Characterizations of Semigroups by Their Anti Fuzzy Ideals

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Abstract

In this paper we have defined anti fuzzy interior ideal in semigroups. We characterize regular, intra-regular and left (right) quasi-regular semigroups by the properties of their anti fuzzy ideals, anti fuzzy bi-ideals, anti fuzzy generalized bi-ideals, anti fuzzy interior ideals and anti fuzzy quasi-ideals.

Keywords: Anti fuzzy ideals, Regular, Intra-regular, Quasi-regular semigroups

Introduction

The fundamental concept of a fuzzy set was first introduced by L. A. Zadeh in 1965. A. Rosenfeld was the first who studied fuzzy sets in the structure of groups (Rosenfeld, A. 1971). The concept of fuzzy ideals in semigroups was developed by N. Kuroki (1979). On the other hand, the concept of anti fuzzy subgroups of groups are introduced by R. Biswas in (1990). S. M. Hong and Y. B. Jun (1998) defined anti fuzzy ideals in BCK-algebra. In (2009) M. Shabir and Y. Nawaz introduced the concept of anti fuzzy ideals in semigroups and characterized different classes of semigroups by the properties of their anti fuzzy ideals. Here in this paper, which is the continuation of the work carried out by M. Shabir and Y. Nawaz in (2009) for semigroups in terms of their anti fuzzy ideals, we characterize regular, intra-regular and left (right) quasi-regular semigroups by the properties of their anti fuzzy ideals. Some preliminaries are given below.

Let f and g be any fuzzy subsets of a semigroup S, then their anti product $f * g$ is defined by

$$(f * g)(a) = \begin{cases} \bigwedge\limits_{a=bc} \{f(b) \vee g(c)\}, & \text{if there exist } a, b \in S\text{, such that } a = bc \\ 1, & \text{otherwise.} \end{cases}$$

Let A be a subset of a semigroup S, then the characteristic function of the complement of A, that is C_{A^C} is defined by

$$C_{A^C}(x) = \begin{cases} 0, \text{ if } x \in A, \\ 1, \text{ if } x \notin A. \end{cases}$$

A fuzzy subset f of a semigroup S is called a anti fuzzy sub-semigroup of S if $f(xy) \leq f(x) \vee f(y)$ for all $x, y \in S$.

A fuzzy subset f of a semigroup S is called anti fuzzy left(right) ideal of S if $f(xy) \leq f(y)$ $(f(xy) \leq f(x))$ for all $x, y \in S$.

A fuzzy subset f of a semigroup S is called anti fuzzy ideal of S if it is both anti fuzzy left ideal and anti fuzzy right ideal of S.

A fuzzy subset f of a semigroup S is called anti fuzzy generalized bi-ideal of S if $f(xyz) \leq f(x) \vee f(z)$, for all x, y and $z \in S$.

A fuzzy sub-semigroup f of a semigroup S is called anti fuzzy bi-ideal of S if $f(xyz) \leq f(x) \vee f(z)$, for all x, y and $z \in S$.

A fuzzy subset f of a semigroup S is called anti fuzzy interior ideal of S if $f(xaz) \leq f(a)$, for all x, a and $y \in S$.

For any fuzzy subset f of a universe S and $t \in [0, 1]$ we define the set

$$L[f;\ t] = \{x \in S : f(x) \leq t\}$$

which is called the anti level cut of f.

It is obvious that every anti fuzzy left (right) ideal of a semigroup is an anti fuzzy quasi-ideal, every anti fuzzy quasi-ideal is an anti fuzzy bi-ideal and every anti fuzzy bi-ideal is an anti fuzzy generalized bi-ideal, but the converse is not true.

It has been proved in (Shabir, M. 2009) that a non-empty set A of a semigroup S is a left (right, bi, generalized bi, quasi) ideal of S if and only if the characteristic function of the compliment of A, that is, C_{A^C} is an anti fuzzy left (right, bi, generalized bi, quasi) ideal of S.

Theorem 1. *A fuzzy subset f of a semigroup S is an anti fuzzy interior ideal of S if and only if $L[f; t] \neq \phi$ is an interior ideal of S for all $t \in [0, 1]$.*

Proof. Let f be any an anti fuzzy interior ideal of S. Let $a \in L[f; t]$ and $x, y \in S$. Then $f(a) \leq t$ and hence $f(xay) \leq f(a) \leq t$. Thus $xay \in L[f; t]$ and so $L[f; t]$ is an interior ideal of S.

Conversely, assume that $L[f; t] \neq \phi$ is an interior ideal of S for all $t \in [0, 1]$. Suppose there exist $x, a, y \in S$ such that $f(xay) > f(a)$. Choose $t \in [0, 1]$ such that $f(xay) > t \geq f(a)$.Then $a \in L[f; t]$ but $xay \notin L[f; t]$, which is a contradiction, hence $f(xay) \leq f(a)$. Thus f is an anti fuzzy interior ideal of S.

Theorem 2. *A non-empty subset A of a semigroup S is an interior ideal of S if and only if the fuzzy subset f of S is defined by*

$$f(x) = \begin{cases} t \text{ if } x \in S - A \\ r \text{ if } x \in A \end{cases}$$

is an anti fuzzy interior ideal of S, where $t, r \in [0, 1]$ such that $t \geq r$.

Proof. Suppose A is an interior ideal of S and $x, a, y \in S$. If $a \in A$ then $xay \in A$, hence $f(xay) = f(a) = r$. If $a \notin A$ then $f(a) = t \geq f(xay)$. Hence f is an anti fuzzy interior ideal of S.

Conversely, assume that f is an anti fuzzy interior ideal of S. Let $a \in A$ and $x, y \in S$, then $f(xay) \leq f(a) = r$. This implies that $f(xay) \leq r$. Thus $xay \in A$ and hence A is an interior ideal of S.

Remark 1. *From above theorem we conclude that a non-empty subset A of a semigroup S is an interior ideal of S if and only if the characteristic function of the complement of A, that is, C_{A^C} is an anti fuzzy interior ideal of S.*

Example 1. *(Shabir, M. 2009) Consider the semigroup $S = \{a, b, c, d\}$.*

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Every fuzzy subset of S which contains a is an interior ideal of S but $\{a, c\}$ and $\{a, d\}$ are not two-sided ideals of S.

Let f be the fuzzy subset of S defined by

$f(a) = 0.3$, $f(b) = 0.9$, $f(c) = 0.5$, $f(d) = 0.7$. Then

$$L(f;\ t) = \begin{cases} \phi \text{ if } 0 \leq t < 0.3 \\ \{a\} \text{ if } 0.3 \leq t < 0.5 \\ \{a, c\} \text{ if } 0.5 \leq t < 0.7 \\ \{a, c, d\} \text{ if } 0.7 \leq t < 0.9 \\ \{a, b, c, d\} \text{ if } 0.9 \leq t < 1 \end{cases}$$

Thus by theorem 1, f is an anti fuzzy interior ideal of S but not an anti fuzzy two-sided ideal of S, because $\{a, c\}$ is not a two-sided ideal of S.Note that Θ can be considered as a fuzzy subset of a semigroup S such that $\Theta(x) = 0$, for all $x \in S$.

The proof of the following four theorems are available in (Shabir, M. 2009).

Theorem 3. *A fuzzy subset f of a semigroup S is an anti fuzzy sub-semigroup of S if and only if $f * f \supseteq f$.*

Theorem 4. *A fuzzy subset f of a semigroup S is an anti fuzzy left(right) ideal of S if and only if $\Theta * f \supseteq f$ ($f * \Theta \supseteq f$).*

Theorem 5. *A fuzzy subset f of a semigroup S is an anti fuzzy generalized bi-ideal of S if and only if $f * \Theta * f \supseteq f$.*

Theorem 6. *A fuzzy subset f of a semigroup S is an anti fuzzy bi-ideal of S if and only if $f * f \supseteq f$ and $f * \Theta * f \supseteq f$.*

Theorem 7. *A fuzzy subset f of a semigroup S is an anti fuzzy interior ideal of S if and only if $\Theta * f * \Theta \supseteq f$.*

Proof. Let f is an anti fuzzy interior ideal of S and $p \in S$ such that $p = xay$. Then

$$\begin{aligned}(\Theta * f * \Theta)(p) &= \bigwedge_{p=xay} \{\Theta(x) \vee f(a) \vee \Theta(y)\} \\ &= \bigwedge_{p=xay} \{0 \vee f(a) \vee 0\} \\ &= \bigwedge_{p=xay} f(a) \geq f(xay) = f(p).\end{aligned}$$

Therefore $(\Theta * f * \Theta) \supseteq f$. If $p \neq xay$. Then

$$(\Theta * f * \Theta)(p) = 1 \geq f(p).$$

Hence $\Theta * f * \Theta \supseteq f$.

Conversely, assume that $\Theta * f * \Theta \supseteq f$. Let $x, a, y \in S$. Then

$$\begin{aligned}f(xay) &\leq (\Theta * f * \Theta)(xay) = \bigwedge_{xay=xay} \{\Theta(x) \vee f(a) \vee \Theta(y)\} \\ &= \bigwedge_{xay=xay} \{0 \vee f(a) \vee 0\} = \bigwedge_{xay=xay} \{f(a)\} \leq f(a).\end{aligned}$$

Hence f is an anti fuzzy interior ideal of S.

Lemma 1. *Let A and B be any non-empty subsets of a semigroup S, then $C_{A^C} * C_{B^C} = C_{A^C B^C}$.*

Proof. It is straight forward. ∎

A fuzzy subset f of a semigroup S is a anti fuzzy quasi-ideal of S if $(f * \Theta) \cup (\Theta * f) \supseteq f$.

Lemma 2. *Let f be an anti fuzzy right ideal and g be an anti fuzzy left ideal of a semigroup S. Then $f \cup g$ is an anti fuzzy quasi-ideal of S.*

Proof. Let f be an anti fuzzy right ideal and g be an anti fuzzy left ideal of S. Then

$$((f \cup g) * \Theta) \cup (\Theta * (f \cup g)) \geq (f * \Theta) \cup (\Theta * g) \geq f \cup g.$$

Thus $f \cup g$ is an anti fuzzy quasi-ideal of S.

Lemma 3. *Let S a semigroup, then the given properties hold.*

(1) *The union $f \cup g$ of two anti fuzzy sub-semigroups f and g of S, is also an anti fuzzy sub-semigroup of S.*

(2)*The union $f \cup g$ of two anti fuzzy left(right, two-sided) ideals f and g of S, is also an anti fuzzy left(right, two-sided) ideal of S.*

Proof. (1) Let f and g be any anti fuzzy sub-semigroups of S and $x, y \in S$. Then

$$\begin{aligned}(f \cup g)(xy) &= f(xy) \vee g(xy) \leq (f(x) \vee f(y)) \vee (g(x) \vee g(y)) \\ &= (f(x) \vee g(x)) \vee (f(y) \vee g(y)) \\ &= (f \cup g)(x) \vee (f \cup g)(y).\end{aligned}$$

Hence $f \cup g$ is an anti fuzzy sub-semigroup of S.

(2) Let f and g be any anti fuzzy left ideals of S and $x, y \in S$. Then

$$(f \cup g)(xy) = f(xy) \vee g(xy) \leq f(y) \vee g(y) = (f \cup g)(y).$$

Hence $f \cup g$ is an anti fuzzy left ideal of S.

Similarly if f and g are any anti fuzzy right ideals of S and $x, y \in S$. Then

$$(f \cup g)(xy) = f(xy) \vee g(xy) \leq f(x) \vee g(x) = (f \vee g)(x).$$

Hence $f \cup g$ is an anti fuzzy right ideal of S.

Now if f and g are anti fuzzy ideals of S, then $f \cup g$ is obviously an anti fuzzy ideal by above proof.

Theorem 8. *Let f and g be anti fuzzy bi-ideals (generalized bi-ideal) of a semigroup S. Then $f \cup g$ is an anti fuzzy bi-ideal (generalized bi-ideal) of S.*

Proof. Let f and g be anti fuzzy bi-ideals of S and $x, y, z \in S$. Then

$$\begin{aligned}(f \cup g)(xy) &= f(xy) \vee g(xy) \\ &\leq (f(x) \vee f(y)) \vee (g(x) \vee g(y)) \\ &= (f(x) \vee g(x)) \vee (f(y) \vee g(y)) \\ &= (f \cup g)(x) \vee (f \cup g)(y).\end{aligned}$$

Also,

$$\begin{aligned}(f \cup g)(xyz) &= f(xyz) \vee g(xyz) \\ &\leq (f(x) \vee f(z)) \vee (g(x) \vee g(z)) \\ &= (f(x) \vee g(x)) \vee (f(z) \vee g(z)) \\ &= (f \cup g)(x) \vee (f \cup g)(y).\end{aligned}$$

Hence $f \cup g$ is an anti fuzzy bi-ideal of S.

Lemma 4. *Let f and g be any anti fuzzy interior ideals of a semigroup S, then $f \cup g$ is also an anti fuzzy interior ideal of S.*

Proof. Let x, a and y be any arbitrary elements of S, then

$$(f \cup g)(xay) = f(xay) \vee g(xay) \leq f(a) \vee g(a) = (f \cup g)(a).$$

Hence $f \cup g$ is an anti fuzzy interior ideal of S.

Now we characterized regular semigroups by the properties their anti fuzzy ideals.

A semigroup S is called regular if for every element $a \in S$, there exists an element $x \in S$ such that $a = axa$ or, equivalently $a \in aSa$ for every $a \in S$.

Lemma 5. *A fuzzy subset f of a regular semigroup S is an anti fuzzy two-ideal of S if and only if it is an anti fuzzy interior ideal of S.*

Proof. Let f be an anti fuzzy two-sided ideal of S, then obviously f is an anti fuzzy interior ideal of S.

Conversely, assume that f is an anti fuzzy interior ideal of S. Let $a, b \in S$, since S is regular so there exist elements x, $y \in S$ such that $a = axa$ and $b = byb$, we have

$$f(ab) = f((axa)b) = f((ax)ab) \leq f(a) \text{ and}$$

$$f(ab) = f(a(byb)) = f(ab(yb)) \leq f(b).$$

Hence f is anti fuzzy two-ideal of S.

Lemma 6. *In a regular semigroup S, every anti fuzzy two-sided ideal is idempotent.*

Proof. Let f be an anti fuzzy two-sided ideal of a semigroup S, then obviously $f * f \supseteq f$. Since S is regular, so for each element $a \in S$ there exists an element $x \in S$ such that $a = axa$, we have

$$(f * f)(a) = \bigwedge_{a=axa} \{f(ax) \vee f(a)\} \leq f(ax) \vee f(a) \leq f(a) \wedge f(a) = f(a).$$

Hence $f * f = f$. ■

Corollary 1. *In a regular semigroup S, every anti fuzzy right ideal is idempotent.*

Lemma 7. *In a regular semigroup S, every anti fuzzy interior ideal is idempotent.*

Proof. Assume that f is an anti fuzzy interior ideal of a semigroup S, then obviously $f * f \supseteq f$. Let a and b be an arbitrary elements of S. Since S is regular so for each a in S there exists an element x in S such that $a = axa$, we have

$$a = axa = axaxa = ((ax)ax)a$$

$$\begin{aligned}(f \circ f)(a) &= \bigwedge_{a=((ax)ax)a} \{f((ax)ax) \vee f(a)\} \\ &\leq f((ax)ax) \vee f(a) \\ &\leq f(a) \vee f(a) = f(a).\end{aligned}$$

Hence $f * f = f$. ■

Lemma 8. *If f and g are any anti fuzzy two-sided ideals of a regular semigroup S, then $f * g = f \cup g$.*

Proof. Let f and g be any fuzzy two-sided ideals of S, then obviously $f * g \supseteq f \cup g$. Since S is regular so for each element a in S there exists an element x in S such that $a = axa$, we have

$$(f * g)(a) = \bigwedge_{a=axa} \{f(ax) \vee g(a)\} \leq f(ax) \vee g(a) \leq f(a) \vee g(a) = (f \cup g)(a).$$

Hence $f \circ g = f \cup g$. ■

Lemma 9. *In a regular semigroup S, for every anti fuzzy bi-ideal (generalized bi-ideal) f of S, we have $f * \Theta * f = f$.*

Proof. Let f be a fuzzy bi-ideal of S, then obviously $f * \Theta * f \supseteq f$.

Let a be an arbitrary element of S. Since S is regular so there exists an element x in S such that $a = axa$, we have

$$\begin{aligned}(f * \Theta * f)(a) &= \bigwedge_{a=axa} \{(f * \Theta)(ax) \vee f(a)\} \leq (f * \Theta)(ax) \vee f(a) \\ &= \left(\bigwedge_{a=ax} \{f(a) \vee \Theta(x)\}\right) \vee f(a) \leq (f(a) \vee \Theta(x)) \vee f(a) \\ &= (f(a) \vee 0) \vee f(a) = f(a),\end{aligned}$$

which implies that $f * \Theta * f \subseteq f$. Hence $f * \Theta * f = f$.

Theorem 9. *The set of anti fuzzy ideals of a regular semigroup S forms a semilattice structure.*

Proof. It follows from lemma 8. ■

A anti fuzzy ideal f of a semigroup S is said to be strongly irreducible if and only if for anti fuzzy ideals g and h of S, $g \cup h \supseteq f$ implies that $g \supseteq f$ or $h \supseteq f$.

The set of anti fuzzy ideals of a semigroup S is called totally ordered under inclusion if for any anti fuzzy ideals f and g of S either $f \supseteq g$ or $g \supseteq f$.

A anti fuzzy ideal h of a semigroup S is called anti fuzzy prime ideal of S, if for any anti fuzzy ideals f and g of S, $f * g \supseteq h$, implies that $f \supseteq h$ or $g \supseteq h$.

Theorem 10. *In a regular semigroup S, an anti fuzzy ideal is strongly irreducible if and only if it is anti fuzzy prime.*

Proof. It follows from lemma 8. ■

Theorem 11. *Every anti fuzzy ideal of a regular semigroup S is anti fuzzy prime if and only if the set of anti fuzzy ideals of S is totally ordered under inclusion.*

Proof. It follows from lemma 8. ■

The proof of the following two theorems are available in (Shabir, M. 2009).

Theorem 12. *A semigroup S is regular if and only if $(f \cup g) = (f * g)$ for every anti fuzzy right ideal f and every anti fuzzy left ideal g of S.*

Theorem 13. *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.

(2) $f = f * \Theta * f$, for every anti fuzzy quasi-ideal f of S.

(3) $f = f * \Theta * f$, for every anti fuzzy generalized bi-ideal f of S.

Theorem 14. *For a semigroup S,the following conditions are equivalent.*

(1) S is regular.

(2) $f \cup g \supseteq f * g$, for every anti fuzzy right ideal f and every anti fuzzy quasi-ideal g of S.

(3) $f \cup g \supseteq f * g$, for every anti fuzzy right ideal f and every anti fuzzy bi-ideal g of S.

(4) $f \cup g \supseteq f * g$, for every anti fuzzy right ideal f and every anti fuzzy generalized bi-ideal g of S.

Proof. (1) $\Rightarrow$ (4)

Let f be any anti fuzzy right ideal and g be any anti fuzzy generalized bi-ideal of S. Let $a \in S$, then since S is regular so there exists $x \in S$ such that $a = axa$, we have

$$\begin{aligned}(f * g)(a) &= \vee_{a=axa}\{f(ax) \vee g(a)\} \\ &\leq f(ax) \vee g(a) \\ &\leq f(a) \vee g(a) \\ &= (f \cup g)(a),\end{aligned}$$

and so $f * g \subseteq f \cup g$. It is clear that (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1)

Let f be any anti fuzzy right ideal and g be any anti fuzzy left ideal of S, then obviously $f * g \supseteq f \cup g$. Since every anti fuzzy left ideal is an anti fuzzy quasi-ideal of S. So by assumption, we have $f * g \subseteq f \cup g$. Therefore $f * g = f \cup g$ for every anti fuzzy right ideal f and every anti fuzzy left ideal g of S. Hence by theorem 12, S is regular. ∎

Theorem 15. *For a semigroup S,the following conditions are equivalent.*

(1) S is regular.

(2) $f \cup g = f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy two sided ideal g of S.

(3) $f \cup g = f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy interior ideal g of S.

(4) $f \cup g = f * g * f$, for every anti fuzzy bi-ideal f and every anti fuzzy two sided ideal g of S.

(5) $f \cup g = f * g * f$, for every anti fuzzy bi-ideal f and every anti fuzzy interior ideal g of S.

(6) $f \cup g = f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy two sided ideal g of S.

(7) $f \cup g = f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy interior ideal g of S.

Proof. (1) $\Rightarrow$ (7)

Let f be any anti fuzzy generalized bi-ideal and g be any anti fuzzy interior ideal of S. Then $f * g * f \supseteq f * \Theta * f \supseteq f$ and $f * g * f \supseteq \Theta * g * \Theta \supseteq g$ which implies that $f * g * f \supseteq f \cup g$. Let $a \in S$, since S is regular so there exists $x \in S$ such that $a = axa$, we have

$$a = axa = axaxa = a(xaxa). \text{ Therefore}$$

$$\begin{aligned}(f * g * f)(a) &= \bigwedge_{a=a(xaxa)} \{f(a) \vee (g * f)(xaxa)\} \\ &\leq f(a) \vee (g * f)(xaxa) \\ &= f(a) \vee \left(\bigwedge_{a=xaxa} \{g(xax) \vee f(a)\} \right) \\ &\leq f(a) \vee \{g(xax) \vee f(a)\} \\ &\leq f(a) \vee g(a) \vee f(a) \\ &= f(a) \vee g(a) = (f \cup g)(a),\end{aligned}$$

and so $f * g * f \subseteq f \cup g$. Hence $f * g * f = f \cup g$.

It is clear that (7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2)

(2) $\Rightarrow$ (1)

Let f be any anti fuzzy quasi-ideal of S. Then, since Θ is itself an anti fuzzy two sided ideal of S. So by assumption, we have

$$f = f \cup \Theta = f * \Theta * f.$$

Hence by theorem 13, S is regular. ■

Now we characterized intra-regular semigroups by the properties of their anti fuzzy ideals.

An element a of a semigroup S is called intra-regular if there exist elements $x, y \in S$ such that $a = xa^2y$ and S is called intra-regular if every element of S is intra-regular.

The proof of the following theorem is available in (Shabir, M. 2009).

Theorem 16. *A semigroup S is intra-regular if and only if $(f \cup g) \supseteq (f * g)$, for every anti fuzzy left ideal f and every anti fuzzy right ideal g of S.*

Theorem 17. *For a semigroup S, the following conditions are equivalent.*

(1) S is regular and intra-regular.

(2) $f \cup g \supseteq f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy left ideal g of S.

(3) $f \cup g \supseteq f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy right ideal g of S.

(4) $f \cup g \supseteq f * g * f$, for all anti fuzzy quasi-ideals f and g of S.

(5) $f \cup g \supseteq f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy bi-ideal g of S.

(6) $f \cup g \supseteq f * g * f$, for every anti fuzzy quasi-ideal f and every anti fuzzy generalized bi-ideal g of S.

(7) $f \cup g \supseteq f * g * f$, for every anti fuzzy bi-ideal f and every anti fuzzy left ideal g of S.

(8) $f \cup g \supseteq f * g * f$, for every anti fuzzy bi-ideal f and every fuzzy right ideal g of S.

(9) $f \cup g \supseteq f * g * f$, for every anti fuzzy bi-ideal f and every anti fuzzy quasi-ideal g of S.

(10) $f \cup g \supseteq f * g * f$, for all anti fuzzy bi-ideals f and g of S.

(11) $f \cup g \supseteq f * g * f$, for every anti fuzzy bi-ideal f and every fuzzy generalized bi-ideal g of S.

(12) $f \cup g \supseteq f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy left ideal g of S.

(13) $f \cup g \supseteq f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy right ideal g of S.

(14) $f \cup g \supseteq f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy quasi-ideal g of S.

(15) $f \cup g \supseteq f * g * f$, for every anti fuzzy generalized bi-ideal f and every anti fuzzy bi-ideal g of S.

(16) $f \cup g \supseteq f * g * f$, for all anti fuzzy generalized bi-ideals f and g of S.

Proof. (1) $\Rightarrow$ (16)

Let f and g be any anti fuzzy generalized bi-ideals of S and $a \in S$. Then, since S is regular so there exists $x \in S$ such that $a = axa$. Since S is intra-regular so there exist $y, z \in S$ such that $a = ya^2z$. Thus we have

$$a = axa = axaxaxa = ax(yaaz)x(yaaz)xa = (axya)(azxya)(azxa).$$

Therefore, we have

$$\begin{aligned}
(f * g * f)(a) &= \bigwedge_{a=(axya)(azxya)(azxa)} \{f(axya) \vee (f * g)((azxya)(azxa))\} \\
&\leq f(axya) \vee (f * g)((azxya)(azxa)) \\
&\leq f(axya) \vee \left(\bigwedge_{a=(azxya)(azxa)} g(azxya) \vee f(azxa) \right) \\
&\leq f(axya) \vee g(azxya) \vee f(azxa) \\
&\leq f(a) \vee g(a) \vee f(a) = (f \cup g)(a),
\end{aligned}$$

and so $f \cup g \supseteq f * g * f$.

It is clear that (11) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3), (16) $\Rightarrow$ (15) $\Rightarrow$ (14) $\Rightarrow$ (13) $\Rightarrow$ (8) $\Rightarrow$ (3), (14) $\Rightarrow$ (12) $\Rightarrow$ (7) $\Rightarrow$ (2) and (16) $\Rightarrow$ (11) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (8).

(3) $\Rightarrow$ (2)

Let f be any anti fuzzy quasi-ideal of S. Then, since Θ is itself an anti fuzzy right ideal of S, we have

$$f = f \cup \Theta = f * \Theta * f.$$

Hence by theorem 13, S is regular. Let f be any anti fuzzy left ideal and g be any anti fuzzy right ideal of S. Then, since f is an anti fuzzy quasi-ideal of S, we have

$$f \cup g \supseteq f * g * f \supseteq f * (g * \Theta) \supseteq f * g.$$

Hence by theorem 16, S is intra-regular.

Finally we characterized left (right) quasi-regular semigroups by the properties their anti fuzzy ideals.

A semigroup S is called left (right) quasi-regular, if every left (right) ideal of S is idempotent and S is called quasi-regular if every left ideal and right ideal of S are idempotent. It is easy to prove that S is left (right) quasi-regular if and only if $a \in SaSa$ ($a \in aSaS$), this implies that there exist elements $x, y \in S$ such that $a = xaya$ ($a = axay$).

Theorem 18. *A semigroup S is left (right) quasi-regular if and only if every anti fuzzy left (right) ideal of S is idempotent.*

Proof. Assume that f be any anti fuzzy left ideal of a left quasi-regular semigroup S, then obviously $f * f \supseteq f$. Let $a \in S$, then there exist elements $x, y \in S$ such that $a = xaya$, we have

$$\begin{aligned}(f * f)(a) &= \bigwedge_{a=xaya} \{f(xa) \vee f(ya)\} \leq \{f(xa) \vee f(ya)\} \\ &\leq f(a) \vee f(a) = f(a),\end{aligned}$$

and so $f * f \subseteq f$. Hence f is idempotent.

Conversely, assume that every anti fuzzy left ideal of S is idempotent. Let $a \in S$, then since $L[a]$ is a principal left ideal of S, then $C_{L[a]^c}$ is anti fuzzy left ideal of S, therefore by assumption and lemma 1, we have

$$C_{L[a]^c L[a]^c}(a) = (C_{L[a]^c} * C_{L[a]^c})(a) = C_{L[a]^c}(a) = 0,$$

which implies that

$$\begin{aligned} a &\in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) \\ &= \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq SaSa.\end{aligned}$$

Hence S is left quasi-regular. ∎

Theorem 19. *For a semigroup S, the following conditions are equivalent.*

(1) S is quasi regular.

(2) $f = (f * S)^2 \cup (S * f)^2$, for every anti fuzzy quasi-ideal f of S.

Proof. (1) $\Rightarrow$ (2)

Let f be any anti fuzzy quasi-ideal of S. Since S is quasi-regular, so by theorem 18, every anti fuzzy right ideal $f * \Theta$ and every anti fuzzy left ideal $\Theta * f$ of S are idempotent. Hence

$$(f * \Theta)^2 \cup (\Theta * f)^2 = (f * \Theta) \cup (\Theta * f) \supseteq f \cup f = f.$$

Let $a \in S$, since S is left quasi regular, there exist elements $x, y \in S$, such that $a = xaya$, we have

$$\begin{aligned}(\Theta * f)^2(a) &= \bigwedge_{a=xaya} \{(\Theta * f)(xa) \vee (\Theta * f)(ya)\} \leq (\Theta * f)(xa) \vee (\Theta * f)(ya) \\ &= \left(\bigwedge_{a=xa} \{\Theta(x) \vee f(a)\}\right) \vee \left(\bigwedge_{a=ya} \{\Theta(y) \vee f(a)\}\right) \\ &\leq (\Theta(x) \vee f(a)) \vee (\Theta(y) \vee f(a)) \\ &= (0 \vee f(a)) \vee (0 \vee f(a)) = f(a).\end{aligned}$$

Therefore $(\Theta \circ f)^2 \subseteq f$. Similarly we can prove that $(f * \Theta)^2 \subseteq f$. Thus

$$(f * \Theta)^2 \cup (\Theta \circ f)^2 \subseteq f.$$

Hence $(f * \Theta)^2 \cup (\Theta \circ f)^2 = f$.

Now (2) $\Rightarrow$ (1)

Let f be any anti fuzzy left ideal of S. Since every anti fuzzy left of S is an anti fuzzy quasi-ideal of S, we have

$$f = (f * \Theta)^2 \cup (\Theta * f)^2 \supseteq (\Theta * f)^2 \supseteq f^2 \supseteq f * \Theta \supseteq f,$$

and so $f = f^2$. Hence by theorem 18, S is a left quasi-regular.

Theorem 20. *For a semigroup S, the following conditions are equivalent.*

(1) S is both intra-regular and left quasi-regular.

(2) $g \cup h \cup f \supseteq g * h * f$, for every anti fuzzy quasi-ideal f, every anti fuzzy left ideal g and every anti fuzzy right ideal h of S.

(3) $g \cup h \cup f \supseteq g * h * f$, for every anti fuzzy bi-ideal f, every anti fuzzy left ideal g and every anti fuzzy right ideal h of S.

(4) $g \cup h \cup f \supseteq g * h * f$, for every anti fuzzy generalized bi-ideal f, every anti fuzzy left ideal g and every anti fuzzy right ideal h of S.

Proof. (1) $\Rightarrow$ (4)

Let f be anti fuzzy generalized bi-ideal, g be any anti fuzzy left ideal and h be any anti fuzzy right ideal of S. Let the element $a \in S$. Since S is intra-regular so there exist elements $x, y \in S$ such that $a = xa^2y$. Since S is left quasi-regular so there exist elements $u, v \in S$ such that $a = uava$, we have

$$a = uava = u(xaay)va = ((ux)a)((a(yv))a).$$

Therefore,

$$\begin{aligned}
(g * h * f)(a) &= \bigwedge_{a=((ux)a)((a(yv))a)} \{g((ux)a) \vee (h * f)((a(yv))a)\} \\
&\leq g((ux)a) \vee (h * f)((a(yv))a) \\
&\leq g(a) \vee \left(\bigwedge_{a=(a(yv))a} \{h(a(yv)) \vee f(a)\} \right) \\
&\leq g(a) \vee (h(a(yv)) \vee f(a)) \\
&\leq g(a) \vee h(a) \vee f(a) = (g \cup h \cup f)(a),
\end{aligned}$$

and so $g * h * f \subseteq g \cup h \cup f$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1)

Let f be any anti right ideal and g be any anti fuzzy left ideal of S. Since every anti fuzzy left ideal g is an anti fuzzy quasi-ideal of S and Θ is itself an anti fuzzy right ideal of S, we have

$$g = g \cup \Theta \cup g \supseteq g * \Theta * g \supseteq g * g \supseteq g * \Theta \supseteq g,$$

and so $g = g * g$. Hence by theorem 18, S is a left quasi-regular.

Now since anti fuzzy right ideal f is an anti fuzzy quasi-ideal of S and since Θ itself an anti fuzzy right ideal of S, we have

$$g \cup f = g \cup \Theta \cup f = g * \Theta * f \supseteq g * f.$$

Hence by theorem 16, S is intra-regular.

References

Biswas, R. (1990). Fuzzy subgroups and anti fuzzy subgroups. *Fuzzy sets and systems*, 35, 121 − 124.

Hong, S. M. & Jun, Y. B. (1998). Anti fuzzy ideals in BCK-algebra. *Kyungpook Math. J.,* 38, 145 − 150.

Kuroki, N.(1979). Fuzzy bi-ideals in semigroups. Comment. *Math. Univ. St. Pauli*, 27, 17 − 21.

Mordeson, J. N., Malik, D. S. and Kuroki, N. (2003). *Fuzzy semigroups*. Springer-Verlag, Berlin Germany.

Rosenfeld, A. (1971). Fuzzy groups. *J. Math. Anal. Appl.*, 35, 512 − 517.

Shabir, M. and Nawaz, Y. (2009). Semigroups characterized by the properties of their anti fuzzy ideals. *Journal of advanced Research in Pure Mathematics*, 3, 42 − 59.

Zadeh, L. A. (1965). Fuzzy sets. *Inform. Control*, 8, 338 − 353.