Journal of Mathematics Research; Vol. 8, No. 1; 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Picard Approximation Method for Solving Nonlinear Quadratic Volterra Integral Equations

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Received: December 5, 2015 Accepted: January 20, 2016 Online Published: January 25, 2016

doi:10.5539/jmr.v8n1p79 URL: http://dx.doi.org/10.5539/jmr.v8n1p79

Abstract

In this paper, we use the Picard method for solving nonlinear quadratic Volterra integral equations by using approach of the self-canceling noise terms which is proposed by Wazwaz (Wazwaz, 2013) . The analytical solutions show that only two iterations are needed to obtain accurate approximate solutions.To illustrate the ability and reliability of the method, some examples are given, revealing its effectiveness and simplicity.

Keywords: integral equations, Volterra equations, Picard method

1. Introduction

Volterra integral equations have many applications in many areas and branches, for instance, mathematical physics, chemistry, electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reaction and population dynamics (Teriele, 1982) and (Lamm et al, 1997).

Hence, different methods are suggested for solving the Volterra integral equation. The Adomian decomposition, (Bougoffa et al, 2011), (Pandey et al, 2009) and Homotopy perturbation (He, 1999) were proposed for obtaining the approximate analytic solution of the integral equation. Huang et al (2008) used the Taylor expansion of unknown function and obtained an approximate solution. Yang (2012) proposed a method for the solution of integral equation using the Chebyshev polynomials, Khodabin et al (2013) solved the stochastic Volterra integral equations by triangular functions and their operational matrix of integration. Yousefi (2006) presented a numerical method for the Abel integral equation by Legendre wavelets. Kamyad (2010) constructed a new method based on the calculus of variations and discretisation method, Yang et (2013) applied Laplace transform and Taylor series to solve the Volterra integral equation with a convolution kernel.

Consider the following non linear Volterra Quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s))ds, \tag{1}$$

where a and g are known functions. The function $f(s, x(s))$ is nonlinear in the unknown function x. The existence of continuous solution of (1) was proved in (El-Sayed et al, 2008).

In (El-Sayed et al, 2010), the classical method of successive approximations (Picard method) and the Adomian decomposition method were used for solving the nonlinear Volterra Quadratic integral equation of the form in (1), the result showed that Picard method gives more accurate solution than ADM. On the other hand, Wazwaz (2013) used a systematic modified Adomian decomposition method (ADM) and the phenomenon of the self-canceling -noise-terms for solving nonlinear weakly-singular Volterra, Fredholm, and Volterra-Fredholm integral equations, he show that the proposed approach minimizes the computation. In this work, we will use the same approach in (Wazwaz, 2013) with Picard method for solving Quadratic Volterra integral equations of the form in (1). The analytical solutions for some examples show that only two iterations are needed to obtain accurate approximate solutions. The elegance of this method can be attributed to its simplistic approach in seeking the exact solution of the problem.

2. Modified Method of Successive Approximations (Picard Method)

Applying Picard method (Curtain, 1977) to the quadratic integral equation (1), the solution is constructed by the sequence

$$\begin{aligned} &x_n(t) = a(t) + g(t, x_{n-1}(t)) \int_0^t f(s, x_{n-1}(s))ds, n = 1, 2, 3, ... \\ &x_0(t) = a(t). \end{aligned} \tag{2}$$

All the functions $x_n(t)$ are continuous functions and the solution will be,

$$x(t) = \lim_{n\to\infty} x_n(t).$$

In this work, we decomposes the function $a(x)$ into two components $f_0(t)$ and $f_1(t)$, where the first part is assigned to the zeroth solution, and the second is added to the first solution $x_1(t)$. ,i.e

$$\begin{aligned} &x_0(t) = f_0(x),\\ &x_1(t) = f_1(t) + g(t, x_0(t)) \int_0^t f(s, x_0(s))ds,\\ &x_{n+1}(t) = g(t, x_n(t)) \int_0^t f(s, x_n(s))ds, n \geq 1. \end{aligned} \tag{3}$$

Notice that if $a(x)$ consists of one term, the modified Picard method cannot be used.

3. Numerical Examples

In this section, we will study some numerical examples by applying Picard and modified Picard method. We begin by using Picard method first, then the modified of Picard method to show how is the new approach give an easily and fast convergence to the exact solution with minimum time and computation cost.

Example 1.

Consider the following nonlinear Volterra equation,

$$x(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5}x(t) \int_0^t s^2 x^2(s)ds, \tag{4}$$

with Exact solution $x(t) = t^2$. If we apply the classical Picard method we get

$$\begin{aligned} &x_n(t) = \left(t^2 - \tfrac{t^{10}}{35}\right) + \tfrac{t}{5}x_{n-1}(t) \int_0^t s^2 {x_{n-1}}^2(s)ds, n = 1, 2, ...\\ &x_0(t) = \left(t^2 - \tfrac{t^{10}}{35}\right)\\ &x_1(t) = \left(t^2 - \tfrac{t^{10}}{35}\right) + \tfrac{t}{5}x_0(t) \int_0^t s^2 {x_0}^2(s)ds\\ &= t^2 - \tfrac{t^{10}}{35} + \tfrac{t^{12}}{35} - \tfrac{44t^{20}}{18375} + \tfrac{1094t^{28}}{14791875} - \tfrac{76t^{36}}{73959375} + \tfrac{t^{44}}{172571875}. \end{aligned}$$

The solution will be,

$$x(t) = \lim_{n\to\infty} x_n(t).$$

Conversely, for solving Example 1 by using the new approach, we first decompose $a(x) = \left(t^2 - \frac{t^{10}}{35}\right)$ into two parts defined as $f_0(x) = t^2, f_1(x) = -\frac{t^{10}}{35}$.

Now, applying Picard method we get the following result

$$\begin{aligned} &x_0(t) = f_0\\ &x_1(t) = f_1 + \tfrac{t}{5}x_0(t) \int_0^t s^2 {x_0}^2(s)ds\\ &= 0. \end{aligned} \tag{5}$$

Therefore, $x_{n+1}(t) = 0, n \geq 0$ and consequently $x(t) = \sum_{n=0}^{\infty} x_n(t) = t^2$, which is the exact solution. As we mentioned before, we get the Exact solution for the integral equation just after calculate $x_1(t)$.

Example 2.

Consider another example of nonlinear Volterra equation as below,

$$x(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}x^2(t) \int_0^t (s+1)x^3(s)\mathrm{d}s, \tag{6}$$

with Exact solution $x(t) = t^3$. First, we Apply Picard method, we get

$$x_n(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}x^2{}_{n-1}(t)\int_0^t (s+1)x_{n-1}{}^3(s)\mathrm{d}s, n = 1, 2, ...,$$
$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)$$
$$x_1(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}x^2{}_0(t)\int_0^t (s+1)x_0{}^3(s)\mathrm{d}s$$
$$= t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} + \frac{1}{10}t^3(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110})^2$$
$$(\frac{t^{10}}{10} + \frac{t^{11}}{11} - \frac{3t^{26}}{2600} - \frac{7t^{27}}{3300} - \frac{3t^{28}}{3080} + \frac{t^{42}}{140000} + \frac{93t^{43}}{4730000} + \frac{3t^{44}}{166375} + \frac{t^{45}}{181500} - \frac{t^{58}}{58000000} - \frac{41t^{59}}{649000000}$$
$$-\frac{21t^{60}}{242000000} - \frac{43t^{61}}{811910000} - \frac{t^{62}}{82522000})$$

and the solution will be,

$$x(t) = \lim_{n\to\infty} x_n(t).$$

Again, for solving Example 2 by the new approach as in Example 1, We decompose
$a(x) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)$ into two parts defined as $f_0(x) = t^3, f_1(x) = -\frac{t^{19}}{100} - \frac{t^{20}}{110}$. After that we apply Picard method to get

$$x_0 = f_0,$$
$$x_1 = f_1 + \frac{t^3}{10}(x_0)^2\int_0^t (s+1)(x_0(s))^3\mathrm{d}s = 0.$$

Therefore, $x_{n+1}(t) = 0, n \geq 0$ and consequently $x(t) = \sum_{n=0}^{\infty} x_n(t) = t^3$, which is the exact solution.

Example 3.

Consider the following nonlinear Volterra equation of second kind,

$$x(t) = 1 + \sin(t) - \cos(t) - \int_0^t x(s)ds, 0 \leq t \leq 2, \tag{7}$$

with Exact solution $x(t) = \sin(t)$. Applying Picard method to equation (7), we get

$$x_n(t) = 1 + \sin(t) - \cos(t) - \int_0^t x_n(s)ds,$$
$$x_0(t) = 1 + \sin(t) - \cos(t),$$
$$x_1(t) = -t + 2\sin(t),$$
$$x_2(t) = -1 + t^2 + cos(t) + sin(t),$$
$$\vdots$$
$$x(t) = \lim_{n\to\infty} x_n(t).$$

On the other hand, we apply the new approach and again we decompose
$a(x) = 1 + \sin(t) - \cos(t)$ into two parts defined as $f_0(x) = \sin(t), f_1(x) = 1 - \cos(t)$.
Now, by applying Picard method we get

$$x_0 = f_0,$$
$$x_1 = f_1 - \int_0^t x_0(s)ds, = 0.$$

Therefore, $x_{n+1}(t) = 0, n \geq 0$ and consequently $x(t) = \sum_{n=0}^{\infty} x_n(t) = \sin(t)$, which is the exact solution for the equation. So, we get the exact solution after few steps.

4. Conclusion

In this paper, we have concerned ourselves with the determination of the exact closed form solutions of several nonlinear Volterra equations, the cost of the calculations was minimized, which validates the efficiency and reliability of the proposed technique.

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