

Application of Meshless Natural Neighbour Petrov-Galerkin Method in Temperature Field

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Abstract

In Meshless natural neighbour Petrov-Galerkin method, The natural neighbour interpolation is used as trial function and a weak form over the local polygonal sub-domains constructed by Delaunay triangular is used to obtain the discretized system of equilibrium equations, and it's a new truly meshless method. This method simplified the formation of the equilibrium equations, facilitates the imposition of essential boundary conditions and the system stiffness matrix in the present method is banded and sparse. Efforts are made to study Meshless natural neighbour Pettrov-Galerkin Method, which is extended to solve the transient heat conduction. The numerical results show that the present method is quite accurate and stable.

Keywords: Meshless local Petrov-Galerkin method, Natural neighbour interpolation, The transient heat conduction

1. Introduction

Meshless local Pettrov-Galerkin (MLPG) Method is a truly meshless method, as it does not need background meshes. For the conventional Galerkin method, the trial and the test function are chosen from the same function space, while for the MLPG method, which is actually a method of local weighted residual, the trial and the test function are chosen from different function space. Atluri has listed six different kinds of MLPG methods named MLPG1-MLPG6 by varying the weight function. In the MLPG implementation, Moving Least Square (MLS) approximation is employed for constructing shape functions. There is an issue of imposition of essential boundary conditions as the shape function does not have Kronecker delta function property. In addition, the major drawback of MLPG is the asymmetry of the system matrices due to the use of the Petrov-Galerkin formulation. Another drawback of MLPG is that the local background integration can be very tricky due to the complexity of the integrand produced by the Petrov-Galerkin approach, especially for domains that intersect with the boundary of the problem domain. All these drawbacks limit the development of the MLPG. In recent years, more and more attention has been paid to natural neighbour interpolation. In this paper, an attempt is made to combine the advantages of the MLPG with the ability of easy imposition of essential boundary condition of the natural element method. A meshless method coined as natural neighbour Petrov-Galerkin method (MNNPG) is derived from the generalized meshless local Petrov-Galerkin method to solve the transient heat conduction problem. In this method, the problem domain and the boundary are discretized by the scattered nodes, and the Voronoi diagram based non-Sibsonian interpolation and is more efficient in computation of the shape functions. The 3-nodal triangular FEM shape functions are used as weight functions in each local sub-domain. The local sub-domains are constructed with Delaunay tessellations, and the Petrov-Galerkin method is used to get the discrete system equations. In included triangular regions of the subdomain, the Gauss quadrature scheme is used to evaluate the domain integrals. Efforts are made to study Meshless natural neighbour Pettrov-Galerkin Method, which is extended to solve the transient heat conduction. The numerical results show that the present method is quite accurate and stable.

2. Natural neighbour interpolation

Natural neighbour interpolation is one kind of multivariate interpolation scheme, and Sibsonian interpolation and non-Sibsonian interpolation have been used in meshless method at present. In this paper, the approximation is constructed according to the natural neighbours of the evaluated point using non-Sibsonian interpolation, which is based on the wellknown Voronoi diagram and Delaunay tessellations. The problem domain is denoted by Ω and set *N* is a partition of plane into regions T_i , where each region T_i is associated with a node*i*, such that any point in T_i is closer to node*i* than to any other nodes, in mathematical form:

$$T_i = \{ x \in R^2 : d(x, x_i) < d(x, x_j) \forall j \neq i \}$$
(1)

Where $d(x, x_i)$ is the distance between x and x_i .

Insert < Figure 1 > here

The important property of the Delaunay triangulation is the empty circumcircle criterion: the circumcircle of the Delaunay triangle contains no other nodes inside it. This criterion is used to find the natural neighbours of a point x (like the Gauss points), if the point x lies within the circumcircle of a Delaunay triangle. The non-Sibsonian interpolation is based on the Voronoi diagram of the evaluated point. To construct the Voronoi diagram, the natural neighbours of the evaluated point x is determined by the circumcircle criterion. The non-Sibsonian interpolation of nodei is calculated by:

$$\phi_{i} = \frac{S_{i}(x)/h_{i}(x)}{\sum_{j=1}^{n} S_{j}(x)/h_{j}(x)}$$
(2)

Where S_i is the Lebesgue measure (length in R^2) of the Voronoi boundary associated with node *i*, and the h_i is the distance between the evaluated point *x* and the node *i*, is the number of natural neighbours of the point *x*. In order to calculate S_i , the centers of the circumcircles of the triangles defined by point *x*, node *i* and another related natural neighbours of the evaluated point *x* should be worked out. The non-Sibsonian interpolation shape functions $\phi_i(x)$ have many properties, among which the main and crucial properties to the meshless method are:

$$0 \le \phi_i(x) \le 1, \ \phi_i(x_j) = \delta_{ij} \tag{3}$$

$$\sum_{i=1}^{n} \phi_i(x) = 1, \ x = \sum_{i=1}^{n} \phi_i(x) x_i \tag{4}$$

The Eq. (3) indicates that the shape function has Kronecker Delta function property and thus the approximation passes through the nodal values, so that the essential boundary condition can be imposed as directly as what it is done in FEM. The Eq. (4) defines a partition of unity and linear completeness, which indicates the shape function can exactly reproduce the constant and linear functions. This is important for the meshless method to get a proper convergence rate. Furthermore, the implementation of the non-Sibsonian interpolation is more efficient because the evaluation of Lebesgue measure is one dimension less than that of the Sibson interpolation. All these properties render the non-Sibsonian interpolation an attractive approximation scheme in meshless methods.

3. MLPG method for heat conduction problem

Transient heat conduction problem dependentes on time. To two-dimensional transient heat conduction problem, the temperature function is denoted by T(x, y, t), according to heat conduction theory, the Transient heat conduction's Poisson equation and boundary conditions can be written as:

$$\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} (k_x \frac{\partial T}{\partial x}) - \frac{\partial}{\partial y} (K_y \frac{\partial T}{\partial y}) - \rho Q = 0$$

$$T = \overline{T} \qquad \text{on } \Gamma_1$$

$$k_x \frac{\partial T}{\partial x} n_x - k_y \frac{\partial T}{\partial y} n_y = q \qquad \text{on } \Gamma_2$$

$$k_x \frac{\partial T}{\partial x} n_x - k_y \frac{\partial T}{\partial y} n_y = h(T_a - T) \qquad \text{on } \Gamma_3$$
(5)

Where ρ is Material density; *c* is specific heat capacity; k_x , k_y are the material conductivities along *x*, *y* direction, respectively; *Q* is Internal density of heat source; n_x , n_y are direction cosines of the unit outward normal vector. \overline{T} is the temperature on Γ_1 ; *q* is the Heat current density on Γ_2 ; *h* is heat emission factor; T_a is the ambient temperature.

The problem domain Ω and its boundary are placed with scattered nodes, and the Delaunay tessellations are used to partition the whole domain into triangular regions. Each node, for example node is associated with a local sub-domain Ω_s , which is constructed by collecting all the surrounding Delaunay triangles with node being their common vertices. In each sub-domain, based on the local weighted residual method, the weak form of governing equations are satisfied, and may be written as:

$$\int_{\Omega} \left[\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} (k_x \frac{\partial T}{\partial x}) - \frac{\partial}{\partial y} (k_y \frac{\partial T}{\partial y}) - \rho Q \right] w d\Omega = 0$$
(6)

Where *w* is the weight or test function, and we use the same weight function for all the equations involved.

Using the divergence theorem, we obtain

$$\int_{\Omega_s} (k_x \frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + k_y \frac{\partial T}{\partial y} \frac{\partial w}{\partial y}) d\Omega + \int_{\Omega_s} w(\rho c \frac{\partial T}{\partial t} - \rho Q) d\Omega - \int_{\partial\Omega_s} w(k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y) d\Gamma = 0$$
(7)

Where $\partial \Omega_s$ denotes the boundary of Ω_s , and usually consists of four parts: the internal boundary L_s , which does not intersect with the global boundary. Γ_{s1} , Γ_{s2} , Γ_{s3} are the local boundary that over the global boundary, which given temperature, rate of heat flow, Convection heat transfer condition, respectively. Therefore, $\partial \Gamma_s = \Gamma_{s1} \cup \Gamma_{s2} \cup \Gamma_{s3} \cup L_s$. Considering the boundary conditions of Γ_{s2} , we can obtain:

$$\int_{\Omega_{s}} (k_{x} \frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + k_{y} \frac{\partial T}{\partial y} \frac{\partial w}{\partial y}) d\Omega + \int_{\Omega_{s}} w(\rho c \frac{\partial T}{\partial t} - \rho Q) d\Omega - \int_{L_{s}} w(k_{x} \frac{\partial T}{\partial x} n_{x} + k_{y} \frac{\partial T}{\partial y} n_{y}) d\Gamma - \int_{\Gamma_{s1}} w(k_{x} \frac{\partial T}{\partial x} n_{x} + k_{y} \frac{\partial T}{\partial y} n_{y}) d\Gamma - \int_{\Gamma_{s1}} w(k_{x} \frac{\partial T}{\partial x} n_{x} + k_{y} \frac{\partial T}{\partial y} n_{y}) d\Gamma$$

$$- \int_{\Gamma_{s2}} wq d\Gamma - \int_{\Gamma_{s3}} wh(T_{a} - T) d\Gamma = 0$$
(8)

For a sub-domain located entirely within the global domain, there is no intersection with the global boundary, the integrals over Γ_{s1} , vanish. To simplify the above equation, the 3-nodal triangular FEM shape function N_i of node *i* is used as weight functions. Therefore, the integrals over internal boundary L_s vanish. Furthermore, the domain integrals over Γ_{s1} have two conditions: For Fig. 2(a), easy know $N_i = 0$, the integral over internal boundary Γ_{s1} vanish; For Fig. 2(b), limited by temperature condition, stiffness matrix vanishes when assemble integral items. So the Eq. (8) can be rewritten as:

$$\sum_{l=1}^{M} \int_{T_{il}} (k_x \frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + k_y \frac{\partial T}{\partial y} \frac{\partial w}{\partial y} + w\rho c \frac{\partial T}{\partial t} - w\rho Q) d\Omega - \int_{\Gamma_{s2}} wq d\Gamma - \int_{\Gamma_{s3}} wh (T_a - T) d\Gamma = 0$$
(9)

Where M is the total number of the Delaunay triangles which constructed local sub-domain Ω_s of node i.

< Figure 2 >

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To obtain the discrete equations from Eq. (9), we use MLS to approximate the test function T as follows:

$$T(x) = \sum_{i=1}^{n} \phi_i(x) T_i$$
 (10)

Where *n* is the total number of nodes for *x*. $\phi_i(x)$ is usually called shape function. Substitution of equation (10) into (9) for all nodes leads to the following simplified discretized system of equations:

$$\sum_{j=1}^{N} K_{ij}T_j + \frac{\partial T_j}{\partial t}C_{ij} = P_i(i=1, 2, \cdots, N)$$
(11)

Where

$$K_{ij} = \sum_{I=1}^{M} \int_{T_{il}} (k_x \frac{\partial \phi_j(x)}{\partial x} \frac{\partial N_i}{\partial x} + k_y \frac{\partial \phi_j(x)}{\partial y} \frac{\partial N_i}{\partial y}) d\Omega + \int_{\Gamma_{s3}} N_i h \phi_j(x) d\Gamma$$

$$C_{ij} = \sum_{I=1}^{M} \int_{\Omega_s} N_i \rho c \phi_j(x) d\Omega$$

$$P_i = \int_{\Gamma_{s2}} N_i q d\Gamma + \int_{\Gamma_{s3}} N_i h T_a d\Gamma + \int_{\Omega_s} N_i \rho Q d\Omega$$
(12)

Where N is the total number of the nodes in the global domain.

4. Numerical example

Consider the heat conduction problem of a rectangular plate (see Fig. 3). The length of the plate is 100. The temperature of the left boundary *AB* is T = 0, the heat current of Underneath boundary *BC* is $q_b = 0$, other boundarys are heat insulation. There is no internal heat source. The following parameters are considered: $k_x = k_y = 1000$, $\rho c = 1.0$, $T_0 = 0$, The regular nodes is discretization (9 × 9) is used in the presented work. We use Gauss integral scheme (3 point) in Delaunay triangles. From Fig. 4; Fig.5 it can be clearly seen that the MNNPG solution is in excellent agreement with the analytical solution provided by reference documentation (Liu, 2002).

<Figure 3-5>

5. Conclusions

MNNPG method proposed in this paper combines the advantages of the non-Sibsonian interpolation with the new meshless Petrov-Galerkin method. In the MNNPG, the local weak form of the equilibrium equation is used, the trail functions are constructed by non-Sibsonian interpolation and the 3-nodal triangular FEM shape functions are chosen as the test function. The global stiffness matrix obtained in MNNPG is sparse and banded, and does not need an assembly process. Natural neighbour interpolation can enforce the essential boundary condition directly. In the presented numerical examples, excellent solutions are obtained. In essence, this method is efficient, accurate and easy to implement, which reveals its potential applications to solve other problems.

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Figure 1. The Voronoi diagram and natural neighbours



Figure 2. Essential boundary condition Γ_{s1} over sub-domain Ω_s



Figure 3. Heat conduction model of two-dimension



Figure 4. Comparison of the solutions obtained by using different methods along x = 80



Figure 5. Comparison of the solutions obtained by using different methods at point C