On Some Properties of $*-\text{annihilators}$ and $*-\text{maximal Ideals}$ in Rings with Involution

Maya A. Shatila

Correspondence: Maya A. Shatila, Department of Mathematics and Computer Science, Faculty of Science, Beirut Arab University, Beirut, Lebanon. E-mail: mayachatila@hotmail.com

Received: November 3, 2015    Accepted: November 19, 2015    Online Published: January 7, 2016

doi:10.5539/jmr.v8n1p1    URL: http://dx.doi.org/10.5539/jmr.v8n1p1

Abstract

We describe the $*-\text{right annihilator}$ ($*-\text{left annihilator}$) of a subset of a ring and we investigate the relationships between the right annihilator and $*-\text{right annihilator}$. These connections permit the transfer of various properties from annihilators to $*-\text{annihilators}$ . It is known that the quotient ring constructed from a ring and a maximal ideal is a field, whereas we prove that the quotient ring constructed from a ring and a $*-\text{maximal}$ ideal is not a $*-\text{field}$. Equivalent definitions to $*-\text{regular}$ ring are given.

Keywords: involution, $*-\text{annihilator}$, $*-\text{maximal ideal}$, $*-\text{regular ring}$

1. Introduction

A ring $A$ is said to be a ring with involution or simply $*-\text{ring}$ if there is a unary operation $*$: $A \rightarrow A$ such that for all $a, b \in A$ we have:

$$a^* = a, (ab)^* = b^*a^*, (a + b)^* = a^* + b^*$$

In this paper, only associative rings are considered. For more details concerning the ring with involution see (Rowen, 1988).

An ideal $I$ of an involution ring $A (I \triangleleft A)$ is called $*-\text{ideal}$ ($I \triangleright A$), if it is closed under involution; that is $I^* = I$. An involution $*$ of a $*-\text{ring}$ $R$ is said to be proper (semiprimitive) if $x^*x = 0 \ (x^*Rx = 0)$ implies $x = 0$ for every $x \in R$. In (Rowen, 1988), the right annihilator of $a \in A$, denoted by $r(a)$, is defined as $r(a) = \{b \in A | ab = 0\}$. Similarly, the left annihilator of $a$ is $l(a) = \{b \in A | ba = 0\}$.

A ring (resp. $*-\text{ring}$) $A$ is semiprime (resp. $*-\text{semiprime}$) if $I^2 = 0$ for every nonzero ideal (resp. $*-\text{ideal}$) $I$ of $A$. A ring $A$ is called reduced if it has no nonzero nilpotent elements ($a^n = 0$ for any $a \in A$ and positive integer $n$). (see (Berberian et al., 1988), (Rowen, 1988)). A ring $A$ is called regular if for every $a \in A, a \in aA$. Equivalently, every principal one-sided ideal of $A$ is generated by an idempotent (see (von Neuman, 1960)).

An element $e$ of $A$ is called idempotent (projection) if $e^2 = e$ (and $e^* = e$). Equivalently, $e = ee^*).

2. Properties of $*-\text{annihilators}$

Let $A$ be a ring with involution which does not necessary have identity. Recall that the right annihilator of a subset $S$ of $A$ is defined as $S^* = \{x \in A | Sx = 0\}$. Now, let $S$ be a non empty subset of the $*-\text{ring}$ $A$, define the $*-\text{of}$ $S$ to be the self adjoint subset $S_* = \{x \in A | Sx = 0 \text{ and } Sx^* = 0\}$. Similarly, the $*-\text{left annihilator}$ can be defined. It is clear that $S_* \subseteq S^*$. However the converse is not true as shown in the following example.

Example 1. Consider the ring $A$ of all $2 \times 2$ matrices rings over the real field $\mathbb{R}$, $M_2(\mathbb{R})$, with transpose of matrices as involution. Let $S = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} | a \in \mathbb{R} \right\}$, then $S^* = \left\{ \begin{pmatrix} b & c \\ -b & -c \end{pmatrix} | b, c \in \mathbb{R} \right\}$ and $S_* = \left\{ \begin{pmatrix} -t^2 & t \\ t & -t \end{pmatrix} | t \in \mathbb{R} \right\}$. It is clear that in this example the right annihilator of $S$ is not a $*-\text{right annihilator}$ of $S$.

In (Anderson et al., 1992), it is proved that the right annihilator of $S$ is a two sided ideal, a similar proof is given in the following proposition to show that the $*-\text{right annihilator}$ of a right ideal $S$ of $A$ is a $*-\text{ideal}$ of $A$.

Proposition 2. If $S$ is a right (resp. left) ideal of a $*-\text{ring}$ $A$, then the $*-\text{right annihilator}$ $S_*^*$ (resp. left) is a $*-\text{ideal}$ of $A$.

Proof. Let $x, y$ be two elements of the $*-\text{right annihilator}$ $S_*^*$, $a \in A$. Then $S(x - y) \subseteq Sx - Sy = 0$ and $S(x - y)^* \subseteq Sx^* - Sy^* = 0$. Also, $S(ax) = (S(a)x) \subseteq Sx = 0, S(ax)^* = S(x^*)a^* = 0$ and similarly $S(xa) = 0 = S(xa)^*$. □
The \textit{*-annihilator} of a non-empty subset \( S \) is defined by \( S_* = S^* \cap S^r \). If \( S \) is self adjoint, then it is clear that \( S_*^r = S_*^l = S_* \). The following is an immediate corollary of the previous proposition.

\textbf{Corollary 3.} If \( S \) is a \(*\)-ideal of \( A \), then \( S_*^r = S_*^l = S_* \) is also a \(*\)-ideal of \( A \).

Our main goal is to give some properties of \(*\)-annihilators.

\textbf{Theorem 4.} Let \( S, T \) be subsets of a ring \( A \), then:

1. \( S_*^r = (S^r)^*_* \)
2. \( S_*^l \cap T_*^l = (S \cup T)_*^l \)
3. \( (S \cup T)_*^l = S_*^l \cap T_*^l \)

\textit{Proof.} 1. Let \( x \in S_*^r \). Then \( ax = 0 \) and \( x^r S^r = 0 \) and \( Sx^* = 0 \) so \( x \in (S^r)^*_* \). Let \( x \in (S^r)^*_* \), \( xS^r = 0 \) and \( xS^r = 0 \) and \( Sx^r = 0 \). So, \( x \in S_*^r \) and \((S^*)_*^r \subseteq S_*^r \), therefore \( S_*^r = (S^r)^*_* \).

2. Let \( x \in S_*^r \). Then \( ax = 0 \) and \( ax^* = 0 \) for every \( a \in S \) then \( x \in (a)_*^l \) for every \( a \in S \) hence \( x \in \cap (a)_*^l \).

3. Let \( x \in (S \cup T)_*^l \). Then \( x \in S_*^l \cap T_*^l \). Let \( x \in S_*^r \cap T_*^r \). Then \( x \in S_*^l \cap T_*^l \). Let \( x \in S_*^r \) and \( x \in T_*^l \). Then \( x \in S_*^l \cap T_*^l \). Let \( x \in (S \cup T)_*^l \). Then \( x \in (S \cup T)_*^l \). Hence \( S_*^r = (S^r)^*_* \).

\textbf{Proposition 5.} If \( A \) is reduced then \( S_*^l = S_*^l \).

\textit{Proof.} Let \( x \in S_*^l \) then \( x = 0 \) and \( x^* = 0 \), \( yx = 0 \) and \( xy^* = 0 \) for every \( y \in S \), we also have \( (xy)^2 = xyxy = 0 \) and \( (xy)^2 = xy^* y = 0 \). But \( A \) is reduced then it has no non zero nilpotent element. Thus, \( xy = 0 \) and \( x^* y = 0 \) for every \( y \in S \). So, \( S \subseteq S_*^l \). Similarly, we get \( S_*^l \subseteq S_*^l \). Hence, \( S_*^l = S_*^l \).

\textbf{Proposition 6.} If \(*\) is a proper (semi proper) involution then \( S \cap S_*^l = 0 \).

\textit{Proof.} Let \( x \in S \cap S_*^l \). Then \( x = 0 \) and \( x^* = 0 \), \( yx = 0 \) and \( xy^* = 0 \) for every \( y \in S \), which implies that \( xS = 0 \) and \( x^* S = 0 \), but \( x \in S \) then \( x^2 = 0 \) and \( x^* x = 0 \). But \(*\) is a proper involution then \( x^* x = 0 \).\(x\) gives \( x = 0 \) (due to (Berberian, 1988)). Hence \( S \cap S_*^l = 0 \).

By a similar reasoning we obtain that \( S \cap S_*^l = 0 \) if \(*\) is a semi proper involution or if \( A \) is a reduced ring.

In general, for any subset \( S \) of \( A \), \( S \subseteq (S^r)^*_* \).

\textbf{Example 7.} \( S = \{ \frac{a}{t} \frac{b}{t}, t \in \mathbb{R} \} \).

\textit{Proof.} Let \( T = S_*^l \). Then \( S \subseteq T_*^l \). To show that \( S \subseteq T_*^l \) we need to show that \( ST = 0 \) and \( T^r S^r = 0 \) but \( S = S_*^l \) then it is enough to show \( ST = 0 \).

But \( T = S_*^l \) gives \( ST = 0 \) and \( ST^r = 0 \), hence \( ST = 0 \) and \( S \subseteq (S^r)_*^l \). Notice that if \( S = S_*^l \) then \( S_*^l = S_*^l \) and \( S \subseteq (S^r)_*^l \).

\textbf{Corollary 9.} If \( A \) is semiprime ring and \( S \triangleleft A \) then \( S_*^l = S_*^l \). (same reasoning as (Herstein), corollary 1, p.6)

\textbf{Corollary 10.} Every element of \( S_*^l \) is a \(*\)-zero divisor. (definition of \(*\)-zero divisor is given in (Anderson, et al., 2010))

\textit{Proof.} Let \( x \in S_*^l \) then \( Sx = 0 \) and \( Sx^* = 0 \) then there exist \( y \in S \) such that \( yx = 0 \) and \( yx^* = 0 \). Hence \( x \) is a \(*\)-zero divisor.

The converse is not true; not every \(*\)-zero divisor of a ring belongs to \( S_*^l \).

\textbf{Example 11.} \( R = A \oplus A^p \) with exchange involution \((a,b)^* = (b,a) \). \( A = Z_6, (2,0) \) is a \(*\)-zero divisor, \((2,0)(3,0) = (0,0) \) and \((2,0)(0,3) = (0,0) \), but \((2,0) \notin S_*^l \).

Since there exist \((1,3) \in S \) such that \((2,0)(1,3) = (0,0) \).
3. *-maximal Ideal

Motivated by a theorem in ring theory which said that an ideal I of a ring A is maximal if and only if the quotient ring A/I is a field, the involutive version will be shown in this section. Birkenmeier has defined *-prime ideal and *-maximal ideal in a ring with involution in (Birkenmeier et al., 1997), he showed that every prime (maximal) ideal is *-prime (*-maximal) ideal.

The ring A considered in this section is commutative.

Every maximal ideal of A is a *-maximal ideal of A but the converse is not true. Indeed, consider the ring R = Z_4 ⊕ Z_4 with exchange involution (a, b)^* = (b, a). I = {0, 2} is a maximal ideal of Z_4, then J = I ⊕ I is a *-maximal ideal of Z_4 ⊕ Z_4 under the exchange involution. But J is not maximal since it is contained in Z_4 ⊕ I.

Proposition 12. Let A be a *-ring, every *-maximal ideal of A is a *-prime ideal of A.

Proof. Let M be a *-maximal ideal of A. If M is a maximal ideal of A then M is a prime ideal and therefore M is a *-prime ideal of A. If M is not a maximal ideal K of A then there exists a maximal K of A such that: K + K^* = A and K ∩ K^* = M (see (Birkenmeier et al., 1997)). K is a maximal ideal of A then K is prime. So K is *-prime and K ∩ K^* is *-prime (see (Birkenmeier et al., 1997)). Then M is *-prime ideal of A.

Proposition 13. Let A be a commutative *- ring with identity and M ⊆ A. If the factor ring A/M is a *-field then M is a *-maximal ideal of A.

Proof. Let A/M is a *-field then A/M is a field then M is a maximal ideal of A and M is a *-maximal ideal of A.

The converse is not always true; the following example shows that if M is a *-maximal ideal of A then A/M is not a *-field.

Example 14. Let A = Z_4 ⊕ Z_4, M = I ⊕ I with I = {0, 2} is a *-maximal ideal of A under the exchange involution (a, b)^* = (b, a), but O ≠ (2, 1) ∈ A/M is not invertible for the reason that (2, 1) is a zero divisor (2, 1)(2, 0) = (0, 0) hence A/M is not a field and not a *-field.

Proposition 15. Every *-field is a *-integral domain.

Proof. Let A be a *-field with a, b and c are non zero elements in A such that ab = ac and a^*b = a^*c, a admits an inverse element a^{-1}, a^{-1}ab = a^{-1}ac and a^{-1}a^*b = a^{-1}a^*c then b = c and A is a *-integral domain since the cancellation property holds true.

4. *-regular Ring

Definition 16. Refer to (vonNeuman, 1960), A *-ring A is called *-regular, if every principal one-sided ideal of A is generated by a projection.

Theorem 17. For every *-ring A, the following statements are equivalent:

1. A is * - regular.
2. a ∈ A a^*a for every a ∈ A
3. a ∈ a^*A for every a ∈ A
4. a ∈ A a^*a ∩ a a^*A for every a ∈ A

Proof. (1) ⇒ (2) Let A be * - regular, then for every a ∈ A, aA = eA for some projection e of A. Hence a = ea and e = ar for some r ∈ A. Thus a = e^*a = r^*a^*A ∈ A a^*a.

(2) ⇒ (3) Let the condition be satisfied. Then for every a ∈ A, we have a^* ∈ A (a^*)^* (a^*) = Aaa^*. Take the involution, then a ∈ aa A.

(3) ⇒ (4) obvious

(4) ⇒ (1) we have a = xa^*a for some x ∈ A. But (xa^*)(xa^*)^* = xa^*ax^* implies (xa^*)(xa^*)^* = (xa^*) which means that xa^* is a projection. Then a = ea for some projection e of A implies aA = eA and hence A is * - regular.
References


Berberian, S. K. (1988). *Baer rings and baer *-rings*, University of Texas at Austin.


Copyrights
Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/3.0/).