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A Priori and A Posteriori Error Estimates for a Crank Nicolson Type Scheme of an Elliptic Problem with Dynamical Boundary Conditions

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Abstract

In this article we claim that we are going to give a priori and a posteriori error estimates for a Crank Nicolson type scheme. The problem is discretized by the finite elements in space. The main result of this paper consists in establishing two types of error indicators, the first one linked to the time discretization and the second one to the space discretization.

Keywords: A priori and a posteriori error estimates, Crank Nicolson type scheme, finite elements

1. Introduction

Let Ω be a bounded smooth sub domain of $\mathbb{R}^{\nvDash}$ and $\gamma(x) = \left[\gamma_{i,j}\right]_{i,j=1}^{n}$ be a real positive definite matrix-valued function. Let $(0, T)$ denote a subinterval of R where $T \in (0, \infty)$ is a fixed final time. Denote by $n(x)$ the unit outward normal vector at $x \in \Gamma$. We intend to work with the following problem,

$$\begin{cases} div(\gamma \nabla u) = 0, \text{in}\,(0, T) \times \Omega \\ \frac{\partial u}{\partial t}(t, x) + \gamma n(x).\nabla u(t, x) = 0, \ \text{on}\,(0, T) \times \Gamma \\ u(0, x) = u_0(x), \ \text{on}\ \Gamma \end{cases} \tag{1}$$

where Γ is the boundary, u is the unknown and u_0 is the initial condition at time $t = 0$.
The solution of the above problem can be represented on the boundary by the Dirichlet-to-Neumann semigroup (Vrabie, 2003) defined as

$$(S(t)f)(x) = u(t, x)|_{\Gamma}$$

In (Cherif, Arwadi, Emmamirad & Sac Epee, 2014), the authors showed that the Lax semigroup is the Dirichlet-to-Neumann semigroup in the particular case where $\Omega = B(0, 1)$ is the unit ball of $\mathbb{R}^{\nvDash}$ and $\gamma(x)$ is the identity matrix. P. Lax showed in his book (Lax, 2002) that the DtN semigroup has an explicit representation. This was a motivation for the authors in (Cherif, Arwadi, Emmamirad & Sac Epee, 2014) and (Emmamirad & Shariftabbar, 2013) to introduce semi discrete implicit and explicit Euler's schemes to approximate the DtN semigroup numerically. They also showed the convergence of these schemes using the Chernoff's product formula.

For more than twenty years, an impressive amount of work has been accomplished concerning a posteriori analysis and mesh adaptivity for the finite element discretization of the elliptic problems. Their main results were to exhibit local error indicators which can be computed explicitly as a function of the discrete solution and the data.

In (Arwadi, Dib & Sayah, 2015), they studied the time dependent linear elliptic problem, and established optimal a priori and a posteriori error estimates using the backward Euler's scheme in time and finite elements in space.

The Crank Nicolson scheme is one of the most popular time-stepping method; however optimal a priori and a posteriori error estimates for elliptic equations have not yet been derived. The aim of this work is to provide optimal a priori and a posteriori estimates and some numerical investigations.

The term "a posteriori error estimator" was first used by Ostrowski (Ostrowski, 1940). It is the quantity which bounds or approximates the error, i.e. an upper bound of the error between an exact solution and a numerical one.

The error estimator is obtained as a sum of local indicators expressed on each element of the mesh (Mishra, 2012). We have two types of computable error indicators, the first being linked to the time discretization and the second to the space discretization.

We say that the a posteriori error estimates are optimal if we are able to bound each one of this indicators by the local error of the solution around the corresponding element. In this work, we propose a low cost discretization relying on the Crank Nicolson's scheme in time combined with the finite elements in space, and then prove a priori and a posteriori error estimates for the discrete problem.

The outline of the paper is as follows. In section 2, we give some notations that will be used in the sequel. Section 3 is devoted to study the discrete problem and the uniqueness of its solution. In section 4, we study the a priori errors and derive optimal estimates. Section 5 is devoted to study the a posteriori errors where two types of error indicators are established.

2. Notations

In this section we will introduce some notations that will be used in the sequel.

- h the maximal diameter of the elements of all τ_{nh}
- h_n the maximal diameter of the elements of τ_{nh} for each n
- h_κ the diameter of κ
- h_e the diameter of the edge e
- Δ_κ the union of elements of τ_{nh} that intersect κ
- Δ_e the union of elements of τ_{nh} that intersect the edge e
- ϵ_κ the set of edges of κ that are not on Γ
- ϵ_κ^m the set of edges of κ that are on Γ
- $[.]_e$ the jump through e for each edge e in ϵ_κ
- ψ_κ the bubble function which is equal to the product of the three barycentric coordinates associated with the vertices of κ
- L_e the lifting operator defined on polynomials on e vanishing on ∂e
- X_{nh} the finite dimensional space of functions such that their restrictions to any element κ of τ_{nh} belong to a space of polynomials of degree one. In other words,

$$X_{nh} = \left\{ v_n^h \in C^0(\bar{\Omega}), v_n^h|_\kappa is affine \forall \kappa \in T_{nh} \right\}$$

- I_h the approximation operator in $L(H^2(\Omega); X_{nh})$ such that for $m = 0, 1$,

$$\forall v \in H^2(\Omega), |I_h(v) - v|_{m,\Omega} \leq Ch^{2-m} |v|_{2,\Omega}$$

- We introduce the Sobolev spaces:

$$H^m(\Omega) = \left\{ v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), \forall |\alpha| \leq m \right\},$$

  equipped with the following semi-norm and norm:

$$|v|_{m,\Omega} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(x)|^2 \, dx \right\}^{\frac{1}{2}}$$

  and

$$||v||_{m,\Omega} = \left\{ \sum_{k \leq m} |v|_{k,\Omega}^2 \right\}^{\frac{1}{2}}$$

3. The Discrete Problem

Assume that Ω is a polyhedron and γ denotes a positive smooth bounded function. We introduce a partition of the interval $[0, T]$ into sub intervals $[t_{n-1}, t_n]$, $1 \le n \le N$, such that $0 = t_0 \le t_1 \le ... \le t_N = T$. Denote by τ_n the length of $[t_{n-1}, t_n]$, by $|\tau|$ the maximum of the τ_n, by τ the N-tuple $(\tau_1, ..., \tau_N)$, and by σ_τ the regularity parameter

$$\sigma_\tau = max_{2\le n\le N} \frac{\tau_n}{\tau_{n-1}}.$$

Theorem 1 *If $u(t) \in H^2(\Omega)$, then Problem (1) is equivalent to the variational problem,*

$$\begin{cases} \text{Find } u(t) \in H^1(\Omega), \text{such that} \\ u(0, x) = u_0(x), \text{ on } \Gamma \\ \int_\Omega \gamma \nabla u \nabla v dx + \int_\Gamma \frac{\partial u}{\partial t}(t, s)v(t, s)ds = 0, \forall v(t) \in H^1(\Omega) \end{cases} \tag{2}$$

Proof. Let $u(t)$ be a solution of problem (1). Multiplying the first equation of problem (1) by $v(t) \in H^1(\Omega)$, integrating over Ω, applying Green's formula and using the second equation of problem (1), we obtain that u is also a solution of problem (2). Conversely, if u is a solution of problem (2), we take $v(t) \in D(\Omega)$ to get the first line of problem (1). Then multiplying the first equation of problem (1) by $v(t) \in H^1(\Omega)$, integrating over Ω, using the Green's formula and comparing with problem (2), we get the second line of problem (1).

Proposition 1 *The solution of Problem (2) satisfies the following bound:*

$$||u||^2_{L^\infty(0,T,L^2(\Gamma))} \le ||u_0||_{L^2(\Gamma)}$$

Now, the full discrete problem associated to the variational problem (2) is:

$$\begin{cases} \text{Given } u_h^n \in X_{nh}, \\ \forall v_h(t) \in X_{nh}, u_h^n(t) \text{ is the solution of} \\ \int_\Omega \gamma \nabla u_h^{n+1}(x) \nabla v_h(t, x)dx + 2\int_\Gamma \frac{u_h^{n+1}-u_h^n}{\tau_n}(x)v_h(t, x)dx - \int_\Omega \gamma \nabla u_h^n \nabla v_h dx = 0 \end{cases} \tag{3}$$

Theorem 2 *The problem (3) admits a unique solution in X_{nh}.*

Proof. We introduce the bilinear form ,

$$a(u_h^{n+1}, v_h) = \int_\Omega \tau_n \gamma \nabla u_h^{n+1} \nabla v_h dx + 2\int_\Gamma u_h^{n+1} v_h d\sigma$$

and the linear form

$$L(v_h) = \int_\Omega \tau_n \gamma \nabla u_h^n \nabla v_h dx + 2\int_\Gamma u_h^n v_h d\sigma$$

Then the previous problem can be written as

$$\forall v_h \in X_{nh}, a(u_h^{n+1}, v_h) = L(v_h)$$

It is obvious that a is bilinear and continuous in $X_{n+1,h} \times X_{n+1,h}$, and that L is linear and continuous in X_{nh} and then, the Lax-Milgram theorem states the existence and the uniqueness of the solution. See (Arwadi, Dib & Sayah, 2015).

4. A Priori Error Estimate

To get an a priori error estimate, we need the following Gronwall's lemma.

Lemma 1 *Gronwall's lemma:*

Let $(a_n)_n \ge 0$, $(b_n)_n \ge 0$ and $(c_n)_n \ge 0$ be three real positive sequences such that $(c_n)_n \ge 0$ is an increasing sequence. Suppose that

$$a_0 + b_0 \le c_0$$

there exists $\lambda > 0$ such that:

$$\forall n \ge 0, a_n + b_n \le c_n + \lambda \sum_{m=0}^{n-1} a_m$$

then we have

$$\forall n \geq 0, a_n + b_n \leq c_n e^{n\lambda}$$

Theorem 3 *If $u \in L^\infty$ we have,*

$$\left\|u(t_{m+1}) - u_h^{m+1}\right\|_{0,\Gamma}^2 + k\left|C_\gamma\right| \sum_{n=0}^{m} \left|u(t_{n+1}) - u_h^{n+1}\right|_{1,\Omega}^2 \leq c(h^2 + k^2)$$

where c is a constant independent of h and k.

Proof. Denote by k the time step, h the parameter of the mesh and X_h the discrete space. Suppose that τ_n and τ_{nh} are constants during time iterations. Consider the equation,

$$\int_\Omega \nabla u(t,x)\nabla v(t,x)dx + 2\int_\Gamma \frac{\partial u}{\partial t}(t,s)v(t,s)ds = 0, \forall v(t) \in H^1(\Omega)$$

For $t \in (t_n, t_{n+1})$ take $v = v_h^{n+1}$, integrate in time

$$\int_{t_n}^{t_{n+1}} \int_\Omega \nabla u(t,x)\nabla v_h^{n+1}(t,x)dx + 2\int_{t_n}^{t_{n+1}} \int_\Gamma \frac{\partial u}{\partial t}(t,s)v_h^{n+1}(t,s)ds = 0 \tag{4}$$

The discrete variation formulation for the Crank Nicolson scheme taken in the time step $n+1$, is

$$\int_\Omega \gamma\nabla u_h^{n+1}\nabla v_h^{n+1}dx + 2\int_\Gamma \frac{u_h^{n+1} - u_h^n}{\tau_n} v_h^{n+1}d\sigma - \int_\Omega \gamma\nabla u_h^n \nabla v_h^{n+1}dx = 0$$

Integrating in time between t_n and t_{n+1} we get,

$$\int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla u_h^{n+1}\nabla v_h^{n+1}dx + 2\int_{t_n}^{t_{n+1}} \int_\Gamma \frac{u_h^{n+1} - u_h^n}{\tau_n} v_h^{n+1}d\sigma - \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla u_h^n \nabla v_h^{n+1}dx = 0 \tag{5}$$

Taking the difference between (4) and (5) we get,

$$\int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(u - u_h^{n+1} + u_h^n)\nabla v_h^{n+1}dx + 2\int_\Gamma [(u(t_{n+1}) - u(t_n)) - (u_h^{n+1} - u_h^n)]v_h^{n+1}d\sigma = 0$$

Now inserting $\pm\nabla(I_h(u(t_{n+1})))$,$\nabla(I_h(u(t_n)))$,$\nabla(u(t_{n+1}))$ and $\nabla(u(t_n))$ into the first term, and $\pm I_h(u(t_{n+1}))$ and $I_h(u(t_n))$ into the second term, we obtain

$$\begin{aligned}
&-\int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(u(t_{n+1}) - u(t_n))\nabla v_h^{n+1}dxdt - \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(I_h(u(t_{n+1})) - u(t_{n+1}))\nabla v_h^{n+1}dxdt \\
&+ \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(I_h(u(t_{n+1})) - u_h^{n+1})\nabla v_h^{n+1}dxdt - \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(I_h(u(t_n)) - u_h^n)\nabla v_h^{n+1}dxdt \\
&+ \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(I_h(u(t_n)) - u(t_n))\nabla v_h^{n+1}dxdt + \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(u(t_n))\nabla v_h^{n+1}dxdt \\
&+2\int_\Gamma (a_{n+1} - a_n)(s)v_h^{n+1}ds - 2\int_\Gamma (I_h(u(t_{n+1})) - u(t_{n+1})) - (I_h(u(t_n)) - u(t_n))v_h^{n+1}ds = 0
\end{aligned}$$

where $a_{n+1} = I_h(u(t_{n+1})) - u_h^{n+1}$ and $a_n = I_h(u(t_n)) - u_h^n$.

Now we will bound the third and fourth terms of the previous equation. Choosing $v_h^{n+1} = a_{n+1}$

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla(I_h(u(t_{n+1})) - u_h^{n+1})\nabla v_h^{n+1}dxdt &= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla a_{n+1}\nabla v_h^{n+1}dxdt \\
&= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma\nabla a_{n+1}^2 dxdt \\
&\leq k\left|C_\gamma\right| |a_{n+1}|_{1,\Omega}^2
\end{aligned}$$

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\int_\Omega \gamma\nabla(I_h(u(t_n))-u_h^n)\nabla v_h^{n+1}dxdt &= \int_{t_n}^{t_{n+1}}\int_\Omega \gamma\nabla a_n\nabla a_{n+1}dxdt\\ &\leq k\left|C_\gamma\right| |a_n|_{1,\Omega}\,|a_{n+1}|_{1,\Omega}\end{aligned}$$

we obtain

$2\int_\Gamma(a_{n+1}-a_n)(s)v_h^{n+1}ds + k\left|C_\gamma\right||a_{n+1}|_{1,\Omega}^2 - k\left|C_\gamma\right||a_n|_{1,\Omega}\,|a_{n+1}|_{1,\Omega}$

$$\begin{aligned}&= 2\int_\Gamma(I_h(u(t_{n+1}))-u(t_{n+1}))-(I_h(u(t_n))-u(t_n))v_h^{n+1}ds\\ &+ \int_{t_n}^{t_{n+1}}\int_\Omega\gamma\nabla(u(t_{n+1})-u(t))\nabla v_h^{n+1}dxdt\\ &+ \int_{t_n}^{t_{n+1}}\int_\Omega\gamma\nabla(I_h(u(t_{n+1}))-u(t_{n+1}))\nabla v_h^{n+1}dxdt - \int_{t_n}^{t_{n+1}}\int_\Omega\gamma\nabla(I_h(u(t_n))-u(t_n))\nabla v_h^{n+1}dxdt\\ &- \int_{t_n}^{t_{n+1}}\int_\Omega\gamma\nabla(u(t_n))\nabla v_h^{n+1}dxdt\end{aligned}$$

We denote by T_1 the first term of the left hand side, T_2 and T_3 the first and second terms of the right hand side, T_4 the third and fourth terms, and T_5 the last term of the equation.

The term T_1 can be expressed as

$$\begin{aligned}T_1 = 2\int_\Gamma(a_{n+1}-a_n)(s)v_h^{n+1}ds &= 2\int_\Gamma(a_{n+1}^2-a_na_{n+1})ds\\ &= \int_\Gamma a_{n+1}^2ds - \int_\Gamma a_n^2ds + \int_\Gamma(a_{n+1}-a_n)^2ds\end{aligned}$$

The term T_2 can be bounded as

$$\begin{aligned}T_2 &= 2\int_\Gamma(I_h(u(t_{n+1}))-u(t_{n+1}))-(I_h(u(t_n))-u(t_n))v_h^{n+1}ds\\ &= 2\int_\Gamma(g(t_{n+1})-g(t_n))a_{n+1}ds\\ &= 2\int_{t_n}^{t_{n+1}}\int_\Omega g'(\tau,s)a_{n+1}dsd\tau\\ &\leq 2\int_{t_n}^{t_{n+1}}\left\|g'(\tau)\right\|_{0,\Gamma}\|a_{n+1}\|_{0,\Gamma}\,d\tau\end{aligned}$$

But

$$\begin{aligned}\left\|g'(\tau)\right\|_{0,\Gamma} &\leq c\left\|g'(\tau)\right\|_{1,\Omega}\\ &\leq \tilde{c}h\left\|u'(\tau)\right\|_{2,\Omega}\\ &\leq c_1h\left\|u'(\tau)\right\|_{L^\infty(0,T,H^2(\Omega))}\end{aligned}$$

then,

$$T_2 \leq c_1hk\left\|u'(\tau)\right\|_{L^\infty(0,T,H^2(\Omega))}\|a_{n+1}\|_{0,\Gamma}$$

Using the inequality $ab \leq \frac{1}{2\epsilon_1}a^2 + \frac{\epsilon_1}{2}b^2$,
with $a = c_1h\sqrt{k}\,\|u'\|_{L^\infty}$ and $b = \sqrt{k}\,\|a_{n+1}\|_{0,\Gamma}$, we get

$$T_2 \leq \frac{1}{2\epsilon_1}c_1^2h^2k\left\|u'(\tau)\right\|_{L^\infty(0,T,H^2(\Omega))}^2 + \frac{\epsilon_1}{2}k\|a_{n+1}\|_{0,\Gamma}^2$$

The term T_3 can be bounded as

$$\begin{aligned} T_3 &= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma \nabla (u(t_{n+1}) - u(t)) \nabla v_h^{n+1} dx dt \\ &= \int_{t_n}^{t_{n+1}} \int_t^{t_{n+1}} \int_\Omega \gamma \nabla (u'(\tau, x)) \nabla a_{n+1} dx d\tau dt \\ &\leq \int_{t_n}^{t_{n+1}} \int_t^{t_{n+1}} \gamma \left\| u'(\tau) \right\|_{1,\Omega} \|a_{n+1}\|_{1,\Omega} d\tau dt \\ &\leq k^2 \left| C_\gamma \right| \left\| u' \right\|_{L^\infty(0,T,H^1(\Omega))} |a_{n+1}|_{1,\Omega} \end{aligned}$$

Using the inequality $ab \leq \frac{1}{2\epsilon_2} a^2 + \frac{\epsilon_2}{2} b^2$,
with $a = k^{\frac{3}{2}} C_\gamma \|u'\|_{L^\infty}$ and $b = \sqrt{k} \|a_{n+1}\|_{1,\Omega}$, we get

$$T_3 \leq \frac{1}{2\epsilon_2} k^3 C_\gamma^2 \left\| u' \right\|^2_{L^\infty(0,T,H^1(\Omega))} + \frac{\epsilon_2}{2} k \, |a_{n+1}|^2_{1,\Omega}$$

Now the term T_4 can be bounded as

$$\begin{aligned} T_4 &= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma \nabla (I_h(u(t_{n+1})) - u(t_{n+1})) \nabla v_h^{n+1} dx dt - \int_{t_n}^{t_{n+1}} \int_\Omega \gamma \nabla (I_h(u(t_n)) - u(t_n)) \nabla a_{n+1} dx dt \\ &= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma \nabla (g(t_{n+1}) - g(t_n)) \nabla a_{n+1} dx dt \\ &= \int_{t_n}^{t_{n+1}} \int_t^{t_{n+1}} \int_\Omega \gamma \nabla (g'(\tau, x)) \nabla a_{n+1} dx d\tau dt \\ &\leq k^2 \left| C_\gamma \right| \left\| g'(\tau) \right\|_{1,\Omega} |a_{n+1}|_{1,\Omega} \\ &\leq k^2 \left| C_\gamma \right| ch \left\| u'(\tau) \right\|_{2,\Omega} |a_{n+1}|_{1,\Omega} \\ &\leq k^2 ch \left| C_\gamma \right| \left\| u' \right\|_{L^\infty(0,T,H^2(\Omega))} |a_{n+1}|_{1,\Omega} \end{aligned}$$

Using the inequality $ab \leq \frac{1}{2\epsilon_3} a^2 + \frac{\epsilon_3}{2} b^2$,
with $a = \left| C_\gamma \right| chk^{\frac{3}{2}} \|u'\|_{L^\infty}$ and $b = \sqrt{k} \|a_{n+1}\|_{1,\Omega}$, we get

$$T_4 \leq \frac{1}{2\epsilon_3} \left| C_\gamma \right|^2 c^2 h^2 k^3 \left\| u' \right\|^2_{L^\infty(0,T,H^2(\Omega))} + \frac{\epsilon_3}{2} k \, |a_{n+1}|^2_{1,\Omega}$$

Finally, the term T_5 can be bounded as

$$\begin{aligned} T_5 &= \int_{t_n}^{t_{n+1}} \int_\Omega \gamma \nabla (u(t_n)) \nabla v_h^{n+1} dx dt \\ &\leq k \left| C_\gamma \right| \|u\|_{1,\Omega} |a_{n+1}|_{1,\Omega} \\ &\leq k \left| C_\gamma \right| \|u\|_{L^\infty(0,T,H^1(\Omega))} |a_{n+1}|_{1,\Omega} \end{aligned}$$

Using the inequality $ab \leq \frac{1}{2\epsilon_4} a^2 + \frac{\epsilon_4}{2} b^2$,
with $a = \left| C_\gamma \right| k^{\frac{1}{2}} \|u\|_{L^\infty}$ and $b = k^{\frac{1}{2}} \|a_{n+1}\|_{1,\Omega}$, we get

$$T_5 \leq \frac{1}{2\epsilon_4} \left| C_\gamma \right|^2 k \, \|u\|^2_{L^\infty(0,T,H^1(\Omega))} + \frac{\epsilon_4}{2} k \, \|a_{n+1}\|^2_{1,\Omega}$$

Now using all the previous bounds, we obtain

$$\int_\Gamma a_{n+1}^2 ds - \int_\Gamma a_n^2 ds + \int_\Gamma (a_{n+1} - a_n)^2 ds + k \left| C_\gamma \right| |a_{n+1}|^2_{1,\Omega} - k \left| C_\gamma \right| |a_n|_{1,\Omega} |a_{n+1}|_{1,\Omega}$$

$$\leq \frac{1}{2\epsilon_1}c_1^2h^2k\left\|u'(\tau)\right\|^2_{L^\infty(0,T,H^2(\Omega))}+\frac{\epsilon_1}{2}k\left\|a_{n+1}\right\|^2_{0,\Gamma}+\frac{1}{2\epsilon_2}k^3\left|\gamma\right|^2\left\|u'\right\|^2_{L^\infty(0,T,H^1(\Omega))}+\frac{\epsilon_2}{2}k\left|a_{n+1}\right|^2_{1,\Omega}$$

$$+\frac{1}{2\epsilon_3}\left|C_\gamma\right|^2c^2h^2k^3\left\|u'\right\|^2_{L^\infty(0,T,H^2(\Omega))}+\frac{\epsilon_3}{2}k\left|a_{n+1}\right|^2_{1,\Omega}-\frac{1}{2\epsilon_4}\left|C_\gamma\right|^2k\left\|u\right\|^2_{L^\infty(0,T,H^1(\Omega))}-\frac{\epsilon_4}{2}k\left\|a_{n+1}\right\|^2_{1,\Omega}$$

Choosing $\epsilon_1=\frac{1}{8T}$, $\epsilon_2=\frac{|C_\gamma|}{2}$, $\epsilon_3=\frac{|C_\gamma|}{2}$ and $\epsilon_4=\frac{|C_\gamma|}{2}$, we get

$$\int_\Gamma a_{n+1}^2ds-\int_\Gamma a_n^2ds+\int_\Gamma(a_{n+1}-a_n)^2ds+\frac{3}{4}k\left|C_\gamma\right|\left|a_{n+1}\right|^2_{1,\Omega}-k\left|C_\gamma\right|\left|a_n\right|_{1,\Omega}\left|a_{n+1}\right|_{1,\Omega}$$

$$\leq ck(h^2+k^2)+\frac{k}{16T}\left|a_{n+1}\right|_{0,\Gamma}$$

Taking sum from $n=0,1,...,m$ and replacing $A_m=4\left\|a_{m+1}\right\|^2_{0,\Gamma}$, $C_m=4c'(h^2+k^2)$ and $B_m=4k\left|\gamma\right|\sum_{n=0}^m(\frac{3}{4}\left|a_{n+1}^2\right|-\left|a_n\right|\left|a_{n+1}\right|)$ with $\frac{k}{16T}\leq\frac{1}{4}$, we get

$$A_m+B_m\leq C_m+\lambda\sum_{n=0}^{m-1}A_n$$

Using Gronwall's Lemma and the properties of I_h we obtain the result.

5. A Posteriori Error Estimate

In this section a posteriori error estimates between the exact solution and the numerical one will be established.

Proposition 2 *(Verfurth, 1996) Denote by $P_r(\kappa)$ the space of polynomials of degree less than r on κ, we have $\forall v\in P_r(\kappa)$*

$$c\left\|v\right\|_{0,\kappa}\leq\left\|v\psi_\kappa^{\frac{1}{2}}\right\|_{0,\kappa}\leq c'\left\|v\right\|_{0,\kappa}$$

$$\left|v\right|_{1,\kappa}\leq ch_\kappa^{-1}\left\|v\right\|_{0,\kappa}$$

Proposition 3 *(Verfurth, 1996) Denote by $P_r(e)$ the space of polynomials of degree less than r on e, we have $\forall v\in P_r(e)$,*

$$c\left\|v\right\|_{0,e}\leq\left\|v\psi_e^{\frac{1}{2}}\right\|_{0,e}\leq c'\left\|v\right\|_{0,e}$$

and for all polynomials in $P_r(e)$ vanishing on ∂e,

$$\left\|L_ev\right\|_{0,\kappa}+h_e\left|L_ev\right|_{1,\kappa}\leq ch_e^{\frac{1}{2}}\left\|v\right\|_{0,e}$$

For the a posteriori error estimates, consider $\forall t\in(t_{n-1},t_n)$ the piecewise affine function $u_h(t)$ which take the values

$$u_h(t)=\frac{t-t_{n-1}}{\tau_n}(u_h^n-u_h^{n-1})+u_h^{n-1}$$

The solutions of Problems (2) and (3) verify the following

$$\begin{aligned}T(v)&=\int_\Omega\gamma\nabla(u-u_h)\nabla v(t,x)dx+2\int_\Gamma\frac{\partial(u-u_h)}{\partial t}(t,x)v(t,x)dx\\&=\int_\Omega\gamma\nabla u\nabla vdx-\int_\Omega\gamma\nabla u_h\nabla vdx-2\int_\Gamma\frac{\partial u_h}{\partial t}(t,x)v(t,x)dx+2\int_\Gamma\frac{\partial u}{\partial t}vdx\\&=-\int_\Omega\gamma\nabla u_h\nabla vdx-2\int_\Gamma\frac{\partial u_h}{\partial t}(t,x)v(t,x)dx\end{aligned}$$

adding and subtracting u_h^n and u_h^{n-1} to the first term, then using the value of u_h we get

$$\begin{aligned}
T(v) &= -\int_\Omega \gamma\nabla(u_h - u_h^n)\nabla v dx - \int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \nabla u_h^{n-1}\nabla v dx - 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(t,x)v(t,x)dx \\
&= -\frac{t-t_n}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla v dx - 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(t,x)v(t,x)dx \\
&= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla v dx - 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(t,x)v(t,x)dx
\end{aligned}$$

adding and subtracting v_h to the second and third terms, we get

$$\begin{aligned}
T(v) &= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla(v - v_h)dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla v_h dx \\
&- 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(v - v_h)dx - 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}v_h dx \\
&= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla(v - v_h)dx - \int_\Omega \gamma\nabla u_h^{n}\nabla v_h dx \\
&- 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}v_h dx
\end{aligned}$$

adding and subtracting v to the third term,

$$\begin{aligned}
T(v) &= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla(v - v_h)dx - \int_\Omega \gamma\nabla u_h^{n}\nabla(v - v_h)dx \\
&- 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(v - v_h)dx + \int_\Omega \gamma\nabla u_h^n\nabla v dx
\end{aligned}$$

Applying Green's theorem on the second and third terms we get

$$\begin{aligned}
T(v) &= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \sum_{k\in\tau_n h}\left(\int_k div(\gamma\nabla u_h^{n-1})(v - v_h)dx - \int_{\partial\kappa}(\nabla u_h^{n-1}.n)(v - v_h)dx\right) \\
&- \sum_{k\in\tau_n h}\left(\int_k div(\gamma\nabla u_h^{n})(v - v_h)dx - \int_{\partial\kappa}(\nabla u_h^{n}.n)(v - v_h)dx\right) + \int_\Omega \gamma\nabla u_h^n\nabla dx \\
&- 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(v - v_h)dx \\
&= -\frac{t-t_{n-1}}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx - \sum_{k\in\tau_n h}\int_{\partial\kappa}(\nabla u_h^{n-1}.n)(v - v_h)dx \\
&- \sum_{k\in\tau_n h}\int_{\partial\kappa}(\nabla u_h^{n}.n)(v - v_h)dx + \int_\Omega \gamma\nabla u_h^n\nabla v dx - 2\int_\Gamma \frac{u_h^n - u_h^{n-1}}{\tau_n}(v - v_h)dx
\end{aligned}$$

We define, for every edge e of the mesh, the function

$$(\varphi_{h,n}^e) = \begin{cases} \gamma\left[\nabla u_h^n.n\right]_e & , e \in \epsilon_k, \\ \gamma\nabla(u_h^n).n + \gamma\nabla(u_h^{n-1}).n + 2\frac{u_h^n - u_h^{n-1}}{\tau_n} & , e \in \epsilon_k^m. \end{cases}$$

We get the following equation

$$\int_\Omega \gamma\nabla(u - u_h)\nabla v(t,x)dx + 2\int_\Gamma \frac{\partial u - u_h}{\partial t}(t,x)v(t,x)dx$$

$$= \frac{t_{n-1}-t}{\tau_n}\int_\Omega \gamma\nabla(u_h^n - u_h^{n-1})\nabla v dx + \int_\Omega \gamma\nabla u_h^{n-1}\nabla v dx - \sum_{k\in\tau_{h,n}}\sum_{e\in\partial k}\int_e \varphi_{h,n}^e(x)(v-v_h)dx$$

For each k in $\tau_{h,n}$ we introduce the indicators

$$\eta_{n,k}^\tau = \sqrt{\frac{\tau_n}{3}}\left\|\nabla(u_h^n - u_h^{n-1})\right\|_{0,k} + \sqrt{\tau_n}\left\|\nabla u_h^{n-1}\right\|_{0,k}$$

$$(\eta_{n,k}^h)^2 = \sum_{e\in\partial k} h_e \left\|\varphi_{h,n}^e\right\|_{0,e}^2$$

5.1 Upper Bounds of the Error

Theorem 4 *For all $m = 1, ..., N$, we have the following upper bound*

$$c\left\|\nabla(u-u_h)\right\|^2_{L^2(0,t_m,L^2(\Omega))} + \left\|u(t_m) - u_h^m\right\|^2_{0,\Gamma} \le c'\left[(\eta_{n,k}^\tau)^2 + \sum_{n=1}^m\sum_k \tau_n(\eta_{n,k}^h)^2 + \left\|u_0 - u_h^0\right\|^2_{0,\Gamma}\right]$$

where c is a constant.

Proof. We denote by $L(v)$ the following,

$$L(v) = \int_\Omega \gamma\nabla(u-u_h)\nabla v(t,x)dx + 2\int_\Gamma \frac{\partial u - u_h}{\partial t} v ds$$

and we define the function $w(t,x)$ by

$$w(t,x) = e^{-t}(u-u_h)(t,x)$$

which verifies the equation

$$\frac{\partial w}{\partial t} + w = e^{-t}\frac{\partial(u-u_h)}{\partial t}$$

Multiplying $L(v)$ by e^{-t} and taking $w = v$,

$$\begin{aligned}
e^{-t}L(v) &= \int_\Omega \gamma\nabla(e^{-t}(u-u_h))\nabla v dx + 2\int_\Gamma e^{-t}\frac{\partial u-u_h}{\partial t} v ds \\
&= \int_\Omega \gamma\nabla w\nabla v dx + 2\int_\Gamma w v ds + 2\int_\Gamma \frac{\partial w}{\partial t} v ds \\
&= \int_\Omega \gamma\left|\nabla w\right|^2 dx + 2\int_\Gamma w^2 ds + \int_\Gamma \frac{\partial(w^2)}{\partial t} ds \\
&\ge \int_\Omega \gamma\left|\nabla w\right|^2 dx + \int_\Gamma \frac{\partial(w^2)}{\partial t} ds \\
&\ge c\left\|\nabla w\right\|^2_{0,\Omega} + \int_\Gamma \frac{\partial w^2}{\partial t} ds
\end{aligned}$$

Note that $e^{-t} \le 1$, so $L(w) \le L(u-u_h)$, then we have the following $c\left\|\nabla w\right\|^2_{0,\Omega} + \int_\Gamma \frac{\partial w^2}{\partial t} ds$

$$\le \int_\Omega \nabla(u-u_h)\nabla(u-u_h)dx + \int_\Gamma \frac{\partial(u-u_h)}{\partial t}(u-u_h)ds$$

Integrating in (t_{n-1}, t_n), we get

$$\int_{t_{n-1}}^{t_n} c\left\|\nabla w\right\|^2_{0,\Omega} dt + \int_{t_{n-1}}^{t_n}\int_\Gamma \frac{\partial(w^2)}{\partial t} ds dt \le \int_{t_{n-1}}^{t_n} L(u-u_h)dt$$

$$\int_{t_{n-1}}^{t_n} c\left\|\nabla w\right\|^2_{0,\Omega} dt + \int_\Gamma w^2(t_n,s)ds - \int_\Gamma w^2(t_{n-1},s)ds \le \int_{t_{n-1}}^{t_n} L(u-u_h)dt$$

Taking the sum from 1 to m, we get

$$c\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}\left\|\nabla e^{-t}(u-u_h)\right\|_{0,\Omega}^2 dt - \int_\Gamma e^{-2t}\left|u-u_h\right|^2(0,s)ds + \int_\Gamma e^{-2t}\left|u-u_h\right|^2(t_m,s)ds$$
$$\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} L(u-u_h)dt$$

$$e^{-2T}\left[c\sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\|\nabla(u-u_h)\|_{0,\Omega}^2 dt + \int_\Gamma |u-u_h|^2(t_m,s)ds\right)\right]$$
$$\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} L(u-u_h)dt + \int_\Gamma |u-u_h|^2(0,s)ds$$

so that

$$\int_0^{t_m} c\|\nabla(u-u_h)\|_{0,\Omega}^2 dt + \left\|u(t_m)-u_h^m\right\|_{0,\Gamma}^2 \leq c'\left[\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} L(u-u_h)dt + \left\|u_0-u_h^0\right\|_{0,\Gamma}^2\right]$$

We decompose $L(v) = L_1(v) + L_2(v)$ and denote $v = u - u_h$.
Now we have to bound $L_1(v)$,

$$\begin{aligned}
L_1(v) &= \frac{t_n-t}{\tau_n}\int_\Omega \gamma\nabla(u_h^n-u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla v dx\\
&= \frac{t_n-t}{\tau_n}\sum_{k\in\tau_{h,n}}\int_k \gamma\nabla(u_h^n-u_h^{n-1})\nabla v dx - \sum_{k\in\tau_{h,n}}\int_k \gamma\nabla u_h^{n-1}\nabla v dx\\
&\leq \left|\frac{t_n-t}{\tau_n}\right|\sum_{k\in\tau_{h,n}} c_\gamma\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k}\|\nabla v\|_{0,k} + \sum_{k\in\tau_{h,n}} c_\gamma\left\|\nabla v_h^{n-1}\right\|_{0,k}\|\nabla v\|_{0,k}
\end{aligned}$$

Integrating in (t_{n-1}, t_n), then taking the sum from 1 to m, we get

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} L_1(v)dt &\leq \sum_{k\in\tau_{h,n}}\left[\int_{t_{n-1}}^{t_n}\left(\left|\frac{t_n-t}{\tau_n}\right| c\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k}\right)^2 dt\right]^{\frac{1}{2}}\left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\\
&+ \sum_{k\in\tau_{h,n}}\left[\int_{t_{n-1}}^{t_n}\left(c_\gamma\left\|\nabla v_h^{n-1}\right\|_{0,k}^2 dt\right]^{\frac{1}{2}}\left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\\
&= \sum_{k\in\tau_{h,n}}\left[c_\gamma^2\frac{\tau_n}{3}\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k}^2\right]^{\frac{1}{2}}\left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\\
&+ \sum_{k\in\tau_{h,n}}\left[c_\gamma^2\tau_n\left\|\nabla v_h^{n-1}\right\|_{0,k}^2 dt\right]^{\frac{1}{2}}\left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\\
&= \left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\sum_{k\in\tau_{h,n}} c_\gamma\left[\sqrt{\frac{\tau_n}{3}}\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k} + \sqrt{\tau_n}\left\|\nabla v_h^{n-1}\right\|_{0,k}\right]\\
&= \left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\sum_{k\in\tau_{h,n}} c_\gamma\left[\eta_{n,k}^\tau\right]\\
&= \left[\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right]^{\frac{1}{2}}\sum_{k\in\tau_{h,n}} c_\gamma^2\left[(\eta_{n,k}^\tau)^2\right]^{\frac{1}{2}}
\end{aligned}$$

Using the inequality $ab \leq \frac{1}{2\epsilon_1}a^2 + \frac{\epsilon_1}{2}b^2$, with $a = \left(c_\gamma^2(\eta_{n,k}^\tau)^2\right)^{\frac{1}{2}}$ and $b = \left(\int_{t_{n-1}}^{t_n}\|\nabla v\|_{0,k}^2 dt\right)^{\frac{1}{2}}$,
we get

$$\int_{t_{n-1}}^{t_n} L_1(v)dt \leq \frac{1}{2\epsilon_1}c_\gamma^2\sum_{k\in\tau_{h,n}}(\eta_{n,k}^\tau)^2 + \frac{\epsilon_1}{2}\|\nabla v\|_{L^2(t_{n-1},t_n,L^2(\Omega))}^2$$

Taking sum from $n = 1, .., m$, we get

$$\sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} L_1(v)dt \leq c_\gamma \sum_{n=1}^{m} \sum_{k\in\tau_{h,n}} (\eta_{n,k}^{\tau})^2 + \frac{\epsilon_1}{2} \|\nabla(u-u_h)\|^2_{L^2(0,t_m,L^2(\Omega))}$$

Next, we will bound $L_2(v)$, using the following proposition (Clement, 1975) The clément regularization operator $R_{n,h} : H^1(\Omega) \to X_h$ has the following property,

$\forall k \in \tau_{n,h}$ and $\forall v \in H^1(\Omega)$, we have the following

$$\left\|v - R_{n,h}v\right\|_{0,k} \leq ch_k \|\nabla v\|_{0,\Delta_k}$$

and

$$\left\|v - R_{n,h}v\right\|_{0,e} \leq ch_e^{\frac{1}{2}} \|\nabla v\|_{0,\Delta_e}$$

$$\begin{aligned} L_2(v) &= -\sum_{k\in\tau_{h,n}} \sum_{e\in\partial k} \int_e \varphi_{h,n}^e(x)(v - v_h)dx \\ &\leq \sum_{k\in\tau_{h,n}} \sum_{e\in\partial k} \left\|\varphi_{h,n}^e\right\|_{0,e} \|v(t) - v_h(t)\|_{0,e} \end{aligned}$$

Now we take $v_h(t) = R_{n,h}(v(t))$, and use the above preposition to get

$$\begin{aligned} L_2(v) &\leq \sum_{k\in\tau_{h,n}} \sum_{e\in\partial k} \left\|\varphi_{h,n}^e\right\|_{0,e} \left\|v(t) - R_{n,h}(v(t))\right\|_{0,e} \\ &\leq \sum_{k\in\tau_{h,n}} \sum_{e\in\partial k} \left\|\varphi_{h,n}^e\right\|_{0,e} c_2 h_e^{\frac{1}{2}} \|\nabla v(t)\|_{0,\Delta e} \end{aligned}$$

Using the inequality $\sum ab \leq (\sum a^2)^{\frac{1}{2}} (\sum b^2)^{\frac{1}{2}}$, with $a = he^{\frac{1}{2}} \left\|\varphi_{h,n}^e\right\|_{0,e}$ and $b = \|\nabla v(t)\|_{0,\Delta e}$, we obtain

$$\begin{aligned} L_2(v) &\leq c_2 \sum_{k\in\tau_{h,n}} \left[\sum_{e\in\partial k} h_e \left\|\varphi_{h,n}^e\right\|^2_{0,e}\right]^{\frac{1}{2}} \left[\sum_{e\in\partial k} \|\nabla v(t)\|^2_{0,\Delta e}\right]^{\frac{1}{2}} \\ &\leq c_2 \left[\sum_{k\in\tau_{h,n}} (\eta_{n,k}^h)^2\right]^{\frac{1}{2}} \left[\sum_{k\in\tau_{h,n}} \sum_{e\in\partial k} \|\nabla v(t)\|^2_{0,\Delta e}\right]^{\frac{1}{2}} \\ &\leq c_3 \left[\sum_{k\in\tau_{h,n}} (\eta_{n,k}^h)^2\right]^{\frac{1}{2}} \|\nabla v(t)\|_{0,\Omega} \end{aligned}$$

Integrating in (t_{n-1}, t_n),

$$\begin{aligned} \int_{t_{n-1}}^{t_n} L_2(v)dt &\leq c_3 \left[\int_{t_{n-1}}^{t_n} \sum_{k\in\tau_{h,n}} (\eta_{n,k}^h)^2 dt\right]^{\frac{1}{2}} \left[\int_{t_{n-1}}^{t_n} \|\nabla v(t)\|^2_{0,\Omega}\, dt\right]^{\frac{1}{2}} \\ &\leq c_3 \left[\sum_{k\in\tau_{h,n}} \tau_n (\eta_{n,k}^h)^2\right]^{\frac{1}{2}} \left[\|\nabla v\|_{L^2(t_{n-1},t_n,L^2(\Omega))}\right] \end{aligned}$$

Taking the sum from $n = 1, ..., m$, we get

$$\sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} L_2(v)dt \leq c_3 \left[\sum_{n=1}^{m} \sum_{k\in\tau_{h,n}} \tau_n (\eta_{n,k}^h)^2\right]^{\frac{1}{2}} \left[\|\nabla v\|_{L^2(0,t_m,L^2(\Omega))}\right]$$

Using $ab \leq \frac{1}{2\epsilon_2}a^2 + \frac{\epsilon_2}{2}b^2$,we get

$$\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} L_2(u-u_h)dt \leq \frac{c_3}{2\epsilon_2}\sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}\tau_n(\eta_{n,k}^h)^2 + \frac{\epsilon_2}{2}\|\nabla(u-u_h)\|^2_{L^2(0,t_m,L^2(\Omega))}$$
$$= c_4\sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}\tau_n(\eta_{n,k}^h)^2 + \frac{\epsilon_2}{2}\|\nabla(u-u_h)\|^2_{L^2(0,t_m,L^2(\Omega))}$$

Using the above bounds, and choosing $\epsilon_1 = \frac{c}{2}$ and $\epsilon_2 = \frac{c}{2}$ we get

$$c\,\|\nabla(u-u_h)\|^2_{L^2(0,t_m,L^2(\Omega))} + \left\|u(t_m)-u_h^m\right\|^2_{0,\Gamma}$$
$$\leq c'\left[\sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}(\eta_{n,k}^h)^2 + \sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}\tau_n(\eta_{n,k}^h)^2 + \left\|u_0-u_h^0\right\|^2_{0,\Gamma}\right]$$

□

Theorem 5 *For all $m = 1,2,..,N$ we have*

$$\left\|\frac{\partial(u-u_h)}{\partial t}\right\|^2_{L^2(0,t_m,H^{-\frac{1}{2}}(\Gamma))} \leq c'\left[\sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}((\eta_{n,k}^\tau)^2 + \tau_n(\eta_{n,k}^\tau)^2) + \left\|u_0-u_h^0\right\|^2_{0,\Gamma}\right]$$

where c' is a constant.

Proof. Define the functions $r(t,x)\in H^{\frac{1}{2}}(\Gamma)$ and $w(t,x) = e^{-t}(u-u_h)(t,x)\in H^1(\Omega)$, and consider the problem

$$\begin{cases} div(\gamma\nabla w(t,x)) = 0 & \text{, in } (0,T)\times\Omega, \\ w(t,x) = r(t,x) & \text{, on } (0,T)\times\Gamma. \end{cases} \tag{6}$$

which admits a unique solution $w(t)\in H^1(\Omega)$ verifying,

$$\|\nabla w(t)\|_{0,\Omega} \leq c_1\|r\|_{\frac{1}{2},\Gamma}$$

Consider the equation

$$\int_\Omega \gamma\nabla(u-u_h)\nabla v(t,x)dx + 2\int_\Gamma \frac{\partial u-u_h}{\partial t}vds$$
$$= \frac{t_n-t}{\tau_n}\int_\Omega \gamma\nabla(u_h^n-u_h^{n-1})\nabla v dx - \int_\Omega \gamma\nabla u_h^{n-1}\nabla v dx - \sum_{k\in\tau_{h,n}}\sum_{e\in\partial k}\int_e \varphi_{h,n}^e(x)(v-v_h)dx$$

Using the inequalities

$$L_1(v) \leq \left|\frac{t_n-t}{\tau_n}\right|\sum_{k\in\tau_{h,n}} c\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k}\|\nabla v\|_{0,k} + \sum_{k\in\tau_{h,n}} c\left\|\nabla v_h^{n-1}\right\|_{0,k}\|\nabla v\|_{0,k}$$

and

$$L_2(v) \leq c_3\left[\sum_{k\in\tau_{h,n}}(\eta_{n,k}^h)^2\right]^{\frac{1}{2}}\|\nabla v(t)\|_{0,\Omega}$$

we get

$$\int_\Omega \gamma\nabla(u-u_h)\nabla v(t,x)dx + 2\int_\Gamma \frac{\partial(u-u_h)}{\partial t}vds$$
$$\leq \|\nabla v(t)\|_{0,\Omega}\left[\left|\frac{t_n-t}{\tau_n}\right|\sum_{k\in\tau_{h,n}} c\left\|\nabla(u_h^n-u_h^{n-1})\right\|_{0,k} + \sum_{k\in\tau_{h,n}} c\left\|\nabla u_h^{n-1}\right\|_{0,k} + c_3(\sum_{k\in\tau_{h,n}}(\eta_{n,k}^h)^2)^{\frac{1}{2}}\right]$$

dividing by $\|v\|_{1,\Omega}$ then using Cauchy-Schwartz inequality, we get

$$\frac{2}{\|v(t)\|_{1,\Omega}}\int_\Gamma \frac{\partial(u-u_h)}{\partial t}v(t,s)ds$$

$$\leq c_\gamma \|\nabla(u-u_h)\|_{0,\Omega} + c_\gamma \left|\frac{t_{n-1}-t}{\tau_n}\right| \left(\sum_\kappa \left\|\nabla(u_h - u_h^{n-1})\right\|_{0,k}^2\right)^{\frac{1}{2}} + c\left(\sum_\kappa (\eta_{n,k}^h)^2\right)^{\frac{1}{2}} + \left(\sum_\kappa c_\gamma \left\|\nabla u_h^n\right\|_{0,\Omega}^2\right)^{\frac{1}{2}}$$

For every $r(t) \in H^{\frac{1}{2}}(\Gamma)$, consider the harmonic lifting in $v \in H^1(\Omega)$satisfying,

$$\begin{cases} div(\gamma\nabla v(t,x)) = 0 & , \text{ in } (0,T)\times\Omega, \\ v(t,x) = r(t,x) & , \text{ on } (0,T)\times\Gamma. \end{cases} \tag{7}$$

where

$$\|v(t)\|_{1,\Omega} \leq c_1 \|r\|_{\frac{1}{2},\Gamma} = c_1 \|v\|_{\frac{1}{2},\Gamma}$$

so

$$\frac{1}{\|v(t)\|_{\frac{1}{2},\Gamma}} \leq \frac{1}{\|v(t)\|_{1,\Omega}}$$

but

$$sup_{v\in H^{\frac{1}{2}}(\Gamma)} \frac{\int_\Gamma \frac{\partial(u-u_h)}{\partial t} v ds}{\|v\|_{\frac{1}{2},\Gamma}} = \left\|\frac{\partial(u-u_h)}{\partial t}\right\|_{-\frac{1}{2},\Gamma}$$

therefore, after integrating over (t_{n-1}, t_n), and taking sum from $n = 1, .., m$ we get

$$\left\|\frac{\partial(u-u_h)}{\partial t}\right\|_{L^2(0,t_m,H^{-\frac{1}{2}}(\Gamma))}^2 \leq c' \left[\sum_{n=1}^m \sum_{k\in\tau_{h,n}} ((\eta_{n,k}^\tau)^2 + \tau_n(\eta_{n,k}^\tau)^2) + \left\|u_0 - u_h^0\right\|_{0,\Gamma}^2\right]$$

□

Theorem 6 *For all $m = 1, ..., N$, we have the following*

$$\|\nabla(u-\pi_\tau u_h)\|_{L^2(0,t_m,L^2(\Omega))}^2 \leq c\left[\sum_{n=1}^m \sum_{k\in\tau_{h,n}} ((\eta_{n,k}^\tau)^2 + \tau_n(\eta_{n,k}^\tau)^2) + \left\|u_0 - u_h^0\right\|_{0,\Gamma}^2\right]$$

where c is a constant.

Proof. We have, using the previous theorem, the following bound

$$\begin{aligned} \|\nabla(u-\pi_\tau u_h)\|_{L^2} &= \|\nabla(u-u_h+u_h-\pi_\tau u_h)\|_{L^2} \\ &\leq \|\nabla(u-u_h)\|_{L^2} + \|\nabla(u_h-\pi_\tau u_h)\|_{L^2} \\ &\leq c'\left[\sum_{n=1}^m \sum_{k\in\tau_{h,n}} ((\eta_{n,k}^\tau)^2 + \tau_n(\eta_{n,k}^\tau)^2) + \left\|u_0 - u_h^0\right\|_{0,\Gamma}^2\right]^{\frac{1}{2}} + \|\nabla(u_h-\pi_\tau u_h)\|_{L^2} \end{aligned}$$

Now we have to bound $\|\nabla(u_h - \pi_\tau u_h)\|_{L^2}$.
For $t \in (t_{n-1}, t_n)$, we have $\pi_\tau u_h(t) = u_h^n$ and $u_h - u_h^n = \frac{t-t_n}{\tau_n}(u_h^n - u_h^{n-1})$, we have

$$\begin{aligned} \|\nabla(u_h-\pi_\tau u_h)\|_{0,\Omega}^2 &\leq \frac{(t-t_n)^2}{\tau_n^2}\left[\sum_k \left\|\nabla(u_h^n - u_h^{n-1})\right\|_{0,k}^2\right] \\ &\leq \frac{(t-t_n)^2}{\tau_n^2}\left[\sum_k \left\|\nabla(u_h^n - u_h^{n-1})\right\|_{0,k}^2\right] + \tau_n^2\left\|\nabla u_h^{n-1}\right\|_{0,k}^2 \end{aligned}$$

integrating over (t_{n-1}, t_n), we get

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \|\nabla(u_h-\pi_\tau u_h)\|_{0,\Omega}^2 dt &\leq \int_{t_{n-1}}^{t_n} \frac{(t-t_n)^2}{\tau_n^2}\left[\sum_k \left\|\nabla(u_h^n - u_h^{n-1})\right\|_{0,k}^2\right] + \tau_n^2\left\|\nabla u_h^{n-1}\right\|_{0,k}^2 \\ &\leq c_1\sum_k (\eta_{n,k}^\tau)^2 \end{aligned}$$

Finally we conclude

$$\|\nabla(u-\pi_\tau u_h)\|_{L^2(0,t_m,L^2(\Omega))} \le c\left[\sum_{n=1}^{m}\sum_{k\in\tau_{h,n}}((\eta^\tau_{n,k})^2+\tau_n(\eta^\tau_{n,k})^2)+\left\|u_0-u_h^0\right\|^2_{0,\Gamma}\right]^{\frac{1}{2}}$$

□

5.2 Upper Bounds of the Indicators

Theorem 7 *For all $m = 1, .., N$ we have the following estimate*

$$(\eta^\tau_{n,k})^2 \le \|\nabla(u-\pi_\tau u_h)\|^2_{L^2(t_{n-1},t_n,L^2(k))}+\|\nabla(u-u_h)\|^2_{L^2(t_{n-1},t_n,L^2(k))}$$

Proof. We have

$$\begin{aligned}\frac{t-t_n}{\tau_n}\nabla(u_h^n-u_h^{n-1}) &= \nabla(u_h-\pi_\tau u_h)\\ &= \nabla(u-u_h)+\nabla(u-\pi_\tau u_h)\end{aligned}$$

then

$$\begin{aligned}\left|\frac{t-t_n}{\tau_n}\nabla(u_h^n-u_h^{n-1})\right|^2 &= |\nabla(u-u_h)+\nabla(u-\pi_\tau u_h)|\\ &\le |\nabla(u-u_h)|^2+|\nabla(u-\pi_\tau u_h)|^2\end{aligned}$$

but

$$(\frac{t-t_n}{\tau_n})^2\left|\nabla(u_h^n-u_h^{n-1})\right|^2 \le (\frac{t-t_n}{\tau_n})^2\left|\nabla(u_h^n-u_h^{n-1})\right|^2+\left|\nabla u_h^{n-1}\right|^2$$

integrating over k and on (t_{n-1}, t_n) we get

$$\int_{t_{n-1}}^{t_n}\int_k(\frac{t-t_n}{\tau_n})^2\left|\nabla(u_h^n-u_h^{n-1})\right|^2+\int_{t_{n-1}}^{t_n}\int_k\left|\nabla u_h^{n-1}\right|^2 \le \int_{t_{n-1}}^{t_n}\int_k|\nabla(u-u_h)|^2+|\nabla(u-\pi_\tau u_h)|^2$$

then

$$\frac{\tau_n}{3}\left\|\nabla(u_h^n-u_h^{n-1})\right\|^2_{0,k}+\tau_n\left\|\nabla u_h^{n-1}\right\|^2_{0,k} \le \|\nabla(u-\pi_\tau u_h)\|^2_{L^2(t_{n-1},t_n,L^2(k))}+\|\nabla(u-u_h)\|^2_{L^2(t_{n-1},t_n,L^2(k))}$$

REMARK: The numerical simulation will be done in a forthcoming paper.

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