On the Irreducibility of Artin’s Group of Graphs

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Abstract

We consider the graph $E_{3,1}$ with three generators $\sigma_1, \sigma_2, \delta$, where $\sigma_1$ has an edge with each of $\sigma_2$ and $\delta$. We then define the Artin group of the graph $E_{3,1}$ and consider its reduced Perron representation of degree three. After we specialize the indeterminates used in defining the representation to non-zero complex numbers, we obtain a necessary and sufficient condition that guarantees the irreducibility of the representation.

Keywords: Artin representation, braid group, Burau representation, graph, irreducibility

1. Introduction

To any undirected simple graph $T$, we introduce the Artin group, $A$, which is defined as an abstract group with vertices of $T$ as its generators and two relations: $xy = yx$ for vertices $x$ and $y$ that have no edge in common and $xyx = yxy$ if the vertices $x$ and $y$ have a common edge.

Let $A_n$ be the graph having $n$ vertices $\sigma_i$’s ($1 \leq i \leq n$) in which $\sigma_i$ and $\sigma_{i+1}$ share a common edge, where $i = 1, 2, ..., n-1$. We notice that the Artin group of $A_n$ is the braid group on $n+1$ strands. That is, $A(A_n) = B_{n+1}$ (J.S. Birman, 1975).

Having defined $A_n$, we consider $E_{n+1,p}$, which is the graph obtained from $A_n$ by adding a vertex $\delta$ and an edge connecting $\sigma_p$ and $\delta$. Here $1 \leq p \leq n$. It is easy to see that the graph $A_n$ embeds in the graph $E_{n+1,p}$. That is, $A(A_n) \subset A(E_{n+1,p})$. This induces an injection on $B_{n+1}$ to $A(E_{n+1,p})$. In other words, a representation of $A(E_{n+1,p})$ yields a representation of $B_{n+1}$.

Knowing the reduced Burau representation of $B_{n+1}$ of degree $n$, Perron extends such a representation to a representation of $B_{n+1}$ of degree $2n$. The representation obtained is referred to as Burau bis representation. Next, Perron constructs for each $\lambda = (\lambda_1, \ldots, \lambda_n)$ a representation $\varphi_{\lambda} : A(E_{n+1,p}) \rightarrow GL_{2n}(Q(t, d_1, \ldots, d_n))$, where $t, d_1, \ldots, d_n$ are indeterminates. We specialize $t, d_1, \ldots, d_n$ to non zero complex numbers, and we study this representation explicitly in the case $n = 2$ and $p = 1$. We then reduce the complex specialization of the representation $\varphi_{\lambda}$ to a representation of degree 3, namely $A(E_{3,1}) \rightarrow GL_3(\mathbb{C})$. A necessary and sufficient condition which guarantees its irreducibility is obtained in that case.

2. Burau bis Representation

Perron’s strategy is to begin with the Burau representation of the braid group and extend it to a representation of $A(E_{n+1,p})$. He begins with the reduced Burau representation: $B_{n+1} \rightarrow GL_{2n}(\mathbb{Z}[t, t^{-1}])$ defined as follows:

$$\sigma_i \rightarrow J_i = \begin{bmatrix}
I_{n-2} & 0 & 0 \\
0 & 1 & 0 \\
0 & -t & 1 \\
0 & 0 & 1 \\
0 & 0 & I_{n-i-1}
\end{bmatrix},$$

where $I_k$ stands for the $k \times k$ identity matrix. Here, $i = 2, \ldots, n-1$. 

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\[ \sigma_1 \to J_1 = \begin{pmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \]

\[ \sigma_n \to J_n = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{pmatrix} \]

Knowing that this representation is of degree \( n \), Perron extends it to a representation of \( B_{n+1} \) of degree \( 2n \). Let \( R_i \) denote an \( n \times n \) block of zeros with a \( t \) placed in the \((i, i)\) th position, and let \( I_n \) denote the \( n \times n \) identity matrix. The obtained representation is referred to as the Burau bis representation. It is defined as follows:

\[ \psi : B_{n+1} \to GL_{2n}(\mathbb{Z}[t, t^{-1}]) \]

\[ \psi(\sigma_i) = \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix}, \quad 1 \leq i \leq n \]

For more details, see (T.E. Brendle, 2002, B. Perron, 1999).

3. Perron Representation

The Burau bis representation extends to \( A(E_{n+1, p}) \) for all possible values of \( n \) and \( p \) in the following way.

Let \( b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \), \( d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \), and \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

We define the following \( n \times n \) matrices:

\[ A = (\lambda_1 b, \lambda_2 b, \ldots, \lambda_n b) \]
\[ B = (0, \ldots, 0, b, 0, \ldots, 0) \]
\[ C = (\lambda_1 d, \lambda_2 d, \ldots, \lambda_n d) \]
\[ D = (0, \ldots, 0, d, 0, \ldots, 0), \]

where \( 0 \) denotes a column of \( n \) zeros.

For each \( i = 1, \ldots, n \), we have that \( b_i \) satisfies the following conditions

\[ tb_i = -td_{i-1} + (1 + t)d_i - d_{i+1}, \quad i \neq p, \]
\[ tb_p = -td_{p-1} + (1 + t)d_p - d_{p+1} + t, \]
\[ \sum_{i=1}^{n} \lambda_i b_i = -(1 + t_p + t), \]

setting any undefined \( d_j \) equal zero.

For any choice \( \lambda = (\lambda_1, \ldots, \lambda_n) \), we get a linear representation

\[ \psi_A : A(E_{n+1, p}) \to GL_{2n}(\mathbb{R}), \]

where \( \mathbb{R} \) is the field of rational fractions in \( n+1 \) indeterminates \( \mathbb{Q}(t, d_1, ..., d_n) \).

\[ \psi_A(\sigma_i) \to \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix}, \]

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\[ \psi_A(\delta) \rightarrow \begin{pmatrix} I_n + A & B \\ C & I_n + D \end{pmatrix}. \]

For more details, see (T.E. Brendle, 2002).

4. Reducibility of \( \psi_A : A(E_{3,1}) \rightarrow GL_4(\mathbb{C}) \)

Having defined Perron’s representation, we set \( n = 2 \) and \( p = 1 \) to get the following vectors. \( b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \), \( d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \), and \( \lambda = (\lambda_1, \lambda_2) \).

We get the following 2 \( \times \) 2 matrices:

\[
A = \begin{pmatrix} \lambda_1 b_1 & \lambda_2 b_1 \\ \lambda_1 b_2 & \lambda_2 b_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda_1 d_1 & \lambda_2 d_1 \\ \lambda_1 d_2 & \lambda_2 d_2 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.
\]

Simple computations show that the parameters satisfy the following equations:

- \( tb_2 = -td_1 + (1 + t)d_2 \)
- \( tb_1 = (1 + t)d_1 - d_2 + t \)
- \( \lambda_1 b_1 + \lambda_2 b_2 = -(1 + t + d_1) \)

Having defined the 2 \( \times \) 2 matrices \( A, B, C \) and \( D \), we obtain the multiparameter representation \( A(E_{3,1}) \). This representation is of degree 4. We specialize the parameters \( \lambda_1, \lambda_2, b_1, b_2, d_1, d_2, t \) to values in \( \mathbb{C} \setminus \{0\} \). **We further assume that** \( t \neq -1 \) **and** \( d_2 = -t \). The representation \( \psi_A : A(E_{3,1}) \rightarrow GL_4(\mathbb{C}) \) is defined as follows:

\[
\psi_A(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & -t & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
\psi_A(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & t & -t \end{pmatrix}
\]

and

\[
\psi_A(\delta) = \begin{pmatrix} 1 + \lambda_1 b_1 & \lambda_2 b_1 & b_1 & 0 \\ \lambda_1 b_2 & \lambda_2 b_2 + 1 & b_2 & 0 \\ \lambda_1 d_1 & \lambda_2 d_1 & 1 + d_1 & 0 \\ -t \lambda_1 & -t \lambda_2 & -t & 1 \end{pmatrix}.
\]

The graph \( E_{3,1} \) has 3 vertices \( \sigma_1, \sigma_2 \) and \( \delta \). Since \( p = 1 \), it follows that the vertex \( \delta \) has a common edge with \( \sigma_p = \sigma_1 \). Therefore, the following relations are satisfied.

\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad (1)
\]

\[
\sigma_2 \delta = \delta \sigma_2 \quad (2)
\]

\[
\sigma_1 \delta \sigma_1 = \delta \sigma_1 \delta \quad (3)
\]
We note that relation (1) is actually Artin’s braid relation of the classical braid group, $B_3$ having $\sigma_1$ and $\sigma_2$ as standard generators. This assures that a representation of $A(E_{3,1})$ yields a representation of $B_3$.

**Lemma 1** The representation $\psi_A : A(E_{3,1}) \rightarrow GL_4(C)$ is reducible.

**Proof.** For simplicity, we write $\sigma_i$ instead of $\psi_A(k)$, where $k$ is a generator of $A(E_{3,1})$. The subspace $S = \langle e_1 + \frac{b_2}{b_1}e_2, e_3, e_4 \rangle$ is an invariant subspace of dimension 3. To see this:

1. $\sigma_1(e_1 + \frac{b_2}{b_1}e_2) = e_1 + \frac{b_2}{b_1}e_2 + te_3 \in S$
2. $\sigma_2(e_1 + \frac{b_2}{b_1}e_2) = e_1 + \frac{b_2}{b_1}e_2 + t\frac{b_2}{b_1}e_3 \in S$
3. $\delta(e_1 + \frac{b_2}{b_1}e_2) = (1 + \lambda_1b_1 + \lambda_2b_2)e_1 + (\lambda_1b_2 + \frac{b_2}{b_1}(\lambda_2b_1 + 1))e_2 + (\lambda_1d_1 + \frac{b_2}{b_1}\lambda_2d_1)e_3 + (-\lambda_1 + \frac{b_2}{b_1}\lambda_2)e_4$
4. $\sigma_1e_3 = -te_3 \in S$
5. $\sigma_2e_3 = e_3 + te_4 \in S$
6. $\delta e_3 = b_1(e_1 + \frac{b_2}{b_1}e_2) + (1 + d_1)e_3 - te_4 \in S$
7. $\sigma_1e_4 = e_1 + e_4 \in S$
8. $\sigma_2e_4 = -te_4 \in S$
9. $\delta e_4 = e_4 \in S$

**5. On the Irreducibility of $\psi_A : A(E_{3,1}) \rightarrow GL_3(C)$**

We consider the representation $\psi_A : A(E_{3,1}) \rightarrow GL_4(C)$ restricted to the basis $e_1, e_1 + \frac{b_2}{b_1}e_2, e_3, e_4$. The matrix of $\sigma_1$ becomes

$$\psi_A(\sigma_1) = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Similarly, we determine the matrices of $\sigma_2$ and $\delta$. It is easy to see that the first column of the matrices of all generators is $(1, 0, 0, 0, 0)^T$, where $T$ is the transpose. We thus reduce our representation to a 3-dimensional one by deleting the first row and the first column to get $\phi_A' : A(E_{3,1}) \rightarrow GL_3(C)$. The representation is defined as follows:

$$\phi_A'(\sigma_1) = \begin{pmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\phi_A'(\sigma_2) = \begin{pmatrix} 1 & 0 & \frac{b_2}{b_1} \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix},$$

and

$$\phi_A'(\delta) = \begin{pmatrix} 1 + \lambda_1b_1 + \lambda_2b_2 & \lambda_1d_1 + \frac{b_2}{b_1}\lambda_2d_1 & -\lambda_1 + \frac{b_2}{b_1}\lambda_2 \\ b_1 & 1 + d_1 & -t \\ 0 & 0 & 1 \end{pmatrix}.$$
We then diagonalize the matrix corresponding to \( \psi'_1(\sigma_1) \) by an invertible matrix, say \( T \), and conjugate the matrices of \( \psi'_1(\sigma_2) \) and \( \psi'_2(\delta) \) by the same matrix \( T \). The invertible matrix \( T \) is given by

\[
T = \begin{pmatrix}
0 & 1 & t \\
0 & 0 & -1 - t \\
1 & 0 & 1
\end{pmatrix}.
\]

In fact, a computation shows that

\[
T^{-1}\psi'_1(\sigma_1)T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -t
\end{pmatrix}.
\]

After conjugation, we get

\[
T^{-1}\psi'_1(\sigma_2)T = \begin{pmatrix}
\frac{-1+i+rt}{1+i} & 0 & \frac{-1(1+i+rt)}{1+i} \\
0 & \frac{tb_1+t+tb_2}{h_1(1+i)} & 1 \\
0 & 0 & \frac{t}{1+i}
\end{pmatrix},
\]

\[
T^{-1}\psi'_2(\delta)T = \begin{pmatrix}
\frac{1}{1+i} & \frac{h_1}{1+i} & \frac{t}{1+i} \\
-t(\lambda_1 + \frac{b_2d_1}{b_1} + \frac{b_1}{1+i}) & 1 + \lambda_2b_2 + b_1(\lambda_1 + \frac{t}{1+i}) & \frac{(-t+\lambda_2t-d_1(1+i)+b_2(1+i)+b_1(\lambda_1+i+\lambda_2i))}{b_1(1+i)} \\
0 & \frac{h_1}{1+i} & \frac{t}{1+i}
\end{pmatrix}.
\]

The entries of the matrices \( T^{-1}\psi'_1(\sigma_2)T \) and \( T^{-1}\psi'_2(\delta)T \) are well-defined since we assume in our work that \( t \neq -1 \). For simplicity, we denote \( T^{-1}\psi'_1(\sigma_1)T \) by \( \psi'_1(\sigma_1) \), \( T^{-1}\psi'_1(\sigma_2)T \) by \( \psi'_1(\sigma_2) \), and \( T^{-1}\psi'_2(\delta)T \) by \( \psi'_2(\delta) \).

We now prove some propositions to determine a sufficient condition for irreducibility of \( \psi'_j : A(E_{3,1}) \to GL_3(\mathbb{C}) \).

**Proposition 2** \( t(b_2 + b_1t + b_2t + 1 + t + t^2)(\lambda_2b_2(1+t) + b_1(\lambda_1 + t + \lambda_2t)) = -(t+1)^2(t^2 + 1) \)

**Proof.** The proof easily follows by considering the following relations:

- \( t b_2 = -td_1 - t(1 + t) \)
- \( t b_1 = (1 + t)d_1 + 2t \)
- \( \lambda_2b_1 + \lambda_2b_2 = -(1 + t + d_1) \)

**Proposition 3** The two expressions \( 1 + t + t^2 \) and \( b_1t + b_2t + b_2 \) cannot be both equal to zeros.

**Proof.** We assume, for contradiction, that both are equal to zeros. By substituting \( t b_2 = -td_1 - t(1 + t) \) and \( t b_1 = (1 + t)d_1 + 2t \), in \( b_1t + b_2t + b_2 = 0 \), we get \(-t(1 + t + t^2) = -t^2 \). By assuming that \( 1 + t + t^2 = 0 \), we get that \( t = 0 \), a contradiction.

**Proposition 4** \(-t + b_1t - d_1(1 + t) \neq 0 \).

**Proof.** Assume, for contradiction, that \(-t + b_1t - d_1(1 + t) = 0 \). Having that \( b_1t = (1 + t)d_1 + 2t \), we get \(-t + b_1t - d_1(1 + t) = -t + (1 + t)d_1 + t + t - d_1(1 + t) = t \). This implies that \( t = 0 \), a contradiction.

We use Proposition 2, Proposition 3 and Proposition 4 to prove the following Lemma. We recall that all the indeterminates used in defining the representations are specialized to non zero complex numbers and, in addition, the complex number associated with \( t \) is not equal to \(-1 \).
Lemma 5 If \( t \neq \pm i \), then any non zero subspace \( S \), which is invariant under the action of the representation \( \psi'_t: A(E_{3,1}) \rightarrow GL_3(\mathbb{C}) \) containing the standard unit vector \( e_3 \), must be the whole space \( \mathbb{C}^3 \).

**Proof.** We have that \( \psi'_t(\sigma_2)(e_3) = \frac{-(1+t+i^2)}{1+t} e_1 + \frac{t(b_2 + b_1 t + b_2 t)}{b_1(1+t)} e_2 + \frac{1}{1+t} e_3 \in S \).

Since \( e_3 \in S \), it follows that \( \frac{-(1+t+i^2)}{1+t} e_1 + \frac{t(b_2 + b_1 t + b_2 t)}{b_1(1+t)} e_2 \in S \).

Moreover,

\[
\psi'_t(\delta)(e_3) = \frac{-t + b_1 t - d_1(1+t)}{1+t} e_1 + \frac{t + b_1 t - d_1(1+t)[(t) + b_1(1+t)]}{b_1(1+t)} e_2 + \frac{1+t+d_1(1+t)(1-t)}{1+t} e_3 \in S.
\]

This also implies that

\[
\frac{-t + b_1 t - d_1(1+t)}{1+t} e_1 + \frac{(-t + b_1 t - d_1(1+t))[\lambda b_2(1+t) + b_1(1+t + \lambda_1 t)]}{b_1(1+t)} e_2 \in S.
\]

Having proved that \( 1 + t + t^2 \) and \( b_1 t + b_2 t + b_2 \) can’t both be zeros, we consider the following cases:

**Case 1.** \( 1 + t + t^2 = 0 \)

By Proposition 3 and (1), we get that \( e_2 \in S \). By Proposition 4 and (2), we get that \( e_1 \in S \). Thus, \( S \) is the whole space.

**Case 2.** \( 1 + t + t^2 \neq 0 \)

Let us multiply (1) by \(-t + b_1 t - d_1(1+t)\) which is proved not to be zero in Proposition 4. We also multiply (2) by \( 1 + t + t^2 \neq 0 \). If we add the obtained equations, we get

\[
\frac{-t + b_1 t - d_1(1+t)}{1+t} e_1 + \frac{(-t + b_1 t - d_1(1+t))[\lambda b_2(1+t) + b_1(1+t + \lambda_1 t)]}{b_1(1+t)} e_2 \in S.
\]

By Proposition 2, we have that \( t(b_2 + b_1 t + b_2 t) + (1 + t + t^2)(\lambda b_2(1+t) + b_1(1+t + \lambda_1 t)) = -(t+1)(t^2 + 1) \).

Assuming that \( t \neq -1 \) and \( t \neq \pm i \), we get \( t(b_2 + b_1 t + b_2 t) + (1 + t + t^2)(\lambda b_2(1+t) + b_1(1+t + \lambda_1 t)) \neq 0 \).

By Proposition 4 and by (3), we get

\[
e_2 \in S.
\]

From (1) we conclude that

\[
e_1 \in S.
\]

Thus, \( S \) is the whole space \( \mathbb{C}^3 \).

Next, we present the following theorem which gives a sufficient condition for irreducibility of \( \psi'_t: A(E_{3,1}) \rightarrow GL_3(\mathbb{C}) \).

**Theorem 6** If \( t \neq \pm i \), then the representation \( \psi'_t: A(E_{3,1}) \rightarrow GL_3(\mathbb{C}) \) is irreducible.

**Proof.** Let \( S \) be a non zero proper subspace of \( \mathbb{C}^3 \), which is invariant under the action of \( \psi'_t \). By Lemma 5, we have that \( e_3 \notin S \).

Then \( S \) is one of the following subspaces:

- \( S = \langle e_1 \rangle \)
- \( S = \langle e_2 \rangle \)
- \( S = \langle e_1 + u e_2 \rangle \), where \( u \in \mathbb{C}^* \)
- \( S = \langle e_1, e_2 \rangle \)

**Case 1.** \( S = \langle e_1 \rangle \). We have that \( \psi'_t(\sigma_2)(e_1) \in S \). This implies that \( \begin{pmatrix} \frac{2}{1+t} \\ 1+t \\ \frac{2}{1+t} \end{pmatrix} \) \( e_3 \in S \). This gives a contradiction because \( t \neq 0 \).
Case 2. $S = \langle e_2 \rangle$. We have that $\psi'_1(\delta)(e_2) \in S$. This implies that 
\[ \left( \begin{array}{c} \frac{b_1}{1+i} \\ 1 + \lambda_2 b_2 + b_1 (\lambda_1 + \frac{t}{1+i}) \end{array} \right) \in S. \]
This gives a contradiction since $b_1 \neq 0$.

Case 3. $S = \langle e_1 + ue_2 \rangle, \ u \in \mathbb{C}^*$. We have that $\psi'_1(\sigma_2)(e_1 + ue_2) \in S$. This implies that 
\[ \left( \begin{array}{c} -\frac{t^2}{1+i} \\ \frac{-t}{1+i} \end{array} \right) \in S. \]
This gives a contradiction since $t \neq 0$.

Case 4. $S = \langle e_1, e_2 \rangle$. We have that $\psi'_1(\sigma_2)(e_1) \in S$. This implies that 
\[ \left( \begin{array}{c} \frac{-t^2}{1+i} \\ \frac{-t}{1+i} \end{array} \right) \in S. \]
This gives a contradiction since $t \neq 0$.

Therefore, we conclude that the representation is irreducible because there is no proper non zero invariant subspace under the action of $\psi'_1$.

We now give a necessary condition for irreducibility.

**Theorem 7** If $t = \pm i$, then the subspace $\langle e_1, e_3 \rangle$ is a proper invariant subspace.

**Proof.**

1. $\psi'_1(\sigma_1)(e_1) = e_1 \in S$.

2. $\psi'_1(\sigma_2)(e_1) = \left( \begin{array}{c} \frac{-t^2}{1+i} \\ \frac{-t}{1+i} \end{array} \right) = ae_1 + be_3$, where $a$ and $b \in \mathbb{C} - \{0\}$.

Here, we have $a = \frac{-t^2}{1+i}, \ b = \frac{-t}{1+i}$, and $\frac{h_2 + h_1 + b_1 t}{h_1 (1+i)} = 0$. This is true since $t^2 = -1$.

3. $\psi'_1(\delta)(e_1) = \left( \begin{array}{c} \frac{1}{1+i} \\ t(\lambda_1 + \frac{b_2 h}{h_1} + \frac{t}{1+i}) \end{array} \right) = ae_1 + be_3$.

where $a$ and $b$ are given by $a = \frac{1}{1+i}$ and $b = \frac{t}{1+i}$.

4. $\psi'_1(\sigma_1)(e_3) = -te_3 \in S$.

5. $\psi'_1(\sigma_2)(e_3) = \left( \begin{array}{c} \frac{-t}{1+i} \\ \frac{h_2 + h_1 + b_1 t}{h_1 (1+i)} \end{array} \right) = ae_1 + be_3$.

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where \( a = \frac{-(1+t+t^2)}{t^2} \) and \( b = \frac{1}{t^2} \).

6. \( \psi'_e(\delta)e_3 = \begin{pmatrix} \frac{t}{t^2} \\ \frac{1}{t^2} \\ \frac{1}{t^2} \end{pmatrix} \) = \( ae_1 + be_3 \),

where \( a \) and \( b \) are given by \( a = \frac{t}{t^2} \) and \( b = \frac{1}{t^2} \).

By Proposition 2, we have \( \frac{t}{t^2} \frac{-(t+b_2c-d_1(1+t))(b_1c_d_1(1+t)+b_1c_d_1+\lambda_4\lambda_5)}{b_1(1+t)} = 0 \).

Thus, we have determined a necessary and sufficient condition for irreducibility.

**Theorem 8** Let \( \lambda_1, \lambda_2, b_1, b_2, d_1, t \in \mathbb{C} - \{0\} \) and \( t \neq -1 \). The representation \( \psi'_e : \mathbb{A}(E_{3,1}) \rightarrow GL_3(\mathbb{C}) \) is irreducible if and only if \( t \neq \pm i \).

**References**

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