Journal of Mathematics Research; Vol. 8, No. 2; April 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Osserman Lightlike Hypersurfaces on a Foliated Class of Lorentzian Manifolds

C. Atindogbe[1] & K. L. Duggal[2]

[1] Institute de Mathématiques et de Sciences Physiques (IMPS) Université d'Abomey-Calavi, 01$B.P.$613, Porto-Novo, Bénin

[2] Department of Mathematics and Statistics, University of Windsor, Windsor, Ontario N9B 3P4, Canada

Correspondence: K. L. Duggal, University of Windsor, Windsor, Ontario, N9B 3P4, Canada. E-mail: yq8@uwindsor.ca

Received: February 5, 2016 Accepted: February 24, 2016 Online Published: March 17, 2016

doi:10.5539/jmr.v8n2p55 URL: http://dx.doi.org/10.5539/jmr.v8n2p55

Abstract

This paper deals with a family of Osserman lightlike hypersurfaces (M_u) of a class of Lorentzian manifolds $\bar{M}$ such that its each null normal vector is defined on some open subset of $\bar{M}$ around M_u. We prove that a totally umbilical family of lightlike hypersurfaces of a connected Lorentzian pointwise Osserman manifold of constant curvature is locally Einstein and pointwise $\mathcal{F}-$Osserman, where our foliation approach provides the required algebraic symmetries of the induced curvature tensor. Also we prove two new characterization theorems for the family of Osserman lightlike hypersurfaces, supported by a physical example of Osserman lightlike hypersurfaces of the Schwarzschild spacetime.

Keywords: lighlike hypersurfaces, Lorentzian manifold, algebraic curvature map, Osserman condition

Subject Clasification: 53C12; 53C50; 53B50

1. Introduction

A primary interest in differential geometry is to determine the curvature and the metric of a given smooth manifold, which distinguishes the geometry of this subject from the others that are analytic, algebraic or topological. It is now well-known that the research on the Osserman condition (which involves sectional curvature and Jacobi operator) has provided substantial information on the curvature and metric tensors of Riemannian manifolds. An up-to-date account on Riemannian and semi-Riemannian Osserman geometry is available in a book (García-Río et. al., 2002). Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. We say that $R \in \otimes^4 T_p^\star \bar{M}$ is an algebraic curvature map (tensor) on $T_p\bar{M}$ if it satisfies the following symmetries:

$$
\begin{aligned}
R(X, Y, Z, W) &= -R(Y, X, Z, W) = R(Z, W, X, Y), \\
R(X, Y, Z, W) &+ R(Y, Z, X, W) + R(Z, X, Y, W) = 0
\end{aligned}
\tag{1.1}
$$

for all $X, Y, Z, W \in T_p\bar{M},\ p \in \bar{M}$. The Riemann curvature tensor is an algebraic curvature tensor on the tangent space $T_p\bar{M}$, for every $p \in \bar{M}$. For an algebraic curvature map R the associated Jacobi operator $J(X)$ is the linear map on $T_p\bar{M}$ characterized by the identity

$$\bar{g}(J(X)Y, Z) = R(Y, X, X, Z), \tag{1.2}$$

$J(X)$ is a self-adjoint map and R is spacelike (respectively timelike) Osserman tensor if Spec$\{J(X)\}$ is constant on the pseudo-sphere of unit spacelike (respectively unit timelike) vectors in $T_p\bar{M}$. These are equivalent notions (Gilkey, 2001) and such a tensor is called an Osserman tensor. The basic problem is to what extent general sectional curvatures can provide information on the curvature and metric tensors.

Since any semi-Riemannian manifold has lightlike subspaces, one reasonably expects a role of Jacobi and Szabó type operators (Gilkey, 2001) in the study of lightlike hypersurfaces say (M, g) of an $(n + 2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. According to (Duggal-Bejancu, 1996) approach, for a lightlike (M, g) there exists a non-vanishing null vector field ξ entirely in TM such that $g(\xi, X) = 0, \forall X \in TM$ and

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{1.3}$$

where $Rad(TM) = \{\xi\}$ and $S(TM)$ are a 1-dimensional radical null distribution and n-dimension non-degenerate complementary screen distribution, respectively, and $\oplus_{orth}$ is a symbol for orthogonal sum. Throughout (Duggal-Bejancu, 1996) approach the null normal vector field ξ is defined entirely on M . For up-to date information on a general study of lightlike hypersurfaces of semi-Riemannian manifolds using the distribution equation (1.3) we refer to books (Duggal-Bejancu, chapter 4, 1996) and (Duggal-Jin, chapter 7, 2007). Since Jacobi and Szabó operators need the use of an inverse of a metric, due to degenerate g of M the definition of these operators is not possible in the usual way for the lightlike case. To deal with this (Atindogbe-Duggal, 2009) used the concept of pseudo-inverse of a degenerate metric (Atindogbe et. el., 2003) and defined pseudo-Jacobi operators. Besides this anomaly, in general, induced Riemann curvature tensors on lightlike hypersurfaces are not algebraic curvature tensors, i.e (1.1) does not hold. For this anomaly, they proved the following result.

Theorem 1. (Atindogbe-Duggal, 2009) *If the induced curvature tensor of M is an an algebraic curvature tensor, then, locally at least one of the following holds.*

(a) M is totally geodesic.

(b) M is null transversally closed, that is, its transversal bundle is parallel along the radical direction.

Also, since in Duggal-Bejancu approach the lightlike geometry depends on a choice of screen distribution $S(TM)$ which is not unique, a well-defined concept of Osserman condition is not possible for an arbitrary lightlike hypersurface. Therefore, they looked for an admissible $S(TM)$ for which the associated induced curvature tensor of M is an algebraic curvature. To the best of our knowledge, at the time of writing this paper, there are the (Atindogbe-Duggal's, 2009) paper and (Atindogbe et. el., 2011, Brunette, 2014) on the Osserman lightlike geometry, using an admissible screen distribution.
Since lightlike (also called degenerate) geometry has been studied by several ways other than using a screen for specific problems (for example, see (Akivis-Goldberg 2000, Leistner, 2006) one may ask the following question. Is there a better way to deal with the lightlike Osserman geometry to improve on previous works on this topic and also find some new results? In this paper we answer both of these questions by using the following different approach: Consider a class of lightlike hypersurfaces M for which we assume that the null normal ξ is defined on some open subset of $\bar{M}$ around M. Following (Carter, 1997), a simple way is to consider a foliation of $\bar{M}$ (in the vicinity of M) by a family (M_u) so that each ξ_u is in the part of $\bar{M}$ foliated by this family such that at each point in this region, ξ_u is a null normal to M_u for some value of u. Although the family (M_u) is not unique, for our purpose we can manage (with some reasonable condition(s)) to involve only those quantities which are independent of the choice of the family (M_u) once evaluated at, say, $M_{u=const.}$. We highlight that since ξ is not entirely in M the distribution equation (1.3) will not hold. However, due to degenerate metric g there is no canonical transversal direction since the normal vector ξ coincides with the tangent vector of M. Thus, a projector mapping $\Pi : T_p\bar{M} \to T_pM$ (needed to obtain well-defined induced objects on M) can not be defined from M alone. There is a need for some extra structure on $\bar{M}$. In this paper we consider an extra structure of a class of Lorentzian manifolds $(\bar{M}, \bar{g})$ whose metric $\bar{g}$ is prescribed by (2.5). This Lorentzian structure and its hypersurfaces have an added advantage of its uses in a variety of physical problems (see some references in Section 2 and 5). In Section 2 we obtain the normalized expression for any ξ_u, its corresponding transversal vector field N_u of $T_p\bar{M}$ (see Theorem 3) and a well-defined projector onto M. Using the transversal vector field and the projector onto M, in Section 3 we write the local extrinsic form of the Gauss and Weingarten equations and the three local Gauss-Codazzi equations (3.6). It is important to notice from these three Gauss-Codazzi equations that the induced Riemannian curvature tensor on M may not be an algebraic curvature tensor, i.e. (1.1) does not hold. In Section 4, we first recall the following Osserman conjecture (see Chi, 1988).

Any Osserman manifold is either a locally flat space or a locally rank-one symmetric space.

In (García-Río et. el., 2002) we have historical development on solving the Osserman conjecture for Riemannian, semi-Riemannian and Lorentzian manifolds. In the Lorentzian case they have presented the following positive answer to this conjecture:
"If $(\bar{M}, \bar{g})$ is a connected (dim($\bar{M} \geq 3$) Lorentzian pointwise Osserman manifold, then it is a real space form." (see García-Río et. el., 2002, Theorem 3.1.2, page 42).
In this paper, we prove *"A totally umbilical family of lightlike hypersurfaces (M_u) of a connected Lorentzian pointwise manifold of constant curvature is locally Einstein and pointwise $\mathcal{F}-$Osserman"*. Contrary to a condition in Theorem 1 of a previous work of (Atindogbe-Duggal, 2009), we show that in Theorem 11 our foliation approach provides the required algebraic symmetries of the induced curvature tensor of any member of (M_u), which is our first new result. Also, we prove two new characterization theorems (see Theorems 12 and 13) on totally geodesic lightlike Osserman hypersurfaces. In Theorem 12, using a symmetry condition we prove that the induced Riemann curvature tensor of the family (M_u) of lightlike hypersurfaces has the required symmetries if and only if the local second fundamental form of its each member is a Codazzi tensor. In Theorem 13 we use stationary coordinate system for a spacetime $\bar{M}$ to present a physical example of Osserman lightlike hypersurfaces of the Schwarzschild spacetime.

2. Projector Mapping onto (M, g)

To make this paper self contained, we have taken some material from a paper (Duggal, 2014). Recall that a hypersurface (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is lightlike if there exists a non-vanishing null vector field ξ in TM which is orthogonal (with respect to g) to all vector fields in TM, that is,

$$g(\xi, X) = 0, \quad \forall X \in TM,$$

where g is the degenerate metric of M. In this paper we assume that the null normal ξ is not entirely in M, but, is defined in some open subset of $\bar{M}$ around M. This will permit to well-define the covariant derivative $\bar{\nabla}\xi$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$. As explained in introduction, we have a foliation of $\bar{M}$ (in the vicinity of M) by a family (M_u) so that each ξ_u is in the part of $\bar{M}$ foliated by this family such that at each point in this region, ξ_u is a null normal to M_u for some value of u. For simplicity, in this paper we consider (M, g) as a member of the family $((M_u), (g_u))$ and its respective metric g for some value of u, with the understanding that the results are same for any other member. The "bending" of M in $\bar{M}$ is described by the *Weingarten map*:

$$\begin{aligned} \mathcal{W}_\xi : T_pM &\rightarrow T_pM \\ X &\rightarrow -\bar{\nabla}_X\xi. \end{aligned} \tag{2.1}$$

$\mathcal{W}_\xi$ associates each X of M a variation of ξ along X, with respect to the connection $\bar{\nabla}$. The second fundamental form, say B, of M is the symmetric bilinear form and is related to the Weingarten map by

$$B(X, Y) = g(\mathcal{W}_\xi X, Y). \tag{2.2}$$

$B(X, \xi) = 0$ for any null normal ξ and for any $X \in TM$ and this implies that B has the same ξ degeneracy as that of the induced metric g. (M, g) is called totally umbilical in $\bar{M}$ if and only if there is a smooth function ρ on M such that

$$B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in TM. \tag{2.3}$$

The totally umbilical notion does not depend on particular choice of ξ. M is proper totally umbilical if and only if ρ is non-zero on entire M. In particular, M is totally geodesic if and only if B vanishes, i.e., if and only if ρ vanishes on M. From (2.2), $B(X, \xi) = 0$ for any null normal ξ and (2.3), we conclude that M is totally umbilical in $\bar{M}$ if and only if, on each neighborhood $\mathcal{U}$ the conformal function ρ satisfies

$$\mathcal{W}_\xi X - \rho X \in \ker(g), \quad \text{for all} \quad X \in \Gamma(TM). \tag{2.4}$$

Following result is important in the study of lightlike hypersurfaces:

Proposition 2. *Let $((M_u), (g_u))$ be a family of hypersurfaces of a semi-Riemannian manifold. Then its each member (M, g) is totally umbilical if and only if its each null normal ξ is a conformal Killing vector of the degenerate metric g.*

Proof. Consider a member (M, g) of $((M_u), (g_u))$. Using the expression $£_\xi g(X, Y) = g(\bar{\nabla}_X\xi, Y) + g(\bar{\nabla}_Y\xi, X)$ and $B(X, Y)$ symmetric in above equation we obtain

$$B(X, Y) = \frac{1}{2}£_\xi g(X, Y), \quad \forall X, Y \in TM,$$

which is well defined up to conformal rescaling (related to the choice of ξ). Suppose M is totally umbilical, i.e., (2.3) holds. Using this in above equation we have $£_\xi g = 2\rho g$ on M. Therefore, ξ is conformal Killing vector of the metric g. Conversely, assume $£_\xi g = 2\rho g$ on M. Then,

$$\begin{aligned} £_\xi g(X, Y) &= g(\bar{\nabla}_X\xi, Y) + g(\bar{\nabla}_Y\xi, X) = 2g(\bar{\nabla}_X\xi, Y) \\ &= 2g(\mathcal{W}_\xi X, Y) = 2\rho g(X, Y), \end{aligned}$$

which implies that (2.3) holds so M is totally umbilical. □

Throughout this paper, for simplicity, we denote by (M, g, ξ) a member of the family of lightlike hypersurfaces $((M_u), (g_u), (\xi_u))$ for some value of u of an $(n+2)$−dimensional Lorentzian manifold $(\bar{M}, \bar{g})$. Due to degenerate metric g of M and not using any screen distribution, there is no canonical way for the degenerate structure alone of M to define a projector mapping $\Pi : T_p\bar{M} \rightarrow T_pM$, to get induced extrinsic objects of M. Thus, there is a need for some extra structure on $\bar{M}$.

For this purpose, consider (M, g, ξ) a lightlike hypersurface of $\bar{M}$ evolved from a spacelike hypersurface H_t, at a coordinate time t to another spacelike hypersurface H_{t+dt} at coordinate time $t + dt$, whose metric $\bar{g}$ is given by

$$ds^2_{|\bar{g}} = \bar{g}_{ij}dx^i dx^j = (-\lambda^2 + |U|^2)dt^2 + 2\gamma_{ab}U^a dx^b dt + \gamma_{ab}dx^a dx^b, \tag{2.5}$$

where $x^0 = t$, and $x^a, a = 1, \cdots, n+1$ are spacelike coordinates of H_t, with γ_{ab} its $(n+1)$-metric induced from $\bar{g}, \lambda = \lambda(t, x^1, \cdots, x^{n+1})$ is the lapse function and U^a are the components of a spacelike vector U, called the shift vector. In this way we also have a family of spacelike hypersurfaces (H_t) of $\bar{M}$. The above choice of spacetime metric (2.5) is physically important frame work. For example, see (Gourgoulhon-Jaramillo, 1006) on event and isolated horizons (also see Duggal, 2008, 2012, 2014), including several related references. The coordinate time vector $\mathbf{t} = \frac{\partial}{\partial t}$ is such that $\bar{g}(dt, \mathbf{t}) = 1$, which we can write

$$\mathbf{t} = \lambda \mathbf{n} + U, \quad \bar{g}(\mathbf{n}, U) = 0,$$

where $\mathbf{n}$ denotes the timelike unit vector field. We assume that each H_t intersects each corresponding lightlike hypersurface M of M_u on some n-dimensional submanifold of $\bar{M}$, we denote by $\mathbf{S}_{t,u} = M_u \cap H_t$ an element of this family $(\mathbf{S}_{t,u})$ of submanifolds of $\bar{M}$. Let $\mathbf{s} \in \Gamma(TH_t)$ be a unit spacelike vector normal to $\mathbf{S}_{t,u}$ defined in some open neighborhood of M_u.

Theorem 3. *Let (M, g, ξ) be a member of the family of lightlike hypersurfaces $((M_u), (g_u), (\xi_u))$ of an $(n+2)$-dimensional Lorentzian manifold $(\bar{M}, \bar{g})$ whose metric $\bar{g}$ is given by (2.5). Then the following holds.*

(a) The vector field N of $T\bar{M}$ along M given by

$$N = \frac{1}{2\lambda}(\mathbf{s} - \mathbf{n}) \tag{2.6}$$

is a null transversal normalizing vector field of M.

(b) The corresponding normalized expression for each ξ of M is

$$\xi = \lambda(\mathbf{n} + \mathbf{s}), \quad \bar{g}(\xi, N) = 1, \quad \forall V \in T_p\mathbf{S}, \ \ \bar{g}(\mathbf{s}, V) = 0. \tag{2.7}$$

Proof. Let $\mathbf{s} \in \Gamma(TH_t)$ be a unit vector normal to $\mathbf{S}_t$ defined in some open neighborhood of M. Taking $(\mathbf{S}_t)$ a foliation of M, the coordinate t can be used as a non-affine parameter along each null geodesic generating M. The question is how to find some (null) direction transverse to M so that the corresponding normalized null tangent vector field ξ is tangent vector associated with this parameterization of the null generators, i.e.,

$$\xi^i = \frac{dx^i}{dt}.$$

This means that ξ is a vector field "dual" to the 1−form dt. Equivalently, the function t can be regarded as a coordinate compatible with ξ, i.e.,

$$g(dt, \xi) = \bar{\nabla}_\xi t = 1.$$

As $\mathbf{n}$ and $\mathbf{s}$ are timelike and spacelike respectively and they both do not belong to M, but also are normal to the n-dimensional spacelike submanifold $\mathbf{S}_t$ in M of $\bar{M}$, it is easy to see that both ξ and N are linear combination of $\mathbf{n}$ and $\mathbf{s}$. Based on above facts, it is easy to see that ξ has the normalized form (2.7) which is tangent to M and it has the property of Lie dragging the family of submanifolds $(\mathbf{S}_{t,u})$ of $\bar{M}$. As any M is defined by u = a constant, the gradient du is its normal, i.e.

$$g(du, X) = 0, \ \ \forall X \in \Gamma(TM).$$

Thus, the 1−form $\underline{\xi}$ associated to the null normal ξ is collinear to du, i.e. $\underline{\xi} = e^\rho du$, where $\rho \in C^\infty(M)$ is a scalar field. It follows that N as given in (2.6) is a null transversal normalizing vector field of M satisfying the second equality in (2.7). This completes the proof. □

We say that two null normals ξ and $\tilde{\xi}$ of M belong to the same equivalence class $[\xi]$ if $\tilde{\xi} = c\xi$ for some positive constant c. Then, it follows from (2.3) that with respect to change of ξ to $\tilde{\xi}$ there is another $\tilde{N} = (1/c)N$ satisfying (2.3). Now we define the projector onto M along N by

$$\begin{aligned} \Pi \ \ : \ \ & T_p\bar{M} \to T_pM \\ & \bar{X} \to X = \bar{X} - \bar{g}(\xi, \bar{X})N. \end{aligned}$$

Above mapping is well defined, i.e., its image is in T_pM. Indeed,

$$\forall \bar{X} \in T_p\bar{M}, \quad \bar{g}(\xi, \Pi(\bar{X})) = \bar{g}(\xi, \bar{X}) - \bar{g}(\xi, \bar{X})\bar{g}(\xi, N) = 0.$$

Π leaves invariant any vector in T_pM and $\Pi(N) = 0$. Moreover, the definition of the projector Π does not depend on the normalization of ξ and N if they satisfy the relation (2.7). In other words, Π is determined only by the foliation of the family $(\mathbf{S}_{t,u})$ of M_u and not by any rescaling of ξ.

Example 4. *Consider a null cone $\Lambda_0^{n+1} \in (M_u)$ of $\mathbf{R}_1^{n+2}$ given by*

$$t = F(x^1, \cdots, x^{n+1}) = r = \sqrt{\sum_{a=1}^{n+1}(x^a)^2},$$

where $(t, x^1, \cdots, x^{n+1})$ are the Minkowskian coordinates with origin $\mathbf{0}$. Exclude $\mathbf{0}$ to keep null cone smooth. Let Λ_0^{n+1} be a member at the level $u = b \neq 0$. Then, the scalar u generates a family of null cones $((\Lambda_0^{n+1})_u)$ given by

$$u(t, x^1, \cdots, x^{n+1}) = r - t + b, \quad \textit{with} \quad t = \sqrt{\sum_{a=1}^{n+1}(x^a)^2}$$

$\nabla_i u = (-1, x^1/t, \cdots, x^{n+1}/t)$. *The components of ξ are*

$$\xi^i = e^\rho(1, x^1/t, \cdots, x^{n+1}/t) \quad \textit{and} \quad \xi_i = e^{-\rho}(-1, x^1/t, \cdots, x^{n+1}/t),$$

where $\lambda = e^\rho$. The two unit normals $\mathbf{n}$ and $\mathbf{s}$ in the expression of ξ are

$$\mathbf{n}^i = (1, 0, \cdots, 0), \quad \textit{and} \quad \mathbf{s}^i = (0, x^1/t, \cdots, x^{n+1}/t).$$

Therefore, it follows from (2.7) that

$$N^i = \frac{e^\rho}{2}(1, x^1/t, \cdots, x^{n+1}/t), \quad N_i = \frac{e^{-\rho}}{2}(-1, x^1/t, \cdots, x^{n+1}/t).$$

Example 5. *Consider a smooth function $F : \Omega \to \mathbf{R}$, where Ω is an open set of $\mathbf{R}^{n+1}$. Then a hypersurface M of $\mathbf{R}_1^{n+2}$ is called a Monge hypersurface (Duggal-Bejancu, 1996, page 129) given by the equation*

$$t = F(x^1, \cdots, x^{n+1}),$$

where $(t, x^1, \cdots, x^{n+1})$ are the standard Minkowskian coordinates with origin $\mathbf{0}$. The scalar u generates a family of Monge hypersurfaces (M_u) as the level sets of u and is given by

$$u(t, x^1, \cdots, x^{n+1}) = F - t + b,$$

where b is a constant and we take $M_b = M$ a member of the family (M_u). $\nabla_i u = (-1, F'_{x^1}, \cdots, F'_{x^{n+1}})$. M is lightlike if and only if, F is a solution of the following partial differential equation

$$\sum_{a=1}^{n+1}(F'_{x^a})^2 = 1.$$

Then, the null normal ξ of M is

$$\xi^i = e^\rho(1, F'_{x^1}, \cdots, F'_{x^{n+1}}) \quad \textit{and} \quad \xi_i = e^{-\rho}(-1, F'_{x^1}, \cdots, F'_{x^{n+1}})$$

Thefore, it follows from (2.7) that

$$N^i = \frac{1}{2}e^\rho(1, F'_{x^1}, \cdots, F'_{x^{n+1}}) \quad \textit{and} \quad N_i = \frac{1}{2}e^{-\rho}(-1, F'_{x^1}, \cdots, F'_{x^{n+1}}).$$

Induced metric on $(\mathbf{S}_{t,u})$. Let h be the induced Riemannian metric on an n-dimensional element $\mathbf{S}_{t,u}$ of the family $(\mathbf{S}_{t,u})$ of co-dimension 2 submanifolds of $\bar{M}$. Using the spacelike normal $\mathbf{s}$ and timelike normal $\mathbf{n}$ it is easy to see that the expression of h is given by

$$h = \gamma - \underline{\mathbf{s}} \otimes \underline{\mathbf{s}} = \bar{g} + \underline{\mathbf{n}} \otimes \underline{\mathbf{n}} - \underline{\mathbf{s}} \otimes \underline{\mathbf{s}}. \tag{2.8}$$

Proposition 6. *The positive definite metric h induced by $\bar{g}$ on $T_p(\mathbf{S}_t)$ coincides with the degenerate metric g induced by $\bar{g}$ on $T_p(M)$.*

Proof. Let x and y be the projections along ξ on $T_p(\mathbf{S}_t)$ of their respective pair of vectors (X, Y) in $T_p(M)$. Then, we have the unique decompositions

$$X = x + a\xi, \quad Y = y + b\xi,$$

for two real numbers a and b. Using $\mathbf{n}.x = \mathbf{n}.y = \mathbf{s}.x = \mathbf{s}.y = 0$ in the righthand side of the relation (2.8) it is straightforward to get

$$h(X, Y) = \bar{g}(X, Y).$$

This means that h and $\bar{g}$ coincide on $T_p(M)$ which further means that h coincides with the degenerate metric g of $T_p(M)$ that completes the proof. □

It follows from above that g can replace the notation h given by

$$g = \bar{g} + \underline{\mathbf{n}} \otimes \underline{\mathbf{n}} - \underline{\mathbf{s}} \otimes \underline{\mathbf{s}}. \tag{2.9}$$

The endomorphism $T_p(\bar{M}) \to T_p(\bar{M})$ canonically associated with h by the metric $\bar{g}$ is the orthogonal projector onto $\mathbf{S}_t$ given by

$$h = I + \mathbf{n}\langle \underline{\mathbf{n}}, \cdot \rangle - \mathbf{s}\langle \underline{\mathbf{s}}, \cdot \rangle. \tag{2.10}$$

3. The Induced Extrinsic Objects

Recall that (Duggal-Bejancu, 1996) used a screen distribution to obtain induced extrinsic objects of a lightlike hypersurface. Although we are not using any screen for M, we do have a vector bundle $T\mathbf{S}$ of the family of n-dimensional co-dimension 2 submanifolds of $\bar{M}$. For the extrinsic structure equations needed in this paper we replace the role of screen by the vector bundle $T\mathbf{S}$ of $\bar{M}$ which has the added advantage that it is obviously integrable. With this understanding, from Theorem 3 we have the following decomposition.

$$T\bar{M}_{|M} = TM \oplus_{orth} tr(TM), \tag{3.1}$$

where $tr(TM) = span\{N\}$ denotes a null transversal vector bundle of rank 1 and is complementary to TM in $T\bar{M}_{|M}$. Using above decomposition and the second fundamental form B, we obtain the following extrinsic *Gauss and Weingarten* formulas:

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{3.2}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X, Y \in \Gamma(TM), \tag{3.3}$$

where A_N is the shape operator of T_pM in $\bar{M}$ and τ is a $1-$form on M and ∇ the linear connection on (M, ξ). In general, ∇ is not a Levi-Civita connection and it satisfies

$$(\nabla_x g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \quad \forall X, Y, Z \in \Gamma(TM_{|\mathcal{U}}), \tag{3.4}$$

where $\eta(X) = \bar{g}(X, N)$ $\forall X \in \Gamma(TM_{|\mathcal{U}})$. Let $\bar{R}$ and R denote the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ and the linear connection ∇ on M, respectively. Using (3.2) and (3.3) we obtain

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)\}N \\ &+ \{\tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N. \end{aligned} \tag{3.5}$$

The local Gauss-Codazzi equations are

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, V) &= g(R(X, Y)Z, V), \quad \forall V \in T\mathbf{S}. \\ \bar{g}(\bar{R}(X, Y)Z, \xi) &= g(R(X, Y)Z, \xi) \\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y). \\ \bar{g}(\bar{R}(X, Y)Z, N) &= g(R(X, Y)Z, N). \end{aligned} \tag{3.6}$$

The induced Ricci tensor of (M, g) is given by the following formula:

$$R(X,\ Y) = trace\{Z \to R(Z,\ X)Y\}, \quad \forall\, X, Y \in \Gamma(TM).$$

Since ∇ on M is not a Levi-Civita connection, in general, Ricci tensor is not symmetric. Indeed, let $F = \{\xi, N, e_1, \cdots, e_n\}$ be a quasi-orthonormal frame on $\bar{M}$. Then, we obtain

$$Ric(X,\ Y) = \sum_{a=1}^{n} g(R(e_a, X)Y, e_a) + g(R(\xi, X)Y, N).$$

Using Gauss-Codazzi equations and the first Bianchi identity we get

$$Ric(X,\ Y) - Ric(Y,\ X) = 2d\tau(X, Y). \tag{3.7}$$

Also, the 1−form τ in (3.3) depends on the the normalizing field N and, requiring that *Ric* is symmetric is important to both geometric and physical purpose. So, the following fact is noteworthy.

Proposition 7. *Let (M, g, N) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with $\overline{g}$ given by* (2.5) *and null transversal vector field (2.7). Then, the induced Ricci tensor of M is symmetric if and only if the connection* 1−*form on M given by*

$$\tau(X) = -\frac{1}{\lambda}\Big[(X \cdot \lambda) - \langle n, \overline{\nabla}_X s\rangle\Big]$$

is closed. In particular, τ vanishes if the lapse function λ satisfies $d\lambda = \langle n, \overline{\nabla}.\mathbf{s}\rangle$, where $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{g}$.

Proof. The connection 1−form τ is given by $\tau(X) = \left\langle \overline{\nabla}_X N, \xi \right\rangle$. Then, using (2.6), (2.7) in Theorem 3 and (3.7) leads to the above fact. □

4. Results

To interpret the relation (1.2) we need to get a pseudo-Jacobi operator associated to an algebraic curvature R of (M, g). Consider on M a normalized pair $\{\xi, N\}$ satisfying Theorem 3 and the 1−form $\eta(X) = \bar{g}(N, X)$, $\forall X \in \Gamma(TM)$. Define an isomorphic map $\flat$ by

$$\begin{aligned} \flat : \Gamma(TM) &\;\to\; \Gamma(T^\star M) \\ X &\;\to\; X^\flat = g(X, \bullet) + \eta(X)\eta(\bullet). \end{aligned}$$

Let $\sharp$ denote the inverse of the isomorphism $\flat$. Define a $(0, 2)$-tensor $\tilde{g} = X^\flat(Y) = g(X, Y) + \eta(X)\eta(Y)$, $\forall X, Y \in \Gamma(TM)$. Clearly, $\tilde{g}$ is a Riemannian metric on M and the following holds:

$$\tilde{g}(\xi, \xi) = 1, \quad \tilde{g}(\xi, X) = \eta(X), \quad \forall X \in \Gamma(TM).$$

Let $\mathcal{S}_p(M) = \{V \in T_pM \| g(V, V)| = 1\}$ be a unit bundle at any point $p \in M$. Then, in terms of the above isomorphisms $\flat_g$ and $\sharp_g$, the following is equivalence to the relation (1.2) for V in the unit bundle, $X, Y \in T_pM$ and g Riemannian.

$$(J_R(V)X)^{\flat_g}(Y) = R(X, V, V, Y), \quad \text{i.e} \quad J_R(V)X = R(X, V, V, \bullet)^{\sharp_g}. \tag{4.1}$$

Definition 8. *Let (M, g, ξ, N) be a member of a family of lightlike hypersurfaces $F = ((M_u), (g_u))$ of a Lorentzian manifold $(\bar{M}, \bar{g})$ defined by the metric (2.5) such that the induced Riemannian curvature tensor R of each $M \in F$ is an algebraic curvature tensor and $V \in \mathcal{S}_p(M)$. By a pseudo-Jacobi operator of R with respect to V we mean the self-adjoint linear map $J_R(V)$ of $V^\perp$ defined by*

$$J_R(V)X = R(X, V, V, \bullet)^{\sharp_g}$$

where $\sharp_g$ is the dual isomorphism on the triplet (M, g, N).

Remark. Let R be the induced (algebraic) Riemann curvature tensor of $(M, \tilde{g})$, $(p \in M)$ and $\xi \in TM^\perp$. Then, we have

$$J_R(V)\xi = 0. \tag{4.2}$$

Indeed, for all $V \in \mathcal{S}_p(M)$, $Z \in T_pM$,

$$\tilde{g}(J_R(V)\xi, Z) = R(\xi, V, V, Z) = g(R(z, V)V, \xi) = 0,$$

and since $\tilde{g}$ is non-degenerate on TM, we have $J_R(V)\xi = 0$.

Definition 9. *A lightlike hypersurface* (M, g) *of a Lorentzian manifold* $(\overline{M}, \overline{g})$ *with metric* $\overline{g}$ *given by (2.5) is said to be timelike (resp. spacelike)* $\mathcal{F}-$*Osserman at* $p \in M$ *if the normalization (2.7) induces an algebraic curvature map and the characteristic polynomial of the associate Jacobi operator* $J_R(x)$ *is independent of* $x \in \mathcal{S}_p^-(M)$ *(resp.* $x \in \mathcal{S}_p^-(M)$*). Moreover, if this holds at each* $p \in M$*, then* (M, g) *is called pointwise* $\mathcal{F}-$*Osserman.*

It is easy to check that timelike $\mathcal{F}-$Osserman and spacelike $\mathcal{F}-$Osserman are equivalent notions and we will use indistinctly $\mathcal{F}-$Osserman for both.
Recall that the Osserman Conjecture is true in Lorentzian manifolds, which we quote

Theorem 10. (Garcı́a-Rı́o et. al., 2002, Th. 3.1.2, page 42) *If* $(\bar{M}, \bar{g})$ *is a connected (dim(*$\bar{M} \geq 3$*) Lorentzian pointwise Osserman manifold, then it is a real space form.*

Thus, the Lorentzian Osserman manifolds have constant curvaure. In our foliated approach fixing the ambiguities inherent to null hypersurfaces normalizations, we have the following important result in characterizing $\mathcal{F}-$Osserman lightlike hypersurfaces of Lorentzian manifolds of constant curvature.

Theorem 11. *Let* $F = ((M_u), (g_u), (\xi_u), (N_u))$ *be a family of totally umbilical lightlike hypersurfaces of an* $n + 2 \geq 3$ *connected Lorentzian pointwise manifold* $(\bar{M}, \bar{g})$ *of constant curvature and defined by the metric (2.5). Then, each lightlike hypersurface of the family* F *is locally Einstein and pointwise* $\mathcal{F}-$*Osserman.*

Proof. We first point out that, as per Theorem 10, for $\dim \overline{M} = n + 2 \geq 3$, the Lorentzian manifold $(\overline{M}, \overline{g})$ given by (2.5) being pointwise Osserman is necessarily a real space form. Pick $M = M_{u_0}$ a member of the family F and let g denote the induced degenerate metric on M. By hypothesis M being totally umbilical in $\bar{M}$ implies that $B(X, Y) = \rho g(X, Y)$ for some smooth function ρ. Let c denote the (constant) ambient sectional curvature and R the induced Riemann curvature on M. Then,

$$\begin{aligned} R(X, Y)Z &= c\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad +\rho\{g(Y, Z)A_N X - g(X, Z)A_N Y\}. \end{aligned} \tag{4.3}$$

For any $X \in \Gamma(TM)$ denote by x its corresponding projection along ξ on TS. Then,

$$x = X - \frac{1}{2}\langle \mathbf{s} - \mathbf{n}, X\rangle(\mathbf{s} + \mathbf{n}).$$

Since each M_u is totally umbilical in $\bar{M}$, it follows from the Proposition 6 that the codimension 2 family ($S_t = M_{u_0} \cap H_t$) of $\bar{M}$ gives a totally umbilical nondegenerate foliation. Hence, from total umbilicity of the foliation (S_t), there is a smooth function μ with

$$A_N = \mu\Big[I - \frac{1}{2}\langle \mathbf{s} - \mathbf{n}, \ \cdot \ \rangle(\mathbf{s} + \mathbf{n})\Big]. \tag{4.4}$$

Then, (4.3) becomes,

$$R(X, Y)Z = (c + \mu\rho)R_0(X, Y)Z - \frac{1}{2}\mu\rho\Big\langle \mathbf{s} - \mathbf{n}, R_0(X, Y)Z\Big\rangle(\mathbf{s} + \mathbf{n}), \tag{4.5}$$

where we set

$$R_0(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

Thus, it is clear that the induced Riemann tensor has required algebraic symmetries. By using (2.6) and (2.7) we compute the $(0, 2)-$tensor of Ricci:

$$\begin{aligned} Ric(X, Y) &= \sum_{i=1}^{n} \langle R(X, W_i)Y, W_i\rangle + \langle R(X, \xi)Y, N\rangle \\ &= c\,[\langle X, Y\rangle - n\langle X, Y\rangle)] + \mu\rho\,[\langle X, Y\rangle - n\langle X, Y\rangle] - c\langle X, Y\rangle \\ &= [(1 - n)\mu\rho - nc]\,\langle X, Y\rangle. \end{aligned}$$

This shows that M is locally Einstein. Let $p \in M$, $x \in T_p\mathcal{S}$ and $Z \in x^{\perp}$. Then,

$$\begin{aligned} J_R(x)Z &= R(Z, x, x, \cdot)^{\sharp_g} \\ &\overset{(4.7)}{=} \Big[(c + \mu\rho)\Big(\langle x, Z\rangle\langle \cdot, Z\rangle - \frac{1}{2}\mu\rho\Big\langle \mathbf{s} - \mathbf{n}, R_0(Z, x)x\Big\rangle\langle \mathbf{s} + \mathbf{n}, \cdot\rangle\Big]^{\sharp_g} \end{aligned}$$

$$\begin{aligned} &= (c+\mu\rho)\langle x,x\rangle\langle\cdot,Z\rangle^{\sharp_g} \\ &= (c+\mu\rho)\langle x,x\rangle z \\ &= (c+\mu\rho)\langle x,x\rangle\Big(Z-\frac{1}{2}\langle \mathbf{s}-\mathbf{n},Z\rangle(\mathbf{s}+\mathbf{n})\Big). \end{aligned}$$

In adapted quasi-orthonormal basis the matrix of $J_R(x)$ has the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ \vdots & (c+\mu\rho)\langle x,x\rangle I_{n-1} & \\ 0 & & \end{pmatrix}.$$

It follows that the characteristic polynomial f_x of $J_R(x)$ is given by

$$f_x(r) = -r\left[(c+\mu\rho)\langle x,x\rangle - r\right]^{n-1}, \quad \langle x,x\rangle = 1.$$

Hence, M is pointwise $\mathcal{F}-$Osserman, which completes the proof. □

Theorem 12. *Let $F = ((M_u),(g_u),(N_u))$ be a family of lightlike hypersurfaces of a Lorentzian manifold $(\bar{M},\bar{g})$ defined by the metric (2.5). Assume that the lapse function satisfies $d\lambda = \langle \mathbf{n}, \overline{\nabla}.\mathbf{s}\rangle$ and the symmetry*

$$£_{\mathbf{n}}\overline{g} = £_{\mathbf{s}}\overline{g} \tag{4.6}$$

holds when restricted to the intersecting data. Then the induced Riemann curvature tensor of each member (M,g,ξ,N) of F has the required symmetries if and only if the local second fundamental form of each member is a Codazzi tensor. Moreover, each member of the family is $\mathcal{F}-$Osserman if and only if the non-degenerate intersecting data $\mathcal{S}_{t,u}$ are Osserman.

Proof. Recall from Proposition 6 that the positive definite induced metrics on S_t by both $\bar{g}$ and (the degenerate metric) g coincide on each $T_p\mathcal{S}_t$, $(p \in S_t)$. We first show that each member $M = M_{u_0}$ of the family is geodesicaly foliated by the (S_t). Let C denote the local second fundamental form of the integrable distribution S_t. By direct computation using (3.2)-(3.4), we have

$$\Big(£_N\overline{g}\Big)(X,Y) = -2C(X, Y-\eta(Y)\xi) + \Big[\tau(X)\eta(Y) + \tau(Y)\eta(X)\Big], \tag{4.7}$$

for all X and Y tangent to M. In particular, as η vanishes on $\Gamma(TS_t)$,

$$\Big(£_N\overline{g}\Big)(X,Y) = -2C(X,Y), \quad \forall X,Y \in \Gamma(TS). \tag{4.8}$$

On the other hand, we compute the left hand side of (4.7) for the normalization $N = \frac{1}{2\lambda}(\mathbf{s}-\mathbf{n})$. We have

$$\Big(£_N\overline{g}\Big) = -\frac{1}{2\lambda^2}\overline{g}[\mathbf{s}-\mathbf{n}\ ,\ d\lambda\otimes id + id\otimes d\lambda] + \frac{1}{2\lambda}\Big(£_{\mathbf{n}}\overline{g} - £_{\mathbf{s}}\overline{g}\Big). \tag{4.9}$$

So, combining (4.8) and (4.9) leads to

$$\Big(£_N\overline{g}\Big)\Big|_{TS} = \frac{1}{2\lambda}\Big(£_{\mathbf{n}}\overline{g} - £_{\mathbf{s}}\overline{g}\Big) = -2C(\cdot,\cdot). \tag{4.10}$$

It follows from above that for a fixed $u = u_0$, the foliation $S_t = M_{u_0} \cap H_t$ is totally geodesic in M. In particular, the Gauss-Codazzi equations reduce to

$$\langle R(X,Y)Z,W\rangle = \Big\langle \overline{R}(X,Y)Z, W - \frac{1}{2}\langle \mathbf{s}-\mathbf{n},W\rangle(\mathbf{s}+\mathbf{n})\Big\rangle.$$

for X,Y,Z and W tangent to M_{u_0} and we have

$$\langle R(Z,W)X,Y\rangle = \langle R(X,Y)Z,W\rangle + \eta(W)\langle\overline{R}(X,Y)Z,\xi\rangle - \eta(Y)\langle\overline{R}(X,\xi)Z,W\rangle.$$

Therefore, as the induced Riemann curvature tensor always satisfies the first Bianchi identity, the only obstruction for R to satisfy the remaining required algebraic symmetries is that $\overline{R}(X,Y)\xi$ be proportional to ξ. Note that the connection $1-$form is given by

$$\tau = -\frac{1}{\lambda}\Big[d\lambda + \langle s, \overline{\nabla}.\mathbf{n}\rangle\Big] = -\frac{1}{\lambda}\Big[d\lambda - \langle \mathbf{n}, \overline{\nabla}.\mathbf{s}\rangle\Big].$$

Hence the hypothesis that the given lapse function satisfies $d\lambda = \langle \mathbf{n}, \overline{\nabla}.\mathbf{s}\rangle$ means that the connection 1−form τ vanishes identically. But by Gauss-Codazzi equations, we also have

$$\bar{g}(\overline{R}(X,Y)\xi, Z) = (\nabla_Y B)(X,Z) - (\nabla_X B)(Y,Z)B(X,Z)\tau(Y) - B(Y,Z)\tau(X),$$

which, using $\tau = 0$ reduce to

$$\bar{g}(\overline{R}(X,Y)\xi, Z) = (\nabla_Y B)(X,Z) - (\nabla_X B)(Y,Z).$$

It follows that the induced Riemann curvature tensor satisfies the required symmetries if and only if the second fundamental form B is a Codazzi tensor.
Now, let R, R' and $\overset{\star}{R}$ denote the algebraic curvature tensor induced on $M = M_{u_0}$ by the normalization (2.7), the restriction of R on S_t and the Riemann curvature tensor given by the Levi-Civita connection $\overset{\star}{\nabla}$ on the S_t, respectively. We show that $R' = \overset{\star}{R}$ at any p in the domain of the foliation. Let x, y, z be tangent to S_t. By straightforward calculation using structure equations of Gauss and Weingarten, we have

$$\begin{aligned} R'(x,y)z = R(x,y)z &= \overset{\star}{R}(x,y)z + \Big[C(x,z)\overset{\star}{A}_\xi\, y - C(y,z)\overset{\star}{A}_\xi\, x\Big] \\ &\quad + \Big[(\nabla_x C)(y,z) - (\nabla_y C)(x,z) + \tau(y)C(x,z) \\ &\quad - \tau(x)C(y,z)\Big]\xi. \end{aligned}$$

Thus, we get $R'(x,y)z = \overset{\star}{R}(x,y)z$ from $\pounds_{\mathbf{n}}\bar{g} - \pounds_{\mathbf{s}}\bar{g} = 0$ using (4.10). But for a given $x \in T_pM$ with $p \in S_t$, $x \in \mathcal{S}_p(M)$ if and only if $\overset{\star}{x} \in \mathcal{S}_p(S_t)$, with $\overset{\star}{x} = x - \frac{1}{2}\langle \mathbf{s} - \mathbf{n}, x\rangle(\mathbf{s} + \mathbf{n})$. Also, $x^\perp = (\overset{\star}{x})^\perp$ and $J_R(x) = J_R(\overset{\star}{x})$. It follows that $J_{\overset{\star}{R}}(\overset{\star}{x})$ is the restriction of $J_R(x)$ to $\overset{\star}{x}^{\perp_{T_pS_t}}$ where $\perp_{T_pS_t}$ denotes the orthogonality symbol restricted to T_pS_t. Hence, $x^\perp = \overset{\star}{x}^{\perp_{T_pS_t}} \oplus_{Orth} Span(\xi)$. Therefore, as from (4.2) we have $J_R(x)\xi = 0$, if we let $f_x(t)$ and $h_{\overset{\star}{x}}(t)$ denote the characteristic polynomials of $J_R(x)$ ($x \in \mathcal{S}^-_p(M)$) and $J_{\overset{\star}{R}}(\overset{\star}{x})$ ($\overset{\star}{x} \in \mathcal{S}^-_p(S_t)$), we have $f_x(t) = t\, h_{\overset{\star}{x}}(t)$. This equality shows that the characteristic polynomial of $J_R(x)$ is independent of $x \in \mathcal{S}^-_p(M)$ (resp. $x \in \mathcal{S}^+_p(M)$) if and only if the characteristic polynomial of $J_{\overset{\star}{R}}(\overset{\star}{x})$ is independent of $\overset{\star}{x} \in \mathcal{S}^-_p(S_t)$ (resp. $\overset{\star}{x} \in \mathcal{S}^+_p(S_t)$). Hence, M_{u_0} is (timelike) $\mathcal{F}$−Osserman at p if and only if the leaf S_t which is a Riemannian due to the Lorentzian signature of $(\bar{M}, \bar{g})$ is (timelike) Osserman at p. □

5. Physical Application

It is well-known that the metric symmetry (an important concept in mathematics and physics) is based on the existence of Killing or conformal Killing vector (CKV) fields (see Duggal-Sharma, 1999) and references therein). Related to this paper, we concentrate on the Killing symmetry extensively used in general relativity, in particular in reference to black hole physics and then prove a characterization theorem for totally geodesic Osserman lightlike hypersurfaces of a stationary spacetime, supported by a physical example. For this purpose we recall the following.
A spacetime admitting a timelike Killing vector field is called stationary. For example, see a paper by (Makoto-Kei-ichi, 2007) in which they found a way to construct the black brane solutions from intersecting M-branes via torus compactification. They proved that five dimensional black brane solutions with asymptotically flatness and regularity at rotating axis are stationary spacetimes.
Also we know (Gromoll-Grove, 1985) that in any space of constant curvature, line fields with bundle-like metric (i.e. Riemannian flows) are always flat (i.e the transversal distribution to the foliation defines a totally geodesic foliation) or homogeneous (i.e. the orbit foliation of a group of isometries). Moreover, time-independent gravitational fields play an important role in General Relativity (Schwarzschild solutions (exterior and interior) and the Kerr metric of a rotating black hole are common examples of stationary space-times. All these exact solutions share two properties: namely asymptotic flatness and time-independence which implies the existence of a timelike Killing vector field. Furthermore, Killing symmetry of totally geodesic lightlike hypersurfaces has been widely studied in the literature in connection with the isolated and event black hole horizons (see Gourgoulhon-Jaramillo, 2006, Hawking, 1972). For a physical application, we need the following stationary coordinate systems of our landing spacetime.

Stationary coordinate systems. Let $(M, g, \xi, \mathbf{N})$ be a member of the family $((M_u), (g_u))$ of lightlike hypersurfaces of a spacetime $(\bar{M}, \bar{g})$ whose metric $\bar{g}$ is given by (2.5). With respect to each $\mathbf{S}_t$, the shift vector U can be expressed as

$$U = \alpha\mathbf{s} - V, \quad \alpha = \mathbf{s}\cdot U, \quad V \in T_p(\mathbf{S}_t). \tag{5.1}$$

Using (2.6) and $\mathbf{t} = \lambda\mathbf{n} + U$ in above we obtain

$$\xi = \mathbf{t} + V + (\lambda - \alpha)\mathbf{s}.$$

We say that a coordinate system $(x^i) = (t, x^a)$ of spacetime $\bar{M}$ is stationary with respect to M if and only if M in this coordinate system involves only the spacial coordinates (x^a) and does not depend on t. This means that the location of the n-dimensional submanifold $\mathbf{S}_t$ is fixed with respect to the coordinate system (x^a) on H_t as t varies. In the sequel, we denote a fixed $\mathbf{S}_t$ by $\mathbf{S}$. For such a stationary coordinate system it is known (Gourgoulhon-Jaramillo, 2006) that

$$\begin{aligned} \alpha &\overset{M}{=} \lambda \quad \text{is non-zero constant on} \quad M, \\ \xi &\overset{M}{=} \mathbf{t} + V. \end{aligned}$$

With above data, consider a coordinate system $(x^A) = (t, x^2, \ldots, x^n)$ on (M, g) defined by $\{x^1 = \text{constant}\}$. Then, the degenerate metric g is

$$ds^2|_M = g_{AB} dx^A dx^B = g_{tt} dt^2 + 2 g_{tk} dt dx^k + g_{km} dx^k dx^m, \tag{5.2}$$

where $2 \leq k, m \leq n$ and

$$g_{tt} = V^k V_k, \quad g_{tk} = U_k = \alpha \mathbf{s}_k - V_k = -V_k,$$

such that the coordinate system (x^A) is stationary on M as above metric is time-independent. Observe that there is a freedom of choice in taking any one of the spacelike coordinates constant. Moreover,

$$\Big((x^A) \quad \text{stationary w.r.t.} \quad M\Big) \Longleftrightarrow \frac{\partial u}{\partial t} = 0 \Longleftrightarrow \mathbf{t} \in T(M).$$

A special case is when $V = 0$, called a coordinate system co-moving with M. This implies that

$$\mathbf{t} \overset{(M,g)}{\Longrightarrow} \xi \quad \text{and} \quad ds^2|_M = g_{km} dx^k dx^m. \tag{5.3}$$

Physically important case is a coordinate system adapted to M of a 4-dimensional spacetime $\bar{M}$ for which its topology is $R \times S^2$ and the coordinate system can be transformed into a spherical type (r, θ, ϕ). For details on above stationary coordinates we refer (Gourgoulhon-Jaramillo, 2006).
Using stationary coordinates we prove a characterization theorem of lightlike Osserman hypersurfaces followed by a physical example.

Theorem 13. *Let (M, g, ξ, N) be a member of the family $((M_u), (g_u), (\xi_u), (N_u))$ of lightlike hypersurfaces of a stationary spacetime $(\bar{M}, \bar{g})$ defined by the metric (2.5) such that the shift spacelike vector field U is given by $U = \lambda \mathbf{s} - V$ where λ is the lapse function and $V \in TS$ is a Killing vector field. Then, M is totally geodesic in $\bar{M}$. Moreover, M is $\mathcal{F}-$Osserman if and only if its fixed nondegenerate* **S** *is Osserman.*

Proof. Since $\bar{M}$ is stationary, the coordinate time vector $\mathbf{t} = \frac{\partial}{\partial t} = \lambda \mathbf{n} + U$ is a Killing vector field which means that $\pounds_{\mathbf{t}} \bar{g} = 0$. Then,

$$\begin{aligned} 0 &= \pounds_{\mathbf{t}} \bar{g} = \langle \mathbf{n}, d\lambda \otimes id + id \otimes d\lambda \rangle + \lambda \pounds_{\mathbf{n}} \bar{g} + \pounds_U \bar{g} \\ &= \langle \mathbf{n} + \mathbf{s}, d\lambda \otimes id + id \otimes d\lambda \rangle + \lambda \Big[\pounds_{\mathbf{n}} \bar{g} + \pounds_{\mathbf{s}} \bar{g} \Big] - \pounds_V \bar{g} \\ &= \langle \mathbf{n} + \mathbf{s}, d\lambda \otimes id + id \otimes d\lambda \rangle + \lambda \Big[\pounds_{\mathbf{n}} \bar{g} + \pounds_{\mathbf{s}} \bar{g} \Big] \ \ (\text{as } \pounds_V \bar{g} = 0\ ). \end{aligned}$$

Consider a stationary coordinate system $(x^i) = (t, x^a)$ of $\bar{M}$ with respect to M so that the location of $\mathbf{S}$ is fixed. Then, we know from above discussion that λ is a non-zero constant. This along with V Killing implies that

$$\pounds_{\mathbf{n}} \bar{g} + \pounds_{\mathbf{s}} \bar{g} = 0. \tag{5.4}$$

On the other hand from (3.2) we know that $\bar{\nabla}_X \xi = \nabla_X \xi$. Using this and Computing $\pounds_\xi g$ from $\xi = \lambda(\mathbf{n} + \mathbf{s})$ for a non-zero constant λ we obtain

$$\pounds_\xi g = \lambda (\pounds_{\mathbf{n}} \bar{g} + \pounds_{\mathbf{s}} \bar{g}). \tag{5.5}$$

So, combining (5.4) and (5.5) leads to $\pounds_\xi g = 0$. Therefore, for a fixed choice of $\mathbf{S}$ with respect to a stationary coordinate system $(x^i) = (t, x^a)$ of $\bar{M}$, it follows that M is totally geodesic in $\bar{M}$. Now, by a standard argument as in Theorem 12 one shows that M is (timelike) $\mathcal{F}-$Osserman at p if and only if the leave $\mathbf{S}$ is (timelike) Osserman at p, which completes the proof. □

Physical Model: Let $\bar{M}$ be a 4-dimensional spacetime with the metric

$$ds^2 = -(1 - \frac{2m}{r})\, dt_s^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.6}$$

which is the exterior Schwarzschild spacetime ($r > 2m$) with m and r as the mass and the radius of a spherical body, and (t_s, r, θ, ϕ) a coordinates system which is singular at $r = 2m$. Consider a new coordinate system (v, r, θ, ϕ), where the coordinate v is constant on each ingoing radial null geodesic, and is related to the Schwarzschild coordinate time t_s by $v = r + t_s + 2m \ln|\frac{r}{2m} - 1|$. The coordinate v is null, but if we introduce another time coordinate $t = v - r = t_s + 2m \ln|\frac{r}{2m} - 1|$ then the system (t, r, θ, ϕ) is called Eddington-Finkelstein coordinates and metric (5.6) transforms into

$$ds^2 = -(1 - \frac{2m}{r})\, dt^2 + \frac{4m}{r} dt\, dr + (1 + \frac{2m}{r}) dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \tag{5.7}$$

which is non-singular Eddington-Finkelstein metric for all values of r. Let $\mathbf{S}$ be an intersection of a hypersurface r = constant with another hypersurface t = constant. Then, $(\mathbf{S}, h)$ is a 2-surface of $\bar{M}$ with its metric h given by

$$ds_h^2 = r^2(d\theta^2 + \sin^2 d\phi^2).$$

In particular for $r = 2m$, as per Proposition 2 and (5.3) this metric h coincides with the degenerate metric g of the corresponding lightlike hypersurface (M, g), rewritten as

$$\mathbf{t} \overset{(M,g)}{\Longrightarrow} \xi \quad \text{and} \quad ds^2|_{_M} = 4m^2(d\theta^2 + \sin^2 d\phi^2), \tag{5.8}$$

where $\mathbf{t} \Longrightarrow \xi$ is a Killing vector associated with the stationary spacetime $\bar{M}$ and, therefore, M is its totally geodesic hypersurface. Moreover, the Eddington-Finkelstein stationary coordinates are adapted as well as comoving with respect to M and it has the topology $R \times S^2$. It is easy to show (see Gourgoulhon-Jaramillo, 2006) that for $r = 2m$ and $\mathbf{t} \Longrightarrow \xi$, the lapse function $\lambda = \alpha = \frac{1}{\sqrt{2}}$ and the components of $\mathbf{n}, \mathbf{s}$ and ξ are

$$\begin{aligned}
\mathbf{n}^a &= (\sqrt{2}, -\frac{1}{\sqrt{2}}, 0, 0), \quad \mathbf{n}_a = (-\frac{1}{\sqrt{2}}, 0, 0, 0), \\
\mathbf{s}^a &= (0, \frac{1}{\sqrt{2}}, 0, 0), \quad \mathbf{s}_a = (\frac{1}{\sqrt{2}}, \sqrt{2}, 0, 0), \\
\xi^a &= (1, 0, 0, 0), \quad \xi_a = (0, 1, 0, 0).
\end{aligned}$$

Now consider a lightlike hypersurface (M, g), defined by the degenerate metric (5.8), of the Eddington-Finkelstein stationary spacetime $(\bar{M}, \bar{g})$ with its metric (5.7). Then, proceeding as in the proof of Theorem 12, it is straightforward to show that M is $\mathcal{F}-$Osserman if and only if its 2-surface $\mathbf{S}$ is Osserman. Thus we have a physical model of Osserman lightlike hypersurfaces of the Schwarzschild spacetime.

6. Discussion

We highlight that (Carter's, 1997) approach of foliations of lightlike hypersurfaces has been very useful in the study of Osserman lightlike hypersurfaces, both for improving on the previous paper (Atindogbe-Duggal, 2009) and for producing new results. Contrary to (Duggal-Bejancu, 1996) approach, we secured a well-defined covariant derivative $\bar{\nabla}\xi$ of the null normal ξ. Theorem 11 is an important step forward in using foliation approach which provides the required algebraic symmetries of the induced curvature tensor as in the case of semi-Riemannian hypersurfaces. Moreover, the condition *" Local second fundamental form of each hypersurface is a Codazzi tensor"* in Theorem 12 is geometrically a desirable condition to get the required algebraic symmetries compared to using a restricted condition stated in Theorem 1 of the previous paper (Atindogbe-Duggal, 2009).

7. Future Prospects

We propose following two open problems.

(A) Recall the following from (Garcı́a-Rı́o et. al., 2002, pages 44-57). Let $\xi \in T_p\bar{M}$ be a null vector of an n-dimensional Lorentzian manifold $(\bar{M}, \bar{g})$ of dimension ≥ 3. Then $\xi^\perp = (span\{\xi\})^\perp$ is a degenerate vector space containing $span\{\xi\}$. Denote by $\bar{\xi}^\perp = \xi^\perp / span\{\xi\}$ the $(n-2)$-dimensional quotient space with the projection $\pi : \xi^\perp \to \tilde{\xi}^\perp$. We quote the following three definitions.

(1) Consider a linear map $\tilde{R}_\xi : \tilde{\xi}^\perp \to \tilde{\xi}^\perp$ defined by $\tilde{R}_\xi \tilde{x} = \pi\left(\bar{R}(x, \xi)\xi\right), where \quad x \in \tilde{\xi}, \quad \pi(x) = \tilde{x}$, and $\bar{R}$ is the curvature tensor on $\bar{M}$. Then, $\tilde{R}_\xi$ is called the Jacobi operator with respect to ξ.

(2) Let z of $\bar{M}$ be a timelike unit vector. The null congruence $N(z)$, determined by z at p, is defined by

$$N(z) = \{\xi \in T_p(\bar{M}) : \bar{g}(\xi, \xi) = 0 \text{ and } \bar{g}(\xi, z) = -1\}.$$

(3) $(\bar{M}, \bar{g})$ is called null Osserman with respect to a unit timelike vector $z \in T_p(\bar{M})$ if the characteristic polynomial of $\tilde{R}_\xi$ is independent of $\xi \in N(z)$. Let L be a timelike line subbundle of $T\bar{M}$. Then $(\bar{M}, \bar{g})$ is called globally null Osserman with respect to L if it is pointwise null Osserman with resepect to L and the characteristic polynomial of $\tilde{R}_\xi$ is independent of unit $z \in L$.

Theorem 14. (García-Río et. al., 2002, page 56) *Let $(\bar{M}, \bar{g})$ be a Lorentzian manifold of dimension ≥ 4. If $(\bar{M}, \bar{g})$ is globally null Osserman with resepect to a timelike line bundle L of $T\bar{M}$ and is not of constant sectional curvature, then it is locally a warped product $(I \times N, -dt^2 \oplus fg')$, whereI $\subseteq R$ is an open interval and (N, g') is a Riemannian real space form.*

Using the methodology as adapted in this paper and above Theorem 14, we propose the study on
Osserman lightlike hypersurfaces of warped product globally null Osserman Lorentzian manifolds.

(B) **General study**. There is a need to extend the results of this paper for a general study on Osserman lightlike hypersurfaces of an ambient semi-Riemannian manifold $(\bar{M}, \bar{g})$. It is quite a challenging open problem to define a well-defined projector mapping $II : T_p\bar{M} \rightarrow T_pM$ for an arbitrary $\bar{M}$, which is needed to get induced extrinsic objects of M and a theorem similar to Theorem 3 of this paper. Initially, one may try to work by choosing a prescribed semi-Riemannian manifold.

Acknowledgements

Sincere thanks to the editor and two referees for their constructive suggestions towards the improvement of the final version of this paper.

References

Akivis, M. A., & Goldberg, V. V. (2000). On some methods of construction of invariant normalizations of lightlike hypersurfaces, *Differential Geometry and its Applications*, *12*(2), 121–143.

Atindogbe, C., & Duggal, K. L. (2009). Pseudo-Jacobi operators and Osserman lightlike hypersurfaces, *Kodai Math. J., 32*(1), 91–108.

Atindogbe, C., Ezin, J.-P., & Tossa, J.(2003). Pseudo-inversion of degenerate metrics, *Int. J. Math. Math. Sci., 55*, 3479–3501.

Atindogbe, C., Lugiambudila, O., & Tossa, J. (2011). Lightlike Osserman submanifolds of semi-Riemannian manifolds, *Afr. Math., 22*, 129–151.

Brunette, L. (2014). Osserman lightlike hypersurfaces of indefinite S-manifolds, *Turkish J. of Math., 34*, 240–352.

Carter, B. (1997). Extended tensorial curvature analysis for embeddings and foliations, *Contempt. Math.*, *203*, 207–210.

Chi, Q. S. (1988). A characterization of certain locally rank-one symmetric spaces. *J. Diff. Geom., 28,* 187-202.

Duggal, K. L. (2008). Time-dependent black hole horizons on spacetime solutions of Einstein's equations with initial data, M. Plaue and M. Scherfner (Eds., *Advances in Lorentzian geometry, Aachen: Shaker Verlag, Berlin, Germany,* 51–61, ISBN 978-3-8322-7786-4.

Duggal, K. L. (2012). Foliations of lightlike hypersurfaces and their physical interpretation,*Cent. Eur. J. Math., 10*(5), 1789–1800.

Duggal, K. L. (2014). Time-dependent Evolving Null Horizons of a Dynamical Spacetime, *ISRN Mathematical Physics, 2014*, Article ID 10291790, 10 pages.

Duggal, K. L. (2015). Maximum principle for totally umbilical null hypersurfaces and time-dependent null horizons, *Journal of Mathematics Research*, *7*(1),, ISSN 1916-9795 E-ISSN 1916-9809, 88–95. http://dx.doi.org/10.5539/jmr.v7n1p88.

Duggal, K. L., & Bejancu, A. (1996) Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. *Kluwer Academic,* 364.

Duggal, K. L., & Jin, D. H. (2007). Null Curves and Hypersurfaces of Semi-Riemannian Manifolds. *World Scientific,* 293.

Duggal, K. L., & Sharma, R. (1999). *Symmetries of Spacetimes and Riemannian Manifolds*, Kluwer Academic Publishers, 487.

Garcı́a-Rı́o, E., Kupeli, D. N., & Váquez-Lorenzo, R. (2002). Osserman Manifolds in Semi-Riemannian Geometry, *Lecture Notes in Mathematics,* 1777, Springer-Verlag, Berlin.

Gilkey, P. (2001). *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor,* World scientific Press.

Gromoll, D., & Grove, K. (1985). One-dimensional metric foliations in constant curvature spaces. *Differential Geometry and Complex Analysis,* SpringerCVerlag, New York, 165C-168.

Gourgoulhon, E., & Jaramillo, J. L. (2006). A $(1+3)$-perspective on null hypersurfaces and isolated horizons. *Phys. Rep.*, *423*(4-5), 159–294.

Hawking, S. W. (1972). *The event horizons, in Black Holes,* edt. C. DeWitt and B. DeWitt, Norht Holland, Amsterdam.

Leistner, T. (2006). Screen bundles of Lorentzian manifolds and some generalizations of pp-waves. *J. Geom. Phys.*, *56*(10), 2117–2134.

Makoto, T., & Kei-ichi M. (2007). Stationary Spacetime from Intersecting M-branes. *J. Phys. A, 40*, 7025–7030. http://dx.doi.org/10.1088/1751-8113/40/25/S56.

Copyrights