Journal of Mathematics Research; Vol. 7, No. 4; 2015
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Quasi-arithmetic Means Inequalities Criteria for Differentiable Functions

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Received: October 8, 2015 Accepted: October 21, 2015 Online Published: November 6, 2015

doi:10.5539/jmr.v7n4p130 URL: http://dx.doi.org/10.5539/jmr.v7n4p130

The research is financed by Faculty of Transport and Traffic Engineering

Abstract

Quasi-arithmetic means are defined for continuous, strictly monotone functions. In the case that functions are twice differentiable, we obtained criteria for inequalities between finite number of quasi-arithmetic means in additional and multiplicative case. Applications for Hölder and Minkowski type inequalities are given.

1. Introduction

The quasi-arithmetic mean in discrete instance is defined for a continuous and monotone function $\varphi : J_x \subseteq \mathbb{R} \to \mathbb{R}$, real sentence $\mathbf{x} = (x_1, \ldots, x_n) \in J_x$ and a probability weight sentence of non-negative real numbers $\mathbf{a} = (a_1, \ldots, a_n)$, with $\sum_{k=1}^{n} a_k = 1$ by the formula:

$$M_\varphi(\mathbf{x}; \mathbf{a}) = \varphi^{-1}\left(\sum_{k=1}^{n} a_k \varphi(x_k)\right). \tag{1}$$

If φ is a differentiable function, then we call it differentiable quasi-arithmetic mean in this article. Here the twice differentiability is considered.

For continuous and monotone functions $\psi : J_y \to \mathbb{R}$ and $\chi : J_w \to \mathbb{R}$ that are defined on intervals $J_y, J_w \subseteq \mathbb{R}$, sentence $\mathbf{y} = (y_1, \ldots, y_n) \in J_y$ and $f : J_x \times J_y \to J_w$, the inequality

$$f(M_\varphi(\mathbf{x}; \mathbf{a}), M_\psi(\mathbf{y}; \mathbf{a})) \geq M_\chi(\mathbf{f(x,y)}; \mathbf{a}) \tag{2}$$

was investigated by E. Beck in 1970 for additive case where $\mathbf{f(x,y)} = \mathbf{x+y}$ and multiplicative case with $\mathbf{f(x,y)} = \mathbf{xy} = (x_1y_1, \ldots, x_ny_n)$. Criteria were obtained for φ, ψ and χ being twice differentiable.

Enlargement with differentiable, continuous and monotone function $\rho : J_z \to \mathbb{R}$, where $J_z \subseteq \mathbb{R}$ and sentence $\mathbf{z} = (z_1, \ldots, z_n) \in J_z$, for a function $f : J_x \times J_y \times J_z \to J_w$, was given in (Ivanković, 2015). The conditions for inequality

$$f\left(M_\varphi(\mathbf{x}; \mathbf{a}), M_\psi(\mathbf{y}; \mathbf{a}), M_\rho(\mathbf{z}; \mathbf{a})\right) \geq M_\chi(\mathbf{f(x,y,z)}; \mathbf{a}) \tag{3}$$

were proven in additive and multiplicative cases.

The inequality (3) is equivalent with inequality

$$H\left(\sum_{i=1}^{n} a_i s_i, \sum_{i=1}^{n} a_i t_i, \sum_{i=1}^{n} a_i r_i\right) \geq \sum_{i=1}^{n} a_i H(s_i, t_i, r_i), \tag{4}$$

where $H(s,t,r) = \chi f\left(\varphi^{-1}(s), \psi^{-1}(t), \rho^{-1}(r)\right)$, $s = \varphi(x), t = \psi(y)$ and $r = \rho(z)$. Direction in (4) depends on convexity of $H(s,t,r)$ and tendency of χ.

In this article, conditions for m quasi-arithmetic means inequality are given in additive and multiplicative case.

2. Fundamental Condition

The inequality (3) is enlarged for m continuous, strictly monotone functions $\varphi_i : J_i \to \mathbb{R}$ generating m quasi-arithmetic means:

$$M_{\varphi_i}(\mathbf{x_i};\mathbf{a}) = \varphi_i^{-1}\left(\sum_{j=1}^{n} a_j \cdot \varphi_i(x_{ij})\right),\ i = 1,\ldots,m.$$

The means are calculating for real sequences $\mathbf{x_i} = (x_{i1},\ldots,x_{in})$, $i = 1,\ldots,m$, belonging to $J_i \subseteq \mathbb{R}$. For given n-tuples, the function values $f : J_1 \times J_2 \times \cdots \times J_m \to \mathbb{R}$ are constituting new n-tuple by calculating: $\mathbf{f(x_1,x_2\ldots,x_m)} = (f(x_{11},x_{21},\ldots,x_{m1}), f(x_{12},x_{22},\ldots,x_{m2}),\ldots, f(x_{1n},x_{2n},\ldots,x_{mn}))$

If $f : J_1 \times J_2 \times \cdots \times J_m \to J_w$, then the quasi-arithmetic mean is defined properly:

$$M_\chi(\mathbf{f(x_1,\ldots,x_m)};\mathbf{a}) = \chi^{-1}\left(\sum_{j=1}^{n} a_j \cdot \chi f(x_{1j}, x_{2j},\ldots, f(x_{mj}))\right). \tag{5}$$

For just defined terms the next proposition is declared.

Proposition 2.1. *With respect to the terms defined above, for strictly increasing function χ the inequality*

$$f\left(M_{\varphi_1}(\boldsymbol{x_1};\boldsymbol{a}),\ldots,M_{\varphi_m}(\boldsymbol{x_m};\boldsymbol{a})\right) \geq M_\chi(\boldsymbol{f(x_1,\ldots,x_m)};\boldsymbol{a}) \tag{6}$$

states if and only if the function

$$H(s_{1j},\ldots,s_{mj}) = \chi f\left(\varphi_1^{-1}(s_{1j}),\ldots,\varphi_m^{-1}(s_{mj})\right),\ s_{ij} = \varphi_i(x_{ij}),\ j = 1,\ldots,n \tag{7}$$

is concave and χ increases or if (7) is convex and χ decreases.

The inequality (6) is opposite if the function H defined by (7) is convex and χ increases or if $H(s_{1j},\ldots,s_{mj})$ is concave and χ decreases. Function (7) is defined as well.

Proof. For the benefit of better understanding, the proof with increasing χ is following. Suppose (7) is a concave function. Then for every collection of n-tuples given bellow

$$\mathbf{s_i} = (\varphi_i(x_i)) = (\varphi_i(x_{i1}),\varphi_i(x_{i2}),\ldots,\varphi_i(x_{in})) = (s_{i1},s_{i2},\ldots,s_{in}),\ \ i = 1,\ldots,m \tag{8}$$

and every choice of probability weights $\mathbf{a}$, the well-known Jensen-McShane inequality (Pečarić, et al., 1992, p.48-49) holds for m-tuples:

$$H\left(\sum_{j=1}^{n} a_j(s_{1j},s_{2j},\ldots,s_{mj})\right) \geq \sum_{j=1}^{n} a_j H(s_{1j},s_{2j},\ldots,s_{mj}). \tag{9}$$

Linear combination calculating obtains the following

$$H\left(\sum_{j=1}^{n} a_j s_{1j},\sum_{j=1}^{n} a_j s_{2j},\ldots,\sum_{j=1}^{n} a_j s_{mj}\right) \geq \sum_{j=1}^{n} a_j H(s_{1j},s_{2j},\ldots,s_{mj}).$$

According the definiton's relations (8), if $s_{ij} = \varphi_i(x_{ij})$, $j = 1,\ldots,n$, then $\varphi_i^{-1}(s_{ij}) = x_{ij}$. From functon's definition $H = \chi f(\varphi_1^{-1},\ldots,\varphi_m^{-1})$ it follows:

$$H\left(\sum_{j=1}^{n} a_j s_{1j},\sum_{j=1}^{n} a_j s_{2j},\ldots,\sum_{j=1}^{n} a_j s_{mj}\right) = \chi f\left(\varphi_1^{-1}\left(\sum_{j=1}^{n} a_j \cdot s_{1j}\right),\varphi_2^{-1}\left(\sum_{j=1}^{n} a_j s_{2j}\right),\ldots,\varphi_m^{-1}\left(\sum_{j=1}^{n} a_j s_{mj}\right)\right).$$

Consequently $H(s_{1j},s_{2j},\ldots,s_{mj}) = \chi f\left(\varphi_1^{-1}(s_{1j}),\varphi_2^{-1}(s_{2j}),\ldots,\varphi_m^{-1}(s_{mj})\right)$. Now, the (9) states as

$$\chi f\left(\varphi_1^{-1}\left(\sum_{j=1}^{n} a_j s_{1j}\right),\varphi_2^{-1}\left(\sum_{j=1}^{n} a_j s_{2j}\right),\ldots,\varphi_m^{-1}\left(\sum_{j=1}^{n} a_j s_{mj}\right)\right) \geq \sum_{j=1}^{n} a_j \chi f\left(\varphi_1^{-1}(s_{1j}),\varphi_2^{-1}(s_{2j}),\ldots,\varphi_m^{-1}(s_{mj})\right).$$

The consequence of χ being increasing is that χ^{-1} increase itself:

$$f\left(\varphi_1^{-1}\left(\sum_{j=1}^{n} a_j \varphi_1(x_{1j})\right),\varphi_2^{-1}\left(\sum_{j=1}^{n} a_j \varphi_2(x_{2j})\right),\ldots,\varphi_m^{-1}\left(\sum_{j=1}^{n} a_j \varphi_m(x_{mj})\right)\right) \geq \chi^{-1}\left(\sum_{j=1}^{n} a_j \chi f(x_{1j},x_{2j},\ldots,x_{mj})\right).$$

The inequality above is in fact the inequality (6). So the reverse proof is end. □

For twice differentiable m-variables function's convexity and concavity the criteria exist. Noting the second partial derivatives by $H_{ij} = \dfrac{\partial^2 H}{\partial s_i \partial s_j}$, $i, j = 1, \ldots, m$, there is a Theorem from general mathematical analysis given here as Remark.

Remark 2.1. *Function $H(s_1, s_2, \ldots, s_m)$ is convex if and only if the next m inequalities are satisfied:*

$$H_{11} > 0, \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} > 0, \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix} > 0, \ldots, \begin{vmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mm} \end{vmatrix} > 0. \tag{10}$$

In opposite, function $H(s_1, s_2, \ldots, s_m)$ is concave if and only if the next m inequalities are satisfied:

$$H_{11} < 0, \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} > 0, \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix} < 0, \ldots, (-1)^m \cdot \begin{vmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mm} \end{vmatrix} > 0. \tag{11}$$

Inequalities (10) and (11) will be of crucial interest in what is followning.

3. Additive Case

The additive case appears when function from (6) is an addition: $f(x_1, \ldots, x_m) = x_1 + \cdots + x_m$. The criteria for inequality (6) are proven through the next Theorem.

Theorem 3.1. *Suppose that $\varphi_1, \ldots, \varphi_m$ and χ are twice differentiable strictly monotone functions with second derivations differ from zero on their domains $J_1, \ldots, J_m$ and J_w. Suppose that each n-tuple $\boldsymbol{x_i}$ is assembled by values from J_i, $i = 1, \ldots, m$ and suppose that sum $\sum_{i=1}^{m} x_{ij}$ belongs to J_w for every $j = 1, \ldots, n$. Then there exist functions:*

$$F_i = \frac{\varphi_i'}{\varphi_i''},\ i = 1, \ldots, m \quad \text{and} \quad F = \frac{\chi'}{\chi''}. \tag{12}$$

Take $\boldsymbol{a} = (a_1, \ldots, a_n)$, $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$. Connote n-tuple: $\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} \boldsymbol{x_i} = \left(\sum_{i=1}^{m} x_{i1}, \sum_{i=1}^{m} x_{i2}, \ldots, \sum_{i=1}^{m} x_{in} \right)$. The inequality

$$\sum_{i=1}^{m} M_{\varphi_i}(\boldsymbol{x_i}; \boldsymbol{a}) \geq M_\chi \left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} \boldsymbol{x_i}; \boldsymbol{a} \right), \tag{13}$$

holds if and only if any of the following conditions is fulfilled:

(i) all $F, F_1, \ldots, F_m$ are positive and $F \geq F_1 + F_2 + \cdots + F_m$.

(ii) F is negative and all $F_1, \ldots, F_m$ are positive

The inequality in (13) is opposite if and only if any of the following is fulfilled:

(i) all $F, F_1, \ldots, F_m$ are negative and $F \leq F_1 + F_2 + \cdots + F_m$.

(ii) F is positive and all $F_1, \ldots, F_m$ are negative

Proof. Since the Proposition 2.1 is proven, it is enough to prove concavity for the function $H(s_{1j}, s_{2j}, \ldots, s_{mj}) = \chi\left(\varphi_1^{-1}(s_{1j}) + \ldots + \varphi_m^{-1}(s_{mj})\right)$, respecting Remark 2.1. Elements in (10) and (11) are given with $H_{ii} = \dfrac{\partial^2 H}{\partial s_i^2} = \frac{\chi'}{(\varphi_i')^2} \left(\frac{\chi''}{\chi'} - \frac{\varphi_i''}{\varphi_i'} \right) =$

$\frac{\chi'}{(\varphi_i')^2}\left(\frac{1}{F}-\frac{1}{F_i}\right)$ and $H_{ij}=\dfrac{\partial^2 H}{\partial si \partial s_j}=\dfrac{\chi''}{\varphi_i'\varphi_j'}=\dfrac{\chi'}{\varphi_i'\varphi_j'}\dfrac{1}{F}$ for $i\neq j$. The condition on the k-th determinant in (11) is:

$$(-1)^k\cdot\begin{vmatrix}
\frac{\chi'}{(\varphi_1')^2}\left(\frac{1}{F}-\frac{1}{F_1}\right) & \frac{\chi'}{\varphi_1'\varphi_2'}\frac{1}{F} & \frac{\chi'}{\varphi_1'\varphi_3'}\frac{1}{F} & \cdots & \frac{\chi'}{\varphi_1'\varphi_k'}\frac{1}{F}\\
\frac{\chi'}{\varphi_2'\varphi_1'}\frac{1}{F} & \frac{\chi'}{(\varphi_2')^2}\left(\frac{1}{F}-\frac{1}{F_2}\right) & \frac{\chi'}{\varphi_2'\varphi_3'}\frac{1}{F} & \cdots & \frac{\chi'}{\varphi_2'\varphi_k'}\frac{1}{F}\\
\frac{\chi'}{\varphi_3'\varphi_1'}\frac{1}{F} & \frac{\chi'}{\varphi_3'\varphi_2'}\frac{1}{F} & \frac{\chi'}{(\varphi_3')^2}\left(\frac{1}{F}-\frac{1}{F_3}\right) & \cdots & \frac{\chi'}{\varphi_3'\varphi_k'}\frac{1}{F}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{\chi'}{\varphi_k'\varphi_1'} & \frac{\chi'}{\varphi_k'\varphi_2'} & \frac{\chi'}{\varphi_n'\varphi_3'} & \cdots & \frac{\chi'}{(\varphi_k')^2}\left(\frac{1}{F}-\frac{1}{F_k}\right)
\end{vmatrix}>0.$$

From every, k-th row, the fraction $\dfrac{\chi'}{\varphi_1'\cdots(\varphi_k')^2\cdots\varphi_m'}$ could be extracted. Their product is $\dfrac{(\chi')^m}{(\varphi_1')^{m+1}\cdots(\varphi_m')^{m+1}}$. After that, each k-th column contains factor $\varphi_1\cdots\varphi_{k-1}\cdot\varphi_{k+1}\cdots\varphi_m$ that could be extracted. Their product is $(\varphi_1')^{m-1}\cdots(\varphi_m')^{m-1}$. Multiplying the product together, we have new condition with factor $\dfrac{(\chi')^m}{(\varphi_1')^2\cdots(\varphi_m')^2}$.

Elementary determinant transformations and some algebra entail the following conditions:

$$(\chi')^k\left(\frac{F}{FF_1\cdots F_k}-\frac{F_1}{FF_1\cdots F_k}-\frac{F_2}{FF_1\cdots F_k}-\cdots-\frac{F}{FF_1\cdots F_k}\right)\geq 0,\ k=1,\ldots,m. \tag{14}$$

The proof of the convex case is analogue and we obtain conditions:

$$(-\chi')^k\left(\frac{F}{FF_1\cdots F_k}-\frac{F_1}{FF_1\cdots F_k}-\frac{F_2}{FF_1\cdots F_k}-\cdots-\frac{F}{FF_1\cdots F_k}\right)\geq 0,\ k=1,\ldots,m. \tag{15}$$

Conditions for inequality in (13) were obtained after discussion when $\chi'>0$ in (14) or when $\chi'<0$ in (15).

Conditions for the opposite inequality in (13) followed after discussion when $\chi'<0$ in (14) or when $\chi'>0$ in (15). □

4. Multiplicative Case

In the multiplicative case the function from (6) is a multiplication: $f(x_1,\ldots,x_m)=x_1\cdots x_m$. The criteria for inequality (6) are proven through the next Theorem.

Theorem 4.1. *Suppose that $\varphi_1,\ldots,\varphi_m$ and χ are twice differentiable strictly monotone functions on their domains $J_1,\ldots,J_m$ and J_w. Suppose that each n-tuple $(x_i)=(x_{i1},\ldots,x_{in})$ is positive and consists values from J_i, $i=1,\ldots,m$ such that product $\prod_{i=1}^m x_{ij}$ belongs to J_w for every $j=1,\ldots,n$. Presume functions*

$$D_i(x_i)=\frac{1}{1+x_i\dfrac{\varphi_i''(x_i)}{\varphi_i'(x_i)}},\ i=1,\ldots,m\ \text{and}\ D(u)=\frac{1}{1+u\dfrac{\chi''(u)}{\chi'(u)}} \tag{16}$$

are definable for $u=x_1\cdots x_m$. Take $\boldsymbol{a}=(a_1,\ldots,a_n)$, $a_i\geq 0$ with $\sum_{i=1}^n a_i=1$ and connote n-tuple

$\prod_{\boldsymbol{i=1}}^{\boldsymbol{m}}\boldsymbol{x_i}=\left(\prod_{i=1}^m x_{i1},\prod_{i=1}^m x_{i2},\ldots,\prod_{i=1}^m x_{in}\right)$. *Then the inequality*

$$\prod_{i=1}^m M_{\varphi_i}(\boldsymbol{x_i};\boldsymbol{a})\geq M_\chi\left(\prod_{\boldsymbol{i=1}}^{\boldsymbol{m}}\boldsymbol{x_i};\boldsymbol{a}\right), \tag{17}$$

holds if and only if any of the following conditions is fulfilled:

(i) all $D, D_1, \dots, D_m$ are positive and $D \geq D_1 + D_2 + \cdots + D_m$.

(ii) D is negative and all $D_1, \dots, D_m$ are positive

The inequality in (17) is opposite if and only if any of the following is fulfilled:

(i) all $D, D_1, \dots, D_m$ are negative and $D \leq D_1 + D_2 + \cdots + D_m$.

(ii) D is positive and all $D_1, \dots, D_m$ are negative

Proof. In the case that χ increases, the inequality in (17) is based on the concavity of the function $H(s_{1j}, s_{2j}, \dots, s_{mj}) = \chi\left(\varphi_1^{-1}(s_{1j}) \cdots \varphi_m^{-1}(s_{mj})\right)$ and opposite inequality is based on its convexity. When χ decreases, inequalities are vice versa. Here we give the proof for (17) according Remark 2.1. From $H(s_{1j}, s_{2j}, \dots, s_{mj})$ it follows that $H_{ii} = \frac{\partial^2 H}{\partial s_i^2} = \frac{x_1 \cdots x_m \chi'}{x_i^2 (\varphi_i')^2} \cdot \left(\frac{1}{D} - \frac{1}{D_i}\right)$ and $H_{ij} = \frac{\partial^2 H}{\partial s_j \partial s_i} = \frac{x_1 \cdots x_m \chi'}{x_i x_j \varphi_i' \varphi_j'} \frac{1}{D}$. The conditions (11) is explored on the k-th determinant:

$$(-1)^m \begin{vmatrix} \frac{x_1 \cdots x_m \chi'}{x_1^2 (\varphi_1')^2}\left(\frac{1}{D} - \frac{1}{D_1}\right) & \frac{x_1 \cdots x_m \chi'}{x_1 x_2 \varphi_1' \varphi_2'} \frac{1}{D} & \cdots & \frac{x_1 \cdots x_m \chi'}{x_1 x_m \varphi_1' \varphi_m'} \frac{1}{D} \\ \frac{x_1 \cdots x_m \chi'}{x_2 x_1 \varphi_2' \varphi_1'} \frac{1}{D} & \frac{x_1 \cdots x_m \chi'}{x_2^2 (\varphi_2')^2}\left(\frac{1}{D} - \frac{1}{D_2}\right) & \cdots & \frac{x_1 \cdots x_m \chi'}{x_2 x_m \varphi_2' \varphi_m'} \frac{1}{D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1 \cdots x_m \chi'}{x_m x_1 \varphi_m' \varphi_1'} \frac{1}{D} & \frac{x_1 \cdots x_m \chi'}{x_m x_2 \varphi_m' \varphi_2'} \frac{1}{D} & \cdots & \frac{x_1 \cdots x_m \chi'}{x_m^2 (\varphi_m')^2}\left(\frac{1}{D} - \frac{1}{D_m}\right) \end{vmatrix} > 0.$$

Elementary determinant transformations and simple algebra entails

$$(\chi')^m (x_1 \cdots x_m)^m \left(\frac{D}{DD_1 \cdots D_m} - \frac{D_1}{DD_1 \cdots D_m} - \frac{D_2}{DD_1 \cdots D_m} - \cdots - \frac{D_m}{DD_1 \cdots D_m}\right) > 0. \tag{18}$$

Discussing

To prove the opposite inequality in (17) it is enough to divide the left hand side of the previous condition (18) by $(-1)^m$ and here it is:

$$(-\chi')^m (x_1 \cdots x_m)^m \left(\frac{D}{DD_1 \cdots D_m} - \frac{D_1}{DD_1 \cdots D_m} - \frac{D_2}{DD_1 \cdots D_m} - \cdots - \frac{D_m}{DD_1 \cdots D_m}\right) > 0. \tag{19}$$

Since all $D_1, \dots, D_m$ and χ'' are negative, the sign of common denominator $DD_1 \cdots D_m$ is $(-1)^{m+1}$. In cumulative, it is $(-1)^{2m+1} = -1$ and the inequality in (18) would be opposite. It is equivalent with conditions that has to be proven. Exploring any smaller determinant in the Remark 2.1 gives the analogue. □

5. Minkowski and Hölder Inequality Types

Minkowsky and Hölder inequality are originally given in (Pečarić, et al., 1992). Defining a power mean generalization $M_{n,a}(\mathbf{x})_p := \left(\sum_{i=1}^n x_i^{a+p} / \sum_{i=1}^n x_i^p\right)^{\frac{1}{a}}$, author obtained a generalization of the Minkowski inequality in (Páles, 1982) and a generalization of the Hölder inequality in (Páles, 1983).

Well-known Minkowski inequality for non-negative n-tuples of real numbers is here enlarged for the case of several different potential means:

$$\left(\sum_{j=1}^n a_j x_{1j}^{\mu_1}\right)^{\frac{1}{\mu_1}} + \cdots + \left(\sum_{j=1}^n a_j x_{mj}^{\mu_m}\right)^{\frac{1}{\mu_m}} \geq \left(\sum_{j=1}^n a_i (x_{1j} + \cdots + x_{mj})^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{20}$$

According the (12), for $\mu_i, \lambda \neq 0$, there $m+1$ auxiliary functions are appearing:

$$F_i(x_i) = \frac{x_i}{\mu_i - 1}, i = 1, \ldots, m \text{ and } F(x_1 + \cdots + x_m) = \frac{x_1 + \cdots + x_m}{\lambda - 1}.$$

Proposition 5.1. *The inequality (20) holds if $\lambda < 1$ and all $\mu_i > 1,\ i = 1, \ldots, m$. If all $\mu_i, \lambda > 1$, the (20) holds if for every $j = 1, \ldots, n$:*

$$\frac{x_{1j} + \cdots + x_{mj}}{\lambda - 1} \geq \frac{x_{1j}}{\mu_1 - 1} + \cdots + \frac{x_{mj}}{\mu_m - 1}. \tag{21}$$

The inequality (21) holds if one of the two following conditions is fulfilled:

- *when $\mu_i > \lambda > 1$ for every $i = 1, \ldots, m$*
- *when the sequential queue $\mu_1 > \mu_2 > \cdots > \mu_k > \lambda > \mu_{k+1} > \cdots > \mu_m > 1$ is interrupted by λ as shown and for every $j = 1, \ldots, n$:*

$$\frac{\mu_1 - \lambda}{\mu_1 - 1} x_{1j} + \frac{\mu_2 - \lambda}{\mu_2 - 1} x_{2j} + \cdots + \frac{\mu_k - \lambda}{\mu_k - 1} x_{kj} > \frac{\lambda - \mu_{k+1}}{\mu_{k+1} - 1} x_{(k+1)j} + \cdots + \frac{\mu_m - \lambda}{\mu_m - 1} x_{mj}.$$

The inequality in (20) is opposite if $\lambda > 1$ and $\mu_i < 1$ for all $i = 1, \ldots, m$. If all $\mu_i, \lambda < 1$, the opposite inequality in (20) holds if

$$\frac{x_{1j} + \cdots + x_{mj}}{\lambda - 1} \leq \frac{x_{1j}}{\mu_1 - 1} + \cdots + \frac{x_m}{\mu_m - 1}. \tag{22}$$

The inequality (22) holds if one of the two followings is fulfilled:

- *when $\mu_i < \lambda$ for every $i = 1, \ldots, m$*
- *when the sequential queue $\mu_1 < \mu_2 < \cdots < \lambda < \mu < k + 1 < \cdots < \mu_m < 1$ is interrupted as shown and:*

$$\frac{\mu_1 - \lambda}{\mu_1 - 1} x_{1j} + \frac{\mu_2 - \lambda}{\mu_2 - 1} x_{2j} + \cdots + \frac{\mu_k - \lambda}{\mu_k - 1} x_{kj} < \frac{\lambda - \mu_{k+1}}{\mu_{k+1} - 1} x_{(k+1)j} + \cdots + \frac{\mu_m - \lambda}{\mu_m - 1} x_{mj}.$$

Proof. Apply Theorem 3.1 for the potential functions $\varphi_i(x_i) = x_i^{\mu_i}$. The statement follows immediately. □

Generalized Hölder inequality is presented in the article as the inequality:

$$\left(\sum_{j=1}^{n} a_j x_{1j}^{\mu_1}\right)^{\frac{1}{\mu_1}} \cdots \left(\sum_{j=1}^{n} a_j x_{mj}^{\mu_m}\right)^{\frac{1}{\mu_1}} \geq \left(\sum_{j=1}^{n} a_j (x_{1j} \cdots x_{mj})^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{23}$$

The suitable auxiliary functions are constants with given exponets as their values: $D_i(x_i) = \dfrac{1}{\mu_i}$ and $D(x_1 \cdots x_m) = \dfrac{1}{\lambda}$

Proposition 5.2. *The inequality (23) holds if $\lambda < 0$ and $\mu_i > 0$ for $i = 1, \ldots, m$. If all $\mu_i, \lambda > 0$, then the (23) holds if*

$$\frac{1}{\lambda} \geq \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_m}.$$

The inequality in (23) is opposite when $\lambda > 0$ and $\mu_i < 0$ for $i = 1, \ldots, m$. If all $\mu_i, \lambda < 0$ and

$$\frac{1}{\lambda} \leq \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_m},$$

the inequality in (23) is opposite too.

Proof. According the Theorem 4.1, statement of Proposition slides immediately. □

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