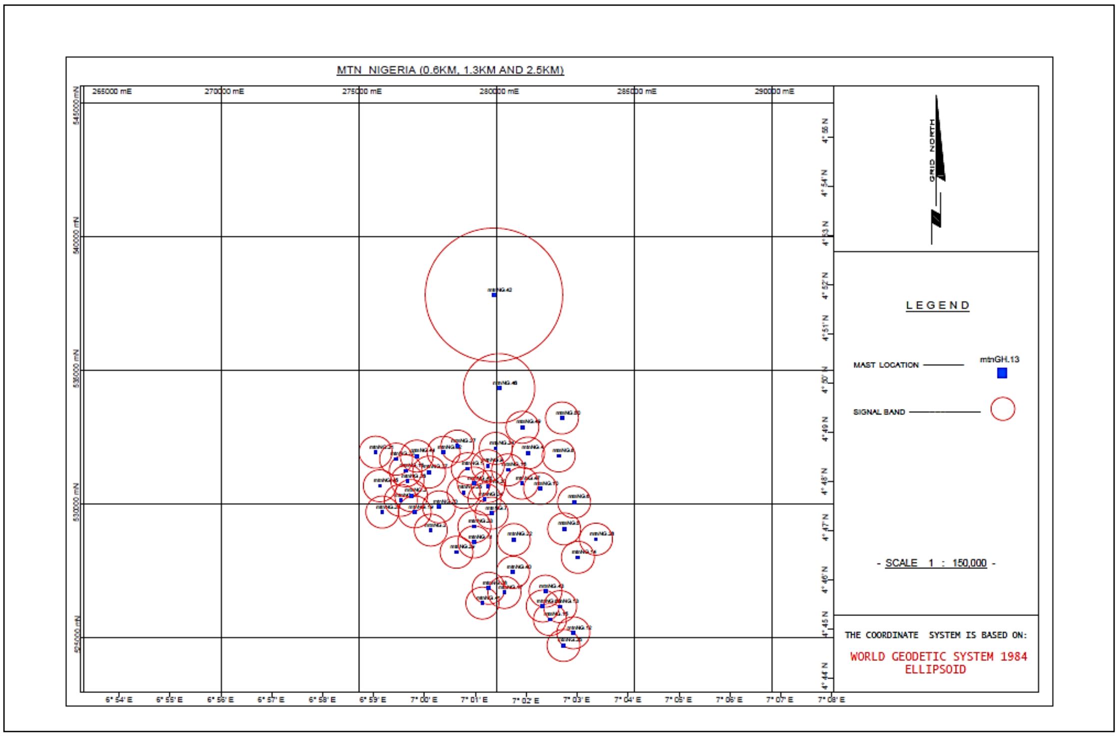


Figure 7(a). Position of 50 MTN GSM cell coordinates with cell band in River State, Nigeria

A Matlab code for 50 MTN GSM Masts position ● , has been covered with 36 hexagons ⬡, in River State-Nigeria. This is written below.

```
>>load set3
>>scatter(set3(:,1),set3(:,2),'fill')
>>n=5.1;
>>m=0.96;
>>figure(1),hold on
>>for i=5.0008:.015:n
      >>for j = 0.81325:0.015:m
            >>hexagon(0.005,i,j)
    >> end
>>end
 >>for i=5.0008:.015:n
       >>for j = 0.81325:0.015:m
             >>hexagon(0.005,i+0.0075,j+0.0075)
       >>end
 >>end
>> axis([5 5.1 0.8 0.96])
 >>xlabel('Easterns (xkm)')
 >>ylabel('Northerns (ykm)')
```

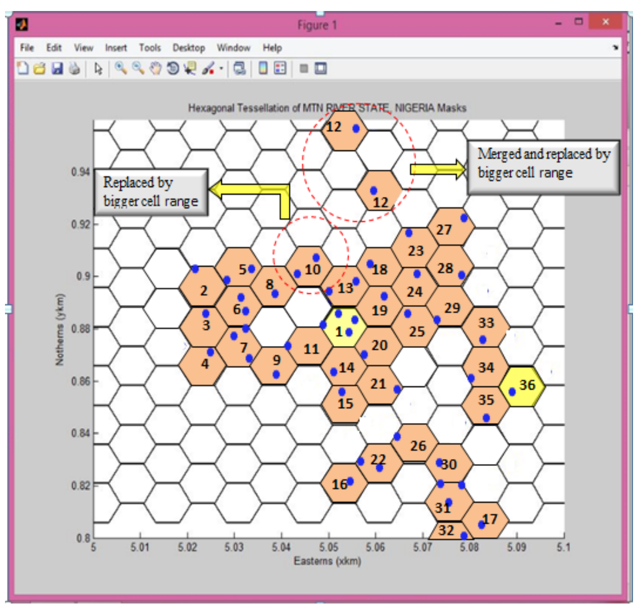


Figure 7(b). Graph of GSM masts nodes

>>title('Hexagonal Tessellation of MTN RIVER STATE, NIGERIA Masts').

This is shown in Figure 7(b). We plot the Figure 7(b) on Autocad and compute the overlap difference and overlap area shown in Table 2. Figure 8 shows the plot.

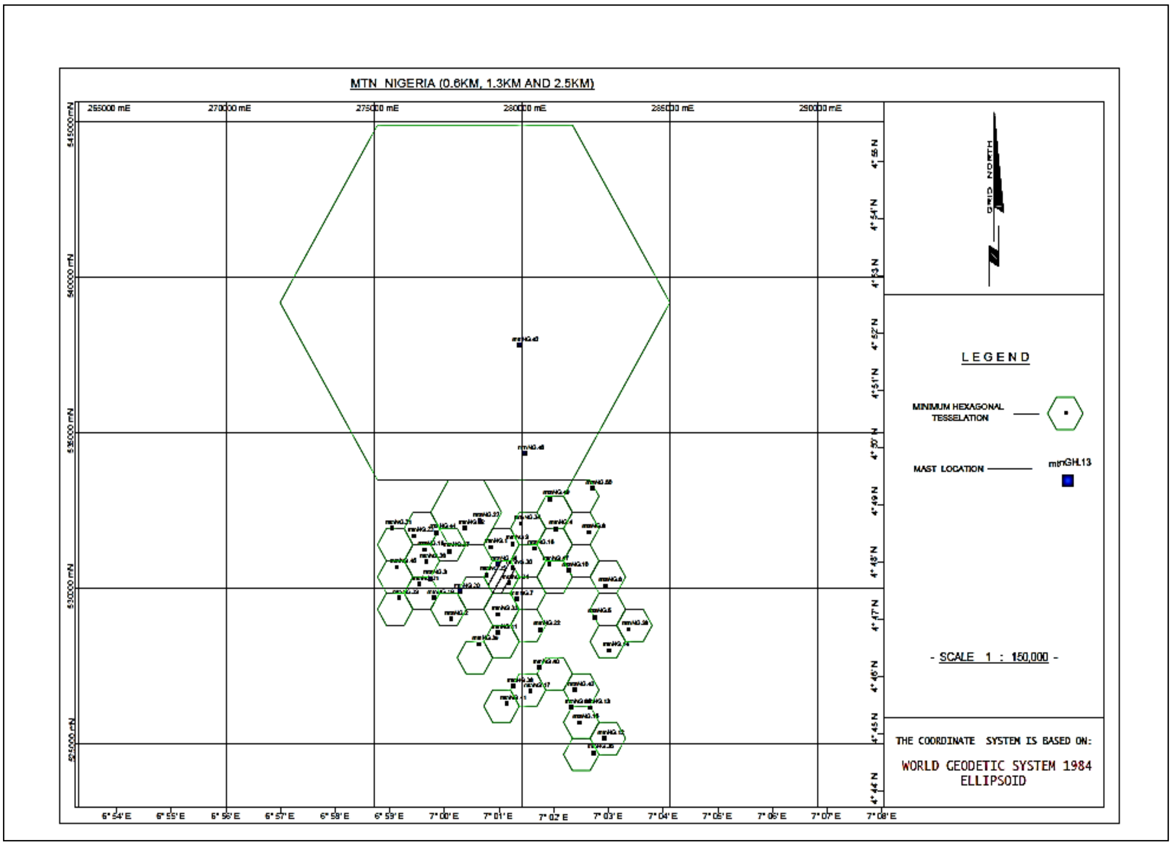


Figure 8. Optimal disks covering for triple size hexagons using MTN masts, River State – Nigeria.

Table 2 shows the overlap difference $(d)$and overlap area (A_d) obtain as a result of Figure 7(b).

Table 2. Overlap difference for 0.6km,1.3km, 2.5km MTN cell range – River State, Nigeria

Serial	Overlap Difference $d = d_m - d_n$	Value (m)	Area of overlap(A_d)	Serial	Overlap Differen $d = d_m - d_n$	Value (m)	Area of overlap(A_d)
1.	$d_{42} - d_{46}$	289.9813	222044.00	22.	$d_{43} - d_{15}$	123.7141	38621.1256
2.	$d_{46} - d_{49}$	215.2717	124885.00	23.	$d_{13} - d_{35}$	567.2068	811838.0168
3.	$d_{49} - d_4$	212.9525	114433.1922	24.	$d_{13} - d_{12}$	109.4508	30229.0436
4.	$d_4 - d_8$	81.6679	16830.20104	25.	$d_{13} - d_{15}$	589.0909	875691.5421
5.	$d_4 - d_{16}$	233.2389	137274.0656	26.	$d_{15} - d_{35}$	567.2068	811838.0168
6.	$d_4 - d_{47}$	47.7039	5742.411344	27.	$d_{15} - d_{12}$	214.6637	116279.6587
7.	$d_4 - d_{34}$	9.6558	235.2681028	28.	$d_{12} - d_{36}$	607.9591	932685.5619
8.	$d_{16} - d_{47}$	493.6746	614990.0552	29.	$d_{31} - d_{23}$	409.7396	423645.3124
9.	$d_{16} - d_{34}$	277.3459	194102.0076	30.	$d_{23} - d_{44}$	438.9687	486243.3069
10.	$d_{16} - d_9$	444.7582	499153.8841	31.	$d_{23} - d_{26}$	810.5369	1657799.9
11.	$d_{16} - d_{30}$	241.1916	146794.8718	32.	$d_{23} - d_{45}$	40.8618	4213.2916
12.	$d_{47} - d_{10}$	505.1518	643917.6609	33.	$d_{44} - d_{32}$	215.4276	117108.7144

13.	$d_6 - d_5$	131.4954	43632.25393	34.	$d_{44} - d_{37}$	453.1748	518224.6067
14.	$d_5 - d_{14}$	27.5591	1916.534266	35.	$d_{44} - d_{26}$	206.3699	107468.0168
15.	$d_{40} - d_{38}$	131.1031	43372.29959	36.	$d_{32} - d_1$	129.4741	42301.16504
16.	$d_{40} - d_{17}$	381.1399	366568.6824	37.	$d_{32} - d_{37}$	292.9307	216529.1175
17.	$d_{43} - d_{35}$	633.2396	1011865.197	38.	$d_{27} - d_1$	273.7579	189112.3313
18.	$d_{43} - d_{13}$	633.2396	1011865.197	39.	$d_{34} - d_9$	487.4520	599584.2833
19.	$d_9 - d_{30}$	418.2921	441515.3955	40.	$d_{24} - d_{25}$	404.3435	412560.3237
20.	$d_9 - d_{48}$	669.0557	1129564.61	41.	$d_{24} - d_{48}$	483.9451	590988.0793
21.	$d_9 - d_1$	464.3242	544037.9123	42.	$d_{29} - d_{21}$	399.7777	403295.761
43.	$d_{45} - d_{26}$	182.0447	83626.24686	63.	$d_{29} - d_{19}$	23.2047	1358.7465
44.	$d_{45} - d_{21}$	264.4319	176446.9753	64.	$d_{19} - d_{20}$	294.0752	218224.4106
45.	$d_{45} - d_{29}$	212.4948	113941.817	65.	$d_{19} - d_{21}$	534.0102	719590.8493
46.	$d_{26} - d_{37}$	341.6531	294548.8004	66.	$d_{19} - d_{26}$	13.6140	467.6899
47.	$d_{26} - d_3$	636.5535	1022483.593	67.	$d_{20} - d_2$	280.9723	199211.103
48.	$d_{26} - d_{19}$	13.6140	467.6899306	68.	$d_7 - d_{33}$	394.9396	393593.4759
49.	$d_3 - d_{21}$	783.8926	1550599.56	69.	$d_{33} - d_{11}$	624.8460	985218.4021
50.	$d_3 - d_{20}$	99.3801	24922.14313	70.	$d_{33} - d_{39}$	40.3124	4100.7551
51.	$d_3 - d_{19}$	576.9297	839909.1656	71.	$d_{39} - d_{11}$	448.9873	508691.6872
52.	$d_3 - d_{37}$	100.0536	25261.08301	72.	$d_{41} - d_{38}$	597.6027	901180.1747
53.	$d_3 - d_{45}$	5.3750	72.90267503	73.	$d_{41} - d_{17}$	305.7118	235836.4513
54.	$d_{25} - d_1$	298.0264	224127.9406	74.	$d_{38} - d_{40}$	131.1031	43372.2996
55.	$d_{25} - d_{48}$	686.8085	1190303.882	75.	$d_{18} - d_{44}$	517.7495	676434.7394
56.	$d_{25} - d_{30}$	288.3549	209817.2539	76.	$d_{18} - d_{23}$	644.1559	1047052.613
57.	$d_{25} - d_{20}$	152.7068	58844.14814	77.	$d_{18} - d_{37}$	345.0220	300386.2877
58.	$d_{24} - d_{30}$	712.9229	1282542.18	78.	$d_{18} - d_{26}$	810.5369	1657799.9
59.	$d_{24} - d_7$	611.8305	944601.809	79.	$d_{18} - d_3$	247.7574	154895.8552
60.	$d_{18} - d_{45}$	105.7063	28196.04989	80.	$d_{18} - d_{21}$	77.0424	14977.7347
	Total $(\sum d) = 26{,}412.518$m				Total $(\sum A_d) = 32{,}834{,}104.29$m^2		

Let R_i be disks with radius , N_{R_i} be number of cells with radius i then

Area of cells = sum of area of all disks− sum of all overlap areas of cells

$$\text{Area } = \sum_{i=1}^{n} N_{R_i} \times \pi R_i^2 - \sum A_d \tag{29}$$

Where the overlap area is calculated using the inclusive exclusive formula

$$\left|\bigcup_{i=1}^{n} A_i\right| = Area\sum_{i}|A_i| - Area\sum_{i<j}|A_i \cap A_j| + Area\sum_{i<j<k}|A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}Area\left|\bigcap_{i=1}^{n} A_i\right|$$

where the first sum is over all i the second sum is over all pairs $i.j$ with $i<j$, the third sum is over all triples i,j,k with $i<j<k$ and so fourth.

$$= (48 \times \pi \times 600^2 + 1 \times \pi \times 1300^2 + 1 \times \pi \times 6600^2 - 32{,}834{,}104.29)m^2$$

$$\text{Area } = 21.48km^2$$

We compute the total coverage area of the proposed multiple size hexagonal tessellation employed in the design and compare with the existing coverage area.

Total area of hexagon = sum of Number of hexagons × unit area

$$= \sum_{i=1}^{n} N_{R_i} \times \frac{3\sqrt{3}}{2} R_i^2 \quad (30)$$

$$= \left(34 \times \frac{3\sqrt{3}}{2} \times 600^2 + 1 \times \frac{3\sqrt{3}}{2} \times 1200^2 + 1 \times \frac{3\sqrt{3}}{2} \times 6600^2\right) m^2$$

$$\text{Area} = 148.71 km^2$$

Table 3. Comparative Analysis of the MSHT model to the Original Layout method for MTN River State, Nigeria.

Case Study	Original Layout		Multiple Size Hexagonal Tessellation	
	Number of masts	Area covered (km²)	Number of masts	Area covered (km²)
MTN Nigeria, River State R_1 =0.6km, R_2 =1.3km, R_3 =2.5km	50	21.48	36	148.71
Number of overlaps	80		k = 1, m = 0, p = 1, n = 52	
$d = \left[R_3\left(1-\frac{\sqrt{3}}{2}\right) + R_2\left(2-\frac{\sqrt{3}}{2}\right) + R_1\left(3-\frac{\sqrt{3}}{2}\right)\right] + (2-\sqrt{3})kR_2 + \frac{1}{2}(2-\sqrt{3})(\mathrm{R_1}+\mathrm{R_2})\mathrm{p} + \frac{1}{2}(2-\sqrt{3})(\mathrm{R_1}+\mathrm{R_2})\mathrm{m} + (2-\sqrt{3})\mathrm{nR_1}$	26.413km		12.368km	
Ratio	1mast	0.43km^2	1mast	4.13km^2
Number of GSM masts based on cell range	$N_{R_1=0.6km}$-48, $N_{R_2=1.3km}$-1, $N_{R_3=2.5km}$-1		$N_{R_1=0.6km}$- 34, $N_{R_2=1.2km}$ − 1, $N_{R_3=6.6km}$ - 1	

5. Discussion

Designing of multiple size hexagonal tessellation with minimum overlap difference is a new area in cell planning in telecommunication network design. Our study conjectures an algorithm for efficient masting with least overlap difference for multiple cell range. Application of this formula to MTN River State GSM network solution resulted in an overlap difference of 12.368km which is a 53.2% reduction over the original overlap difference of 26.413km. The formula also uses 36GSM masts, covering an area of 148.715km^2 compared to the cell engineers original design of 50 masts covering an area of 21.48km^2. This is equivalent to using 1GSM masts to cover 4.13km^2 in the multiple size hexagonal tessellation model instead of 1GSM masts for 0.43km^2 using the original design. Table 3 shows the results of the computation.

6. Conclusion

Our study provide an optimal multiple size hexagonal tessellation design with least overlap difference of 12.368km and total coverage area of 148.71km^2. The number of GSM masts obtained from the MSHTM is 36 as compared to the original design of 50 GSM masts. This gives a 28% reduction over the original number of GSM masts. We used geometry of hexagonal tessellation approach to geometric disks covering for multiple cell range to reach optimality and it is the first study that uses multiple size hexagonal tessellation for covering of point sets to arrive at minimum overlap difference and overlap area.

Acknowledgment

The authors would like to acknowledge the Operations Manager of ATC Ghana Limited, Mr. Alexander Owuahene, and Mr. Isaac Yankson of the Geodetic department of KNUST, who enlightened us on the use of GIS and AutoCAD software.

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