Journal of Mathematics Research; Vol. 8, No. 2; April 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Standard Ideals in BCL^+ Algebras

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Received: December 23, 2015 Accepted: January 16, 2016 Online Published: March 10, 2016

doi:10.5539/jmr.v8n2p37 URL: http://dx.doi.org/10.5539/jmr.v8n2p37

Abstract

We show some useful properties of these ideals that give various methods how to get ideals from them, and so our main aim is to study their properties. Here, we introduce these ideals i.e., the natural ideal, normal ideal, former ideal (and its doublet, latter ideal), proper ideal, normal extension ideal, normal uptake ideal. In particular, we introduce Boolean ideal and normal Boolean ideal to grasp the diversity of ideal for BCL^+ algebras. As a means, we can define quotient BCL^+ algebras only in terms of ideal, and we discuss its structure.

Keywords: *BCL*-algebras, BCL^+ algebras, ideal

1. Introduction

The author (Liu, 2011) first studied that the *BCL*-algebras, which is a new class of algebra of type (2, 0) and a wider class than BCK/BCI algebras. The *BCL*-algebras progress by Al-Kadi and Hosny (2013), and soft *BCL*-algebras were treated by Al-Kadi (2014). The author (Liu, 2012) was introduced BCL^+ algebras. A BCL^+ algebras can be considered as a fragment of propositional logic containing only a logical connective implication a binary operation "*" and 1 which is interpreted as the value "true". And in recent years, the author (Liu, 2013-15) has launched a series of original research to improve the general development of BCL^+ algebras. In algebra we can say that the concept of the ideal is important such as rings (Isaacs, 1993) and semigroups (Anjaneyulu, 1980); in linear algebras such as Leibniz algebras and Lie algebras (Geoffrey & Gaywalee, 2013); in logical algebras such as Hilbert algebras (Dudek, 1999; Sergio & Daniela, 2012), BCC-algebras (Dudek *et al*., 2011), and BCK/BCI algebras (Huang, 2006; Borzooei & Zahiri, 2012). It might be the best motivation to do something about the notions of ideals in BCL^+ algebras. To that end, we will introduce the standard ideals for BCL^+ algebras in this paper. The key here is to identify ideal, as well as study for their interesting properties. Certainly, we want to find some ideals that specifically in ideals of poset, serve as some standard ideals to express all the ideals, for example, from the *normal ideal* to the *normal extension ideal*. Because of an ideal is precisely an object that is both *former ideal* and *latter ideal*, and also relationships relevant to the deductive systems.We will work on the structure of algebra by ideal in the quotient BCL^+ algebras. All these means have become of the distinctive features of the ideals in BCL^+ algebras.

2. Preliminary

In this section, we recall some basic facts about BCL^+ algebras which will be needed for this paper.

Definition 2.1 (Liu, 2012) A BCL^+ algebra is a triple (Y; $*$, 1), where Y is a nonempty set, "$*$" is a binary operation on Y, and $1 \in Y$ is an element such that the following three axioms hold for any $x, y, z \in Y$:

(BCL^+1) $x * x = 1$.

(BCL^+2) $x * y = 1$ and $y * x = 1$ imply $x = y$.

(BCL^+3) $((x * y) * z) * ((x * z) * y) = (z * y) * x$.

Theorem 2.1 (Liu, 2012) *Assume that* (Y; $*$, 1) *is a* BCL^+ *algebra. Then the following hold for any* $x, y, z \in Y$:

(i) $(x * (x * y)) * y = 1$.

(ii) $x * 1 = x$ *implies* $x = 1$.

(iii) $((x * y) * (x * z)) * (z * y) = 1$.

Definition 2.2 (Liu, 2012) Suppose that (Y; $*$, 1) is a BCL^+ algebra, the ordered relation if $x \le y$ if and only if $x * y = 1$, for all $x, y \in Y$, then $(Y; \le)$ is partially ordered set and (Y; $*$, 1) is an algebra of partially ordered relation.

Definition 2.3 (Liu, 2015) If D be a nonempty subset of a BCL^+ algebra (Y; $*$, 1). Then we say that D be a deductive system if

(DS1) $1 \in D$.

(DS2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 2.1 (Liu, 2015) *Let D be a deductive system of a BCL⁺ algebra* (*Y*; ∗, 1), *and suppose* $a \le x$ *whenever* $a \in D$. *Then* $x \in D$.

Lemma 2.2 (Liu, 2015) *Let Y be commutative and let for all* $x, y, z \in Y$. *Then the following equalities are satisfied*:

(Y1) $x*(y*z) = y*(x*z)$.

(Y2) $(x*y)*z = (x*z)*y$.

Theorem 2.2 (Liu, 2015) *Suppose that B is subalgebra of Y. Then B is a filtration and* $x \in B$ *if and only if* $1 * x \in B$.

Theorem 2.3 (Liu, 2015) *Let H be a nonempty subset of a* BCL^+ *algebra* (*Y*; ∗, 1). *Then H is a filtration if and only if it is a deductive system.*

3. Main Results

Definition 3.1 Let *I* be a nonempty subset of BCL^+ algebras (*Y*; ∗, 1), let $a \in I$, $x \in Y$. We say that *I* is a *natural ideal* of *Y* if

(NAI) $x * a \in I$ implies $x \in I$.

Example 3.1 Let *Y* = {1, *a*, *b*, *c*} and the binary operation ∗ on *Y* by the following Cayley Table 1 (Liu, 2015) and Figure 1 display Hasse diagram.

Table 1. BCL^+ operation

∗	1	*a*	*b*	*c*
1	1	1	1	1
a	*a*	1	1	*c*
b	*b*	1	1	*c*
c	*c*	1	*c*	1

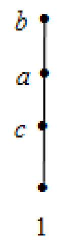

Figure 1. The Hasse diagram of ({1, *a*, *b*, *c*}).

Then *I* = {1, *b*} is a *natural ideal.*

Solution. Let $x = 1 \in Y$ and let $b \in I$, by Definition 3.1, we get $1 * b = 1 \in I$ for instance.

Definition 3.2 Let *I* be a nonempty subset of BCL^+ algebras (*Y*; ∗, 1), we say that *I* is a *normal ideal* of *Y* if

(NOI1) $1 \in I$, and

(NOI2) For all $x, y, z \in Y$, $y*(z*x) \in I$ and $z \in I$ imply $y * x \in I$.

Example 3.2 In Example 3.1, let $1 \in I$. If

$$b*(c*a) = b*1 = b,$$

by Definition 3.2 (NOI2), since $b, c \in I$, $a \notin I$, we have

$$b*a = 1 \in I.$$

Then *I* = {1, *b*, *c*} is a *normal ideal.*

Theorem 3.1 *Let I be a normal ideal of BCL*$^+$ *algebras* (*Y*; ∗, 1). *Then the following hold for all* $x, y, z \in Y$:

$x * y \in I$ *implies* $(y * x) * ((x * y) * z) \in I$.

Proof. Let $(y * x) * z \in I$. By Definition 3.2 (NOI2) and Lemma 2.2 (Y2), we have

$$(y * x) * z = (y * z) * x \in I .$$

By Definition 3.1 (NAI), since $x * y \in I$, we have $x \in I$ and $y * z \in I$, and so $y * x \in I$ and $z \in I$, we see that $(x * y) * z \in I$. Let $x \le z$, then

$$y * x \le y * z \le (x * y) * z ,$$

by Definition 2.2 and Definition 3.2 (NOI1). Thus $(y * x) * ((x * y) * z) = 1 \in I$. □

Definition 3.3 Let *E* be a nonempty subset of *BCL*$^+$ algebras (*Y*; ∗, 1). Assume that for all $x, y, z \in Y$. We say that *E* is a *former ideal* (FI) and it is a *latter ideal* (LI) if

(EI) $1 \in E$,

(FI) $y * (z * x) \in E$ and $y * x \in E$ imply $y * z \in E$, and

(LI) $y * (z * x) \in E$ and $y * z \in E$ imply $y * x \in E$.

Theorem 3.2 *If* (*Y*; ∗, 1) *is a BCL*$^+$ *algebra, then former ideal* (*latter ideal*) *of Y is a normal ideal*.

Proof. (i) By Definition 3.2 (NOI2) we have that $y * x \in I$. If $z = y * x$, then by Lemma 2.2 (Y1) we can write

$$y * ((y * x) * x) = (y * x) * (y * x) = 1 \in I .$$

Let $y * (y * x) \in I$, then by Definition 3.2 (NOI2) we have $y \in I$ implies $y * x \in I$, since $y * x = z \in I$, then $y * z \in I$.

(ii) Let $y * z \in I$ and $z \in I$. Then $y \in I$. By Definition 3.2 (NOI2) we have

$$y * ((y * z) * x) = (y * z) * (y * x) \in I ,$$

since $y * z \in I$, then $y * x \in I$. The complete the proof. □

Definition 3.4 Let *E* be a nonempty subset of *BCL*$^+$ algebras (*Y*; ∗, 1), we say that *E* is a *proper ideal* of *Y* if

(PI1) $1 \in E$, and

(PI2) For all $x, y \in Y$, $y * x \in E$ and $x \in E$ imply $y \in E$.

Theorem 3.3 *Let E be a proper ideal of BCL*$^+$ *algebras* (Y; $*$, 1). *If* $x \le y \in E$ *, then* $x \in E$.

Proof. By Definition 2.2, since $x \le y$. We now have $x * y = 1 \in E$. Let *E* be a *proper ideal* and let $y \in E$.

Then $x \in E$, and the proof that Definition 3.4 (PI2) is satisfied. □

Remark 3.1 Hint that Theorem 3.3, the *proper ideal* of *BCL*$^+$ algebras is its ideal of *partially ordered set*.

Remark 3.2 If $S \subseteq P$, we say that $a \in S$ is a *maximal* element of *S* if there is no element $b \in S$ with $b > a$. Similarly, $a \in S$ is a *minimal* if there is no $b \in S$ with $b < a$. If *S* is nonempty but finite, it necessarily contains both *maximal* element and *minimal* element. In fact, the poset *P* satisfies the *maximal condition* if every nonempty subset has a *maximal* element, and dually, it satisfies the *minimal condition* if every nonempty subset has a *minimal* element.

Corollary 3.1 *Let* $I \subseteq Y$ *be a proper ideal*. *Then I is contained in some maximal ideal*.

Corollary 3.2 *Let* $I \subseteq Y$ *be a former* (*or latter*) *ideal*. *Then I is contained in maximal former* (*or latter*) *ideal*.

Definition 3.5 Let *N* be a nonempty subset of *BCL*$^+$ algebras (Y; $*$, 1), we say that *N* is a *normal extension ideal* of *Y* if

(NE1) $1 \in N$,

(NE2) For all $x, y \in Y$, $y * x \in N$ and $x \in N$ imply $y \in N$, and

(NE3) For all $x, y, z \in Y$, $(z * (y * x)) * x \in N$.

Theorem 3.4 *Let* $\{I_k \mid k \in K\}$ *is a normal extension ideal variety of BCL*$^+$ *algebras* (Y; $*$, 1). *Then*

(i) *If* $\{I_k \mid k \in K\}$ *is a normal extension ideal chain* (*i.e. between any two elements can compare in* I_k). *Then* $\bigcup_{k \in K} I_k$ *is a normal extension ideal of Y*.

(ii) $\bigcap_{k \in K} I_k$ *is a normal extension ideal of Y*.

Proof. For part (i), of course, we have $1 \in \bigcup_{k \in K} I_k$. Now suppose that $y * x \in \bigcup_{k \in K} I_k$ and $x \in \bigcup_{k \in K} I_k$, we have $k_1, k_2 \in K$. Then $y * x \in I_{k_1}$ and $x \in I_{k_2}$. We may assume that $I_{k_1} \subseteq I_{k_2}$, then $y * x \in I_{k_2}$ and $x \in I_{k_2}$, where I_{k_2} is a *normal extension* ideal of *Y*. Since $y \in I_{k_2} \subseteq \bigcup_{k \in k} I_k$. We prove Definition 3.5 (NE2). We know that we can write

$$\begin{aligned} z * ((z * (y * x)) * x) &= (z * (y * x)) * (z * x) \\ &\ge (y * x) * x \\ &\ge y, \end{aligned}$$

and

$$y, z \in \bigcup_{k \in K} I_k \text{, imply } (z * (y * x)) * x \in \bigcup_{k \in K} I_k .$$

In fact, since $y*x\in\bigcup_{k\in K}I_k$, we have $k_1,k_2\in K$, and so $z*(y*x)\in I_{k_1},I_{k_2}$. Since $x\in I_{k_1},I_{k_2}$. Then

$$(z*(y*x))*x\in I_{k_1},I_{k_2}\subseteq\bigcup_{k\in K}I_k,$$

and the proof that Definition 3.5 (NE3) is satisfied. Thus $\bigcup_{k\in K}I_k$ is a *normal extension* ideal of *Y*.

For part (ii), assume $k\in K$ and let $1\in I_k$. Then we can choose $1\in\bigcap_{k\in K}I_k$. Let $y*x\in\bigcap_{k\in K}I_k$ and $x\in\bigcap_{k\in K}I_k$. Since for each $k\in K$. Then $y*x\in I_k$ and $x\in I_k$. If I_k is a *normal extension* ideal of *Y*. Than $y\in I_k$. This yields $y\in\bigcap_{k\in K}I_k$, and thus $\bigcap_{k\in K}I_k$ is a *normal extension* ideal of *Y*. We prove Definition 3.5 (NE2), is also. We also use the part (i) method to prove $(z*(y*x))*x\in I_{k_1},I_{k_2}\subseteq\bigcap_{k\in K}I_k$. Thus $\bigcap_{k\in K}I_k$ is a *normal extension* ideal of *Y*. □

Theorem 3.5 *A nonempty subset K of a BCL*$^+$ *algebra* (*Y*; ∗, 1) *is a normal extension ideal if and only if it is a deductive system*.

Proof. Let *K* be a *normal extension* ideal, we show now that if $1\in K$. Then Definition 2.3 (DS1) is satisfied. To prove Definition 2.3 (DS2) suppose $a\in K$ and $x*a=a_1\in K$ for some $x\in Y$. Then by Definition 3.1 (NAI), since $1\in K$, we have

$$a_2=(x*a)*x\in K,$$

and so

$$\begin{aligned}x&=1*x\\&=(((x*a)*x)*((x*a)*x))*x\\&=(a_2*(a_1*x))*x\in K,\end{aligned}$$

Thus $a\in K$ and $a*x\in K$ imply $x\in K$. We prove Definition 2.3 (DS2). Then *K* is a deductive system.
Conversely, if *K* is a deductive system, then $1\in K$, we have

$$\begin{aligned}a*(a*x)&\le(a*a)*(a*x)\\&=1*(a*x)\\&=1*(a*a)\\&=1*1\\&=1\in K.\end{aligned},$$

This for $a\in K$ implies $a*x\in K$ for every $x\in K$. Hence Definition 3.1 (NAI) is satisfied. □

Definition 3.6 Let *I* be a nonempty subset of *BCL*$^+$ algebras (*Y*; ∗, 1), we say that *I* is a *normal uptake ideal* of *Y* if

(NU1) $1\in I$ and

(NU2) For all $x,y,z\in Y$, $(y*x)*z\in I$ and $z\in I$ imply $y*(x*(x*y))\in I$.

Theorem 3.6 *Let* (*Y*; ∗, 1) *is a BCL*$^+$ *algebra and let U and I be any tow proper ideals of Y, where* $I\subset U$. *If I is a normal uptake ideal*. *Then U is also a normal uptake ideal*.

Proof. Let $y * x \in U$, we have

$$y * ((y * x) * x) = 1 \in I .$$

We know that *I* is a *normal uptake ideal*. By Definition 3.6 (NU2), we have

$$(y * (y * x)) * (x * (x * (y * (y * x)))) \in U .$$

By Lemma 2.2 (Y2), we have

$$(y * (x * (x * (y * (y * x))))) * (y * x) \in U ,$$

since $y * x \in U$, and *U* is a *normal uptake ideal*. Then

$$y * (x * (x * (y * (y * x)))) \in U ,$$

and since

$$\begin{aligned}(y * (x * (x * y))) * (y * (x * (x * (y * (y * x))))) &\leq (x * (x * (y * (y * x)))) * (x * (x * y)) \\ &\leq (x * y) * (x * (y * (y * x))) \\ &\leq (y * (y * x)) * y = 1,\end{aligned}$$

and so

$$y * (x * (x * y)) \leq y * (x * (x * (y * (y * x)))).$$

By Theorem 3.3, we have $y * (x * (x * y)) \in U$, and so *U* satisfies Definition 3.6 (NU2) and hence is a *normal uptake ideal*. □

Definition 3.7 Let *I* be a nonempty subset of BCL^+ algebras (*Y*; $*$, 1), we say that *I* is a *Boolean ideal* of *Y* if

(BI) For all $x, y, z \in Y$, $((z * y) * x) * x \in I$ implies $x \in I$.

Example 3.3 In Example 3.1, let $a, b, c \in Y$ and suppose that $1 \in I$, by Definition 3.7, we get

$$((c * b) * a) * a = 1 \in I \quad \text{implies} \quad a \in I .$$

The subset *I* = {1, *a*} is a *Boolean ideal*.

Definition 3.8 If ideal *I* is both a *normal ideal* and a *Boolean ideal*. Then *I* is a *normal Boolean ideal*.

Theorem 3.7 *Let ideal* {1} *is a normal Boolean ideal of* BCL^+ *algebras. Then any* $a \in Y$ *, we have* $B(a) = \{x \in Y \mid a \leq x\}$ *is a normal ideal of Y.*

Proof. Let (*Y*; $*$, 1) is a BCL^+ algebra, since any $a \in Y$, we have $1 \in B(a)$. Let $x * y \in B(a)$ and let $x \in B(a)$, then $a \leq x$, and so $a \leq x * y$. Let $x \leq a$, we have $x * y \leq a * y$. Since $a \leq a * y$, we have

$$a * (a * y) = 1 \in \{1\}.$$

Since {1} satisfies Definition 3.8 (a *normal Boolean ideal* of *Y*), by Definition 3.2 (NOI2) and Definition 3.7 (BI), we have $a * y \in \{1\}$ (or By Definition 3.3 (LI), since $a * a = 1 \in \{1\}$). We see that $a * y = 1$ with $a \leq y$, and so $y \in B(a)$. We deduce that $B(a)$ is a *normal ideal* of *Y*. □

In BCL^+ algebras (*Y*; ∗, 1), if $I \subseteq Y$ is an ideal, we can define the quotient BCL^+ algebras Y/I. To make Y/I into a BCL^+ algebra, we want $Y \to Y/I$ to be a homomorphism of BCL^+ algebras. Introduce the structure of poset (with respect to containment), we are forced to define $\Im$: = {BCL^+ subalgebras of *Y*}.

Definition 3.9 Let $I = \ker\mu = \{a \mid a \in Y, (a)\mu = 0\}$. Then *I* is an ideal of *Y*, and is also a kernel of zero homomorphism μ.

Remark 3.3 Suppose $a + I, b + I \in Y/I$. Show that

$$\begin{aligned}((a+I)\cdot(b+I))\beta &= (a\cdot b+I)\beta \\ &= (a\cdot b)\mu \\ &= ((a)\mu)\cdot((b)\mu) \\ &= ((a+I)\beta)\cdot((b+I)\beta).\end{aligned}$$

Theorem 3.8 *Let Y be a* BCL^+ *algebra and* $\Im$ *be subalgebra in Y, and suppose* $\gamma : Y \to \Im$. *Then there exist a unique morphism* $\beta : Y/I \to \Im$, *where* $I \subseteq Y$ *is an ideal and define* Y/I *is the coset* $I + r$ (*is a kernel I in Y*) *and* $r \in Y$ *such that* $\beta\alpha = \gamma$ *which means that the following diagram in Figure* 2 *is commutative*.

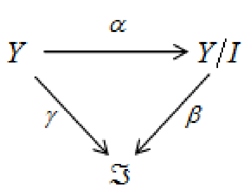

Figure 2.

Proof. Let $Y/I = \{\bar{a} \mid a \in Y\}$ and let $\bar{a} \in Y/I$. Since $(\bar{a})\beta = (a)\gamma$ for some $a \in Y$, we have

$$(a)(\alpha\beta) = ((a)\alpha)\beta = (\bar{a})\beta = (a)\gamma.$$

Thus $\beta\alpha = \gamma$, and Figure 2 is commutative.

Conversely, since β is a unique morphism, suppose $\bar{\beta} : Y/I \to \Im$, we have Figure 2 is commutative in this case. Then

$$(\bar{a})\bar{\beta} = ((a)\alpha)\bar{\beta} = (a)(\alpha\bar{\beta}) = (a)\gamma.$$

Thus $\overline{\beta} = \beta$. □

Corollary 3.3 *Let* $\mu : Y \to H$ *be a morphism of BCL*$^+$ *algebras. Then* μ *is a unique morphism iff* $\ker\mu = \{e\}$, *where e is an identical element.*

Theorem 3.9 *Let* $\mu : Y \to H$ *is an epic morphism of BCL*$^+$ *algebras and let* $I = \ker\mu$. *Then* $H \cong Y/I$.

Proof. By Corollary 3.3 the μ is a unique morphism, and so by Theorem 3.8 the μ is an isomorphism, however, we have $H \cong Y/I$. □

Acknowledgments

The author is grateful to the anonymous referees for useful comments and suggestions.

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