Second-Order Duality for Nondifferentiable Multiobjective Programming Involving ($\Phi, \rho$)-Univexity

Ganesh K. Thakur
Department of Applied Mathematics
Maharaja Agarsain Institute of Technology
Ghaziabad 245304 (UP), India
E-mail: meetgangesh@gmail.com

Bandana Priya
Department of Applied Mathematics
R.D. Engineering College, Ghaziabad India
E-mail: bandanapriya@gmail.com

Abstract
The concepts of ($\Phi, \rho$)-invexity have been given by Carsiti, Ferrara and Stefanescu (Carsiti, G., et al, 2006). We consider a second-order dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate second-order ($\Phi, \rho$)-univexity conditions.

AMS 2002 Subject Classification: 90C29, 90C30, 90C46.

Keywords: Second-order ($\Phi, \rho$)-(pseudo/quasi)-convexity, Multiobjective programming, Second-order duality, Duality theorem.

1. Introduction
For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual (Dorn, W.S. 1960, Hanson, M. et al, 1982) is well known. While studying duality under generalized convexity, Mond and Weir (Mond, B., and Weir, T., 1981) proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used (Gulati, T.R., et al, 2001, Suneja et al, 2003, Yang, X.M. et al, 2003). Mangasarian (Mangasarian, O.L., 1975) considered a nonlinear programming problem and discussed second order duality under inclusion condition. Mond (Mond, B., 1974) was the first who present second order convexity. He also gave in (Mond, B., 1974) simpler conditions than Mangasarian using a generalized form of convexity. which was later called bonvexity by Bector and Chandra (Bector, C.R., Chandra, S., 1987). Further, Jeyakumar (Jeyakumar, V., 1985, Jeyakumar, V., 1986) and (Yang X.M., et al, 2003) discussed also second order Mangasarian type dual formulation under $\rho$-convexity and generalized representation conditions respectively. Zhang and Mond, Mond, B. 1996) established some duality theorems for second-order duality in nonlinear programming under generalized second-order $B$-invexity, defined in their paper. In (Mond, B., 1974) it was shown that second order duality can be useful from computational point of view, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve n-set functions was studied by Preda (preda, V., 1998). Recently, Yang et al. (Yang, X.M., et al, 2003) proposed four second-order dual models for nonlinear programming problems and established some duality results under generalized second-order $F$-convexity assumptions.

For $\Phi(x, a, (y, r)) = F(x, a; y) + rd^2(x, a)$, where $F(x, a; )$ is sublinear on $R^n$, the definition of ($\Phi, \rho$)-invexity reduces to the definition of ($F, \rho$)-convexity introduced by Preda (Preda, V., 1992) which in turn Jeyakumar (jeyakumar, V., 1985) generalizes the concepts of $F$-convexity and $\rho$-invexity.

The more recent literature, (Xu, Z., 1985, Ojha, D.B., 2005, Ojha D.B., and Mukherjee 2006) for duality under generalized ($F, \rho$)-convexity, (Mishra, S.K., 2000) and (Yang et al, 2003) for duality under second order $F$-convexity. (Liang et al. 2003) and (Hachimi, M., 2004) for optimality criteria and duality involving ($F, \alpha, \rho, d$)-convexity or generalized ($F, \alpha, \rho, d$)-type functions are may consulted by interested redear. The ($F, \rho$)-convexity was recently generalized to ($\Phi, \rho$)-invexity by Caristi, Ferrara and Stefanescu (Caristi, G., et al 2006) and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming; Ferrara and Stefanescu (Ferrara, M., Stefanescu, M.V., 2008) showed that...
sufficiency Kuhn-Tucker condition can be proved under $(\Phi,\rho)$-invexity, even if Hanson’s invexity is not satisfied. The results of this paper are real extensions of the similar results known in the literature.

In Section 2 we define the second-order $(\Phi,\rho)$-univexity. In Section 3 we consider a class of Multiobjective programming problems and for the dual model we prove a weak duality result.

2. Notations And Preliminaries

we denote by $R^n$ the n-dimensional Euclidean space, and by $R^n_+$ its nonnegative orthant. Further, $R^n_+ = \{x \in R^n | x > 0\}$. For any vector $x \in R^n$, $y \in R^n$, we denote $x^Ty = \sum_{i=1}^n x_i y_i$. Let $C \subset R^n$ be a convex compact set. The support function of $C$ is defined by $s(x|C) = \max\{x^Ty | y \in C\}$. Being convex and every where finite, it has a subdifferential, that is, there exist $z \in R^n$ such that $s(y|C) \geq s(x|C) + z^T(y-x)$ for all $y \in C$.

The subdifferentials of $s(x|C)$ is given by $\partial s(x|C) = \{z \in C | z^Tx = s(x|C)\}$.

A point $a \in X$ is said to be an efficient solution of problem (P) if there exit no $x \in X_0$ such that $f(a) - f(x) \in R^n_+ \setminus \{0\}$ i.e., $f_i(x) \leq f_i(a)$ for all $i \in [1,...,p]$, and for at least one $j \in [1,...,p]$ we have $f_j(x) < f_j(a)$.

Definition 2.1

A point $a \in X_0$ is said to be a weak efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition 2.2

A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant $K$ such that for each $x \in X_0$ and for each $i \in [1,2,...,p]$ satisfying $f_i(x) < f_i(a)$, there exist at least one $i \in [1,2,...,p]$ such that $f_i(a) - f_i(x) \leq K(f_j(x) - f_j(a))$.

Denoting by $WE(P)$, $E(P)$ and $PE(P)$ the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have $PE(P) \subseteq E(P) \subseteq WE(P)$.

We denote by $\nabla f(a)$ the gradient vector at $a$ of a differentiable function $f : R^n \to R$, and by $\nabla^2 f(a)$ the Hessian matrix of $f$ at $a$. For a real valued twice differentiable function $\psi(x,y)$ defined on an open set in $R^n \times R^n$, we denote by $\nabla_x \psi(a,b)$ the gradient vector of $\psi$ with respect to $x$ at $(a,b)$, and by $\nabla_y \psi(a,b)$ the Hessian matrix with respect to $x$ at $(a,b)$. Similarly, we may define $\nabla_x \psi(a,b), \nabla_y \psi(a,b)$ and $\nabla_{xy} \psi(a,b)$.

For convenience, let us write the definitions of $(\Phi,\rho)$-univexity from [1]. Let $\varphi : X_0 \to R$ be a differentiable function ($X_0 \subseteq R^m,X \subseteq X_0$ and $a \in X_0$). An element of all $(n+1)$- dimensional Euclidean Space $R^{n+1}$ is represented as the ordered pair $(z,r)$ with $z \in R^n$ and $r \in [0,\rho]$ a real number and $\Phi$ is a real valued function defined on $X_0 \times X_0 \times R^{n+1}$, such that $\Phi(\Phi(x,a,\alpha(0,0)),r) \geq 0$, for every $(x,a) \in X_0 \times X_0$ and $r \in [0,\rho]$. Denoted by $b(x,a,\alpha) = \lim_{\alpha \to 0} b(x,a,\alpha)$, $b(x,a,\alpha) \geq 0$, and $b$ does not depend upon $\alpha$ if the corresponding functions are differentiable.

We assume that $\psi_0, \psi_1 : R \to R$ satisfying $u \leq 0 \implies \psi_0(u) \leq 0$ and $u \leq 0 \implies \psi_1(u) \leq 0$, and $b_0(x,a) > 0$ and $b_0(x,a) \geq 0$.

Example 2.1

$$\min f(x) = x - 1$$
$$g(x) = -x - 1 \leq 0, x \in X_0 \in [1,\infty)$$

$$\Phi(x,a;\varphi_1(x)) = 2(2^r - 1)|x - a| + \langle y, x - a \rangle$$

for $\psi_0(x) = x, \varphi_1(x) = -x, \alpha_1 = \frac{1}{2}$ (for $f$) and $\rho = 1$ (for $g$), then this is $(\varphi,\rho)$-univex but it is not $(\varphi,\rho)$-invex.

Definition 2.4
A real-valued twice differentiable function $f(.,.) : X \times X \rightarrow R$ is said to be second-order ($\Phi, \rho$)-univex at $u \in X$ with respect to $p \in R^n$, if for all $b : X \times X \rightarrow R$, $\Phi : X \times X \times R^{r+1} \rightarrow R$, $\rho$ is a real number, we have

$$b(x,u)[\psi[f_i(x,y) - f_i(u,y) + \frac{1}{2}p^T\nabla^2 f_i(u,y)p]] \geq \Phi(x,u; (\nabla f_i(u,y) + \nabla^2 f_i(u,y)p, p))$$

(2.1)

**Definition 2.5**

A real-valued twice differentiable function $f(., .) : X \times X \rightarrow R$ is said to be second-order ($\Phi, \rho$)-psedounivex at $a \in X$ with respect to $p \in R^n$, if for all $b : X \times X \rightarrow R$, $\Phi : X \times X \times R^{r+1} \rightarrow R$, $\rho$ is a real number, we have

$$\Phi(x,u; (\nabla f_i(u,y) + \nabla^2 f_i(u,y)p, p)) \geq 0$$

$$\Rightarrow b(x,u)[\psi[f_i(x,y) - f_i(u,y) + \frac{1}{2}p^T\nabla^2 f_i(u,y)p]] \geq 0$$

(2.2)

**Definition 2.6**

A real-valued twice differentiable function $f(., .) : X \times X \rightarrow R$ is said to be second-order ($\Phi, \rho$)-quasiunivex at $a \in X$ with respect to $p \in R^n$, if for all $b : X \times X \rightarrow R$, $\Phi : X \times X \times R^{r+1} \rightarrow R$, $\rho$ is a real number, we have

$$b(x,u)[\psi[f_i(x,y) - f_i(u,y) + \frac{1}{2}p^T\nabla^2 f_i(u,y)p]] \leq 0$$

$$\Rightarrow \Phi(x,u; (\nabla f_i(u,y) + \nabla^2 f_i(u,y)p, p)) \leq 0$$

(2.3)

**Remark 2.1**

(i) If we consider the case $b \geq 1$, $\Phi(x,u; (\nabla f_i(u), p)) = F(x,u; \nabla f(u))$ (with $F$ is sublinear in third argument, then the above definition reduce to Definition 4 of Chen[4].

(ii) When $h(u,y) = y^T \nabla u f_i(u)$ and $\Phi(x,u; (\nabla f(u), p)) = F(x,u; \nabla f(u)) = \eta(x,u)^T \nabla f(u)$ where $\eta : X \times X \rightarrow R^n$, the above definition reduce to $\eta$-(pseudo/quasi)-bonvexity.

**Example 2.1**

We present here a function which is second-order ($\Phi, \rho$)-univex for $b = 1$. Let us consider $X = (0, \infty)$ and $f : X \rightarrow R, f(x) = \log x, h : X \times R \rightarrow R, h(u,y) = -y \log u$. We have

$$\nabla_u f(u) = 1 + \log u, \nabla_u h(u,y) = -\log u, \Phi : X \times X \times R^{r+1} \rightarrow R$$

Taking $\rho = 0\Phi(x,y; b) = |b| + |b|^2$

It is obvious our mapping is more generalized rather than previous ones.

Hence $f(x) = \log x$ is second order ($\Phi, \rho$)-univex at $u \in X$, with respect to $h(u,y) = -y \log u$.

A real valued twice differentiable function $g$ is second order $F$-pseudoconcave if $-g$ is second order $F$-pseudoconvex.

We shall make use of the following generalized Schwartz inequality:

$$x^TAy \leq (x^TAx)^{\frac{1}{2}} (y^TAY)^{\frac{1}{2}}$$

where $x, y \in R^n$ and $A \in R^n \times R^n$ is a positive semidefinite matrix. Equality holds if and only if $x = Ax = Ay$.

3. **Mond-Weir type second order symmetric duality**

We consider here the following pair of second order nondifferentiable multiobjective with $r$-objectives and establish weak, strong and converse duality theorems.

(MP)

minimize

$$H(x,y,w,p) = [H_1(x,y,w,p), H_2(x,y,w,p), \ldots, H_r(x,y,w,p)]$$

subject to

$$\sum_{j=1}^{r} A_i[\nabla_{x_j} f_i(x,y) - C_j w_i + \nabla_{x_j} f_i(x,y)p_j] \leq 0$$

(3.1)

$$y^T \sum_{j=1}^{r} A_i[\nabla_{x_j} f_i(x,y) - C_j w_i + \nabla_{x_j} f_i(x,y)p_j] \geq 0$$

(3.2)

$$w_i^T C_j w_i \leq 1, i = 1, 2, \ldots, r$$

(3.3)
\[ \lambda > 0 \]  
(3.4)

\[ x \geq 0 \]  
(3.5)

\[ \text{maximize} \]

\[ J(u, v, a, q) = \{ J_1(u, v, a, q), J_2(u, v, a, q), \ldots, J_r(u, v, a, q) \} \]

subject to

\[ \sum_{i=1}^{r} \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_x f_i(u, v) q_i] \geq 0 \]  
(3.6)

\[ u^T \sum_{i=1}^{r} \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_x f_i(u, v) q_i] \leq 0 \]  
(3.7)

\[ a_i^T E_i a_i \leq 1, \quad i = 1, 2, \ldots, r \]  
(3.8)

\[ \lambda > 0 \]  
(3.9)

\[ y \geq 0 \]  
(3.10)

Where \( H_i(x, y, w, p) = f_i(x, y) + (x^T E_i x)^\frac{1}{2} - (v^T C_i x - \frac{1}{2} q_i^T \nabla f_i(x, y) p_i \)

\[ J_i(u, v, a, q) = f_i(u, v) - (v^T C_i x)^\frac{1}{2} + u^T E_i a_i - \frac{1}{2} q_i^T \nabla f_i(u, v) q_i \]

\( \lambda_i \in R, \quad p_i \in R^n, \quad q_i \in R^n, \quad i = 1, 2, \ldots, r \) and \( f_i, a_i = 1, 2, \ldots, r \) are thrice differentiable function from \( R^n \times R^n \rightarrow R \). \( E_i, C_i \) are positive semidefinite matrices. Also, we mean here, \( b_i = R^n \times R^n \times R^n \times R^n \rightarrow R, \quad p = (p_1, p_2, \ldots, p_r), q = (q_1, q_2, \ldots, q_r), w = (w_1, w_2, \ldots, w_r), a = (a_1, a_2, \ldots, a_r) \)

**Remark: 3.1**

Since the objective functions of (MP) and (MD) contain the support functions \( s(x|C_i) \) and \( s(v|D_i) \), \( i = 1, 2, \ldots, p \), these problems are nondifferentiable multiobjective programming problems.

**Theorem 3.1 (Weak duality)**

Let \( (x, y, \lambda, a, w, p) \) be a feasible solution of (MP) and \( (u, v, a, q) \) a feasible solution of (MD). Then the inequalities can not hold simultaneously:

(i) \( \sum_{i=1}^{r} \lambda_i [f_i(x, y) + (x^T E_i x)^\frac{1}{2} - (v^T C_i x - \frac{1}{2} q_i^T \nabla f_i(x, y) p_i) \)

(ii) \( \sum_{i=1}^{r} \lambda_i [f_i(x, y) - (x^T E_i x)^\frac{1}{2} + u^T E_i a_i - \frac{1}{2} q_i^T \nabla f_i(u, v) q_i) \)

(iii) \( \Phi(x, u; (\xi, \rho)) + u^T \xi \geq 0, \text{ for } \xi \in R^n \), and

(iv) \( \Phi(v, y; (\zeta, \rho)) + y^T \zeta \geq 0, \text{ for } \zeta \in R^n \), then

\[ H(x, y, w, p) \leq J(u, v, a, q). \]

**Proof**

With the help of \( \sum_{i=1}^{r} \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_x f_i(u, v) q_i] \), we have

\[ \Phi(x, u; (\sum_{i=1}^{r} \lambda_i [\nabla_x f_i(u, v) + E_i a_i + \nabla_x f_i(u, v) q_i]), (\rho)) \]

\[ + u^T \sum_{i=1}^{r} \lambda_i [\nabla_x f_i(u, v) + w_i + \nabla_x g_i(u, v, \mu_i)] \geq 0 \]

(By hypothesis (iii) and (3.7), also by the second order \((\Phi, \rho)\)-pseudounivexity of \( \sum_{i=1}^{r} \lambda_i [f_i(\cdot, \cdot) + (x^T E_i x)^\frac{1}{2}] \) at \( u \), with property of band \( \psi \), provides
Applying Schwartz inequality, (3.3) and (3.8), we get

$$\sum_{i=1}^r \lambda_i |f_i(x, y) + (x^T E_i a_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(x, y) p_i| \geq 0$$

Combining (3.11) and (3.12), we get

$$\sum_{i=1}^r \lambda_i |f_i(x, y) + (x^T E_i a_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(x, y) p_i| \geq 0$$

Applying Schwartz inequality, (3.3) and (3.8), we get

$$\sum_{i=1}^r \lambda_i |(f_i(x, y) + (x^T E_i a_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(x, y) p_i| \geq \sum_{i=1}^r \lambda_i |(f_i(x, y) + (x^T E_i a_i) - \frac{1}{2} q_i^T \nabla_{xx} f_i(x, y) p_i|$$

Hence

$$H(x, y, w, p) \leq J(u, v, a, q).$$

**Theorem 3.2 (Strong duality)**

Let f be thrice differentiable on \(R^n \times R^n\) and \((x', y', \lambda', w', p')\) be a weak efficient solution for (MP), and \(\lambda = \lambda'\), assume that

1. \(\nabla_{yy} f_i\) is nonsingular for all \(i = 1, 2, \ldots, r;\)
2. the matrix \(\sum_{i=1}^r \lambda_i (\nabla_{yy} f_i p_i')\) is positive or negative definite, and ;
3. the set \(\{\nabla_{yy} f_i - C_i w_i' + \nabla_{yy} f_i p_i', \nabla_{yy} f_2 - C_2 w_2' + \nabla_{yy} f_2 p_2', \ldots, \nabla_{yy} f_r - C_r w_r' + \nabla_{yy} f_r p_r'\}\), are linearly independent;

where \(f_i = f(x', y'); i = 1, 2, \ldots, r\). Then \((x', y', \lambda', w', p') = 0\) is a feasible solution of (MD), \(b_i(x', y', a', \gamma') > 0, i = 1, 2, \ldots, r\), and the two objectives have the same values. Also, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then \((x', y', \lambda', a', \gamma') = 0\) is an efficient solution for (MD).

**Proof**

Since \((x', y', \lambda', w', p')\) is a weak efficient solution of (MP), by Fritz-John condition [7], there exist \(\alpha \in R^n, \beta \in R^n, y \in R, v \in R^n\) and \(\xi \in R^n\) such that

$$\sum_{i=1}^r \alpha_i |\nabla_{xx} f_i + E_i a_i' - \frac{1}{2} (\nabla_{yy} f_i p_i') x p_i'| + \sum_{i=1}^r \lambda_i' |\nabla_{yy} f_i + (\nabla_{yy} f_i p_i') x| (\beta - \gamma y') - \xi = 0$$

$$\sum_{i=1}^r \alpha_i |\nabla_{yy} f_i - C_i w_i' + \frac{1}{2} (\nabla_{yy} f_i p_i') p_i'| + \sum_{i=1}^r \lambda_i' |\nabla_{yy} f_i + (\nabla_{yy} f_i p_i') x| (\beta - \gamma y')$$

$$- y \sum_{i=1}^r \lambda_i' |\nabla_{yy} f_i - C_i w_i' + (\nabla_{yy} f_i p_i')| = 0$$

$$(\beta - \gamma y')^T (\nabla_{xx} f_i - C_i w_i' + \nabla_{yy} f_i p_i') = 0, i = 1, 2, \ldots, r$$

$$\alpha_i C_i y' + (\beta - \gamma y')^T (\nabla_{yy} f_i - C_i w_i', i = 1, 2, \ldots, r$$

$$[(\beta - \gamma y') A_i' \nabla_{yy} f_i = 0, i = 1, 2, \ldots, r$$
\[ x^T E_i a'_i = (x^T E_i x'_i)^2, \quad i = 1, 2, \ldots, r \]  

(3.18)

\[ \beta^T \sum_{i=1}^r \alpha'_i [\nabla_x f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \]  

(3.19)

\[ \gamma y' \sum_{i=1}^r \alpha'_i [\nabla_x f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \]  

(3.20)

\[ \nu_i (w'_i^T C_i w'_i - 1) = 0, \quad i = 1, 2, \ldots, r \]  

(3.21)

\[ \delta^T \chi' = 0 \]  

(3.22)

\[ x^T \xi = 0 \]  

(3.23)

\[ a_i^T E_i a'_i \leq 1, \quad i = 1, 2, \ldots, r \]  

(3.24)

\[ (\alpha, \beta, \gamma, \nu, \delta, \xi) \geq 0 \]  

(3.25)

\[ (\alpha, \beta, \gamma, \nu, \delta, \xi) \neq 0 \]  

(3.26)

Since \( \lambda' > 0 \) and \( \delta \geq 0 \), (3.22) implies \( \delta = 0 \). Consequently, (3.15) gives

\[ (\beta - \gamma y')' [\nabla_x f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \]  

(3.27)

Since \( \nabla_{yy} f_i \) is nonsingular for \( i = 1, 2, \ldots, r \), from (3.17), it follows that

\[ (\beta - \gamma y')' i = \alpha_i p'_i, \quad i = 1, 2, \ldots, r. \]  

(3.28)

From (3.14), we get\( \sum_{i=1}^r (\alpha_i - \gamma \lambda'_i) [\nabla_x f_i - C_i w'_i] + \sum_{i=1}^r \lambda'_i [\nabla_{yy} f_i (\beta - \gamma y') - \gamma y' p'_i] + \sum_{i=1}^r (\nabla_{yy} f_i p'_i)_i [\beta - \gamma y'] - \frac{1}{2} \alpha_i p'_i = 0 \)

using (3.28), we get

\[ \sum_{i=1}^r (\alpha_i - \gamma \lambda'_i) [\nabla_x f_i - C_i w'_i] + \frac{1}{2} \sum_{i=1}^r \lambda'_i [\nabla_{yy} f_i p'_i)_i [\beta - \gamma y'] = 0 \]  

(3.29)

Premultiplying (3.29) by \( (\beta - \gamma y')' \) and using (3.27), we get

\[ (\beta - \gamma y')' \sum_{i=1}^r \lambda'_i [\nabla_{yy} f_i p'_i] \beta - \gamma y' = 0, \]  

by hypothesis (ii) implies

\[ \beta = \gamma y' \]  

(3.30)

Therefore, from (3.29), we get \( \sum_{i=1}^r (\alpha_i - \gamma \lambda'_i) [\nabla_x f_i - C_i w'_i + \nabla_{yy} f_i p'_i] = 0 \), which by hypothesis (iii) gives

\[ \alpha_i = \gamma \lambda'_i, \quad i = 1, 2, \ldots, r \]  

(3.31)

If \( \gamma = 0 \), then \( \alpha_i = 0, i = 1, 2, \ldots, r \) and from (3.31) \( \beta = 0 \). Also from (3.13) and (3.16), we get \( \xi = 0, \nu_i = 0, i = 1, 2, \ldots, r. \) Thus \( (\alpha, \beta, \gamma, \nu, \delta, \xi) = 0 \), a contradiction to (3.26). Hence \( \gamma > 0 \), since \( \chi'_i > 0, i = 1, 2, \ldots, r. \) (3.31) implies \( \alpha_i > 0, i = 1, 2, \ldots, r. \) Using (3.30) in (3.28) \( \alpha p'_i = 0, i = 1, 2, \ldots, r. \) Hence \( \xi = 0, \nu_i = 1, 2, \ldots, r. \) Using (3.30) and \( p'_i = 0, i = 1, 2, \ldots, r. \) From (3.13), it gives \( \sum_{i=1}^r \alpha_i [\nabla_x f_i + E_i a'_i] = \xi, \) which by (3.31) gives

\[ \sum_{i=1}^r \lambda'_i [\nabla_x f_i + E_i a'_i] = \frac{\xi}{\gamma} \geq 0 \]  

(3.32)
\[ x^T \sum_{i=1}^{r} \alpha'_i [\nabla_s f_i + E_i a'_i] = x^T \sum_{i=1}^{r} \xi = 0 \]  

(3.33)

Also, from (3.30), we get

\[ y' = \frac{\beta}{\gamma} \geq 0 \]  

(3.34)

Hence from (3.24) and (3.32-3.34), \((x', y', \lambda', a', q') = 0\)is feasible for (MD).

Let \(2 \frac{v_i}{\alpha_i} = t\). Then \(t \geq 0\)and from (3.16) and (3.30)

\[ C_i y' = t C_i w' \]  

(3.35)

Which is the condition in the Schwartz inequality. Therefore

\[ y^T C_i w' = (y^T C_i y')^\frac{1}{2} (w_i^T C_i w_i')^{\frac{1}{2}} \]

In case, \(v_i > 0\), (3.21) gives \(w_i^T C_i w_i' = 1\) and so \(y^T C_i w' = (y^T C_i y')^\frac{1}{2}\). In case \(v_i = 0\), (3.35) gives \(C_i y' = 0\)and so \(y^T C_i w' = (y^T C_i y')^\frac{1}{2} = 0\).

Thus in either case

\[ y^T C_i w' = (y^T C_i y')^\frac{1}{2} \]  

(3.36)

Hence \(H_i (x', y', w', p') = 0\) gives \(w_i^T C_i w_i' = 1\) and so \(y^T C_i w' = (y^T C_i y')^\frac{1}{2}\). In case \(v_i > 0\), (3.35) gives \(C_i y' = 0\)and so \(y^T C_i w' = (y^T C_i y')^\frac{1}{2} = 0\).

A converse duality theorem may be merely stated as its proof would run analogously to that of Theorem 3.2.

**Theorem 3.3 (Converse duality)**

Let \(f\) be thrice differentiable on \(R^n \times R^n\)and \((u', v', \lambda', a', q')\)be a weak efficient solution for (MD), and \(\lambda = \lambda'\) fixed in (MP).

Assume that

1. \(\nabla_{xx} f_i\) is nonsingular for all \(i = 1, 2, \ldots, r\);
2. the matrix \(\sum_{i=1}^{r} \lambda'_i (\nabla_{xx} f_i q'_i)\) is positive or negative definite, and ;
3. the set \(\{\nabla_s f_i + E_i a'_i, \nabla_{xx} f_i q'_i, \nabla_{ss} f_i, \nabla_{xx} f_i q'_i, \ldots, \nabla_s f_i + E_i a'_i, \nabla_{xx} f_i q'_i\}\), are linearly independent;

where \(f_i = f_i (u', v'), i = 1, 2, \ldots, r\). Then \((u', v', \lambda', w', p') = 0\) is a feasible solution of (MP), \(b_i (x', y', u', v', \lambda', a', q') > 0, i = 1, 2, \ldots, r\), and the two objectives have the same values. Also, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then \((u', v', \lambda', w', p') = 0\) is an efficient solution for (MP).

**4. Special cases**

(i) If \(b = 1, \psi \equiv I, E_i = C_i, 0 = i, 1, 2, \ldots, r, \) and \(\Phi (x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))\) for \(\rho = 0\) then (MP) and (MD) reduce to the second order multiobjective symmetric dual program studied by Sun et a. with omission of non-negativity constraints from (MP) and (MD). If in addition \(p = q = 0, \) and \(r = 1\), then we get the first order symmetric dual programs of Chandra et al.

(ii) If \(b = 1, \psi \equiv I, \) we set \(p = q = 0, \) and \(\Phi (x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))\) for \(\rho = 0\) in (MP) and (MD), then we obtain a pair of first order symmetric dual nondifferentiable multiobjective programs considered by Mond et al.\n
(iii) If we set, \(b = 1, \psi \equiv I, \Phi (x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u))\) for \(\rho = 0\) in (MP) and (MD), then we obtain a pair of second order symmetric dual nondifferentiable multiobjective programs considered by Ahmad et al.

**References**


