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On the Product of the Non-linear of Diamond Operator and $\otimes^k$ Operator

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Abstract

In this paper, we study the solution of nonlinear equation $\otimes^k \diamond_{c_1}^k u(x) = f(x, \Box^{k-1} L^k \diamond_{c_1}^k u(x))$ where $\otimes^k \diamond_{c_1}^k$ is the product of the Otimes operator and Diamond operator where c_1 is positive constants, k is a positive integer, $p + q = n$, n is the dimension of the Euclidean space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,$u(x)$ is an unknown function and $f(x, \Box^{k-1}.L^k \diamond_{c_1}^k u(x))$ is a given function. It was found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\Box^{k-1} L^k \diamond_{c_1}^k u(x)$.

Keywords: The hyperbolic kernel of Marcel Riesz, Diamond operator, Schander's estimates

1. Introduction

The operator $\diamond^k$ has been first by A. Kananthai (1997) and is named as the Diamond operator iterated k times and is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k, \tag{1}$$

$p + q = n$, n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamond^k$ can be expressed in the form $\diamond^k = \triangle^k \Box^k = \Box^k \triangle^k$ where $\triangle^k$ is the Laplacian operator iterated k times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{2}$$

and $\Box^k$ is the ultra-hyperbolic operator iterated k times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k. \tag{3}$$

Next, W. Satsanit has been first introduced $\otimes^k$ operator and $\otimes^k$ is defined by

$$\begin{aligned}
\otimes^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right) \cdot \right. \\
&\quad \left. \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \Box^k \left(\triangle^2 - \frac{1}{4}(\triangle + \Box)(\triangle - \Box)\right)^k \\
&= \left(\frac{3}{4}\diamond\triangle + \frac{1}{4}\Box^3\right)^k
\end{aligned} \tag{4}$$

where $\diamond$, $\triangle$ and $\Box$ are defined by (1), (2) and (3) with $k = 1$ respectively.

Consider the nonlinear equation

$$\otimes^k u(x) = f(x, \Box^{k-1} L^k u(x)) \tag{5}$$

where $\otimes^k$ is the operator iterated k times is defined (4), and L^k is the operator iterated k times is defined by

$$Ł^k = (\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k \tag{6}$$

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and $\square^{k-1}$ is the ultra-hyperbolic operator iterated $k-1$ times defined by (3). Let f defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of R^n and $\partial\Omega$ denotes the boundary of Ω and f be a bounded function, that is

$$|f(x, \square^{k-1}L^k u(x))| \leq N, \quad x \in \Omega \tag{7}$$

with the boundary condition

$$\square^{k-1}L^k u(x) = 0 \ , \quad x \in \partial\Omega. \tag{8}$$

Then, we obtain

$$u(x) = R^H_{2(k-1)}(x) * (R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)) * (S^{*k}(x))^{*-1} * W(x) \tag{9}$$

as a solution of (5) with the boundary condition

$$u(x) = (R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)) * (S^{*k}(x))^{*-1} * (R^H_{2(k-2)}(x))^{(m)} \tag{10}$$

for $x \in \partial\Omega$, $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension for $k = 2, 3, 4, 5, ...$ and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$. The function $R^H_{2(k-2)}(x)$ defined by (15) with $\alpha = 2(k-2)$ and , $R^e_{4k}(x)$ is given by (20) with $\gamma = 4k$.

The purpose o this work to extend the $\diamond^k$ operator defined by (1) to be

$$\diamond^k_{c_1} = \left(\frac{1}{c_1^4} \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k. \tag{11}$$

Where c_1 is positive constant and k is a non-negative integer

Now, we study the nonlinear equation of the form

$$\otimes^k \diamond^k_{c_1} u(x) = f(x, \square^{k-1} L^k \diamond^k_{c_1} u(x)) \tag{12}$$

with f defined and having continuous first derivative for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of R^n and $\partial\Omega$ denoted the boundary of Ω , f is bounded on Ω that is $|f| \leq N$, N is constant ,$\diamond^k_{c_1}$ defined by (11) and $\square^k$ defined by (3) and L^k defined by (6).

We can find the solution $u(x)$ of (12) which unique under the boundary condition $\square^{k-1}L^k\diamond^k_{c_1}u(x) = 0$ for all $x \in \Omega$. By (R.Courant, 1996 p.369) there exists a unique solution $W(x)$ of the equation $\square W(x) = f(x, W(x))$ for all $x \in \Omega$ with the boundary condition $W(x) = 0$ for all $x \in \partial\Omega$ where $W(x) = \square^{k-1}L^k\diamond^k_{c_1}u(x)$. Moreover, if we put $p = k = 1$ in $\square^k\square^k_{c_1}M(x) = W(x)$, we found that $M(x) = I^H_2(x) * N^H_2(x) * W(x)$ is solution of the inhomogeneous wave equation where $I^H_2(x)$ and $N^H_2(x)$ are defined by (18) and (19) with $\alpha = \beta = 2$ respectively.

Before going that points , the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1 Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Denoted by

$$\upsilon = (x_1^2 + x_2^2 + ... + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{13}$$

$$w = c_1^2(x_1^2 + x_2^2 + ... + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \tag{14}$$

where $p + q = n$. The interior of forward cone defined by
$\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, \ \upsilon > 0, w > 0\}$. For any complex number α, define the function

$$R^H_\alpha(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{15}$$

and

$$S^H_\beta(w) = \begin{cases} \frac{w^{\frac{\beta-n}{2}}}{K_n(\beta)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{16}$$

where the constant $K_n(\alpha), K_n(\beta)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})}. \tag{17}$$

The function $R_\alpha^H(u), S_\beta^H(w)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki (1964, p.72). It is well known that $R_\alpha^H(u)$ and $S_\beta^H(w)$ is an ordinary function if $Re(\alpha) \geq n$ and $Re(\beta) \geq n$ and is a distribution if $Re(\alpha) < n$ and $Re(\beta) < n$. By putting $p = 1$ in (13), (14) and (17) using the Legendre's duplication of

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$$

then (15) and (16) reduce to

$$I_\alpha^H(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{18}$$

and

$$N_\beta^H(x) = \begin{cases} \frac{w^{\frac{\beta-n}{2}}}{H_n(\beta)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{19}$$

respectively, where

$$H_n(\alpha) = \pi^{\frac{n-2}{2}}2^{\alpha-1}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{\alpha}{2}).$$
$$u = x_1^2 - x_2^2 - x_3^2 - ... - x_n^2$$

and

$$H_n(\beta) = \pi^{\frac{n-2}{2}}2^{\beta-1}\Gamma(\frac{2+\beta-n}{2})\Gamma(\frac{\beta}{2}).$$
$$w = c_1^2x_1^2 - x_2^2 - x_3^2 - ... - x_n^2.$$

The functions $I_\alpha^H(x)$ and $N_\beta^H(x)$ are precisely called the Hyperbolic kernel of Marcel Riesz.

Definition 2.2 Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and the function $R_\gamma^e(x)$ and $L_\rho^e(x)$ is defined by

$$R_\gamma^e(x) = \frac{X^{\frac{\gamma-n}{2}}}{P_n(\gamma)} \tag{20}$$

and

$$L_\rho^e(x) = \frac{Y^{\frac{\rho-n}{2}}}{P_n(\rho)} \tag{21}$$

where

$$X = x_1^2 + x_2^2 + ... + x_n^2$$

and

$$Y = c_1^2(x_1^2 + x_2^2 + ... + x_p^2) + (x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2)$$
$$P_n(\gamma) = \frac{\pi^{\frac{n}{2}}2^\gamma\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)} \tag{22}$$

γ is a complex parameter and n is the dimension of $\mathbb{R}^n$.

Definition 2.3 Let c_1 be positive number, $p + q = n$ and k is a nonnegative integer. The ultra-hyperbolic operators iterated k times $\Box^k$ and $\Box_{c_1}^k$ are defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k$$

and

$$\Box_{c_1}^k = \left(\frac{1}{c_1^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right)^k. \tag{23}$$

The Laplacian operators iterated k times $\triangle^k$ and $\triangle_{c_1}^k$ are defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k$$

and

$$\triangle_{c_1}^k = \left(\frac{1}{c_1^2}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)\right)^k. \tag{24}$$

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Lemma 1 *Given the equation*

$$\Box^k u(x) = \delta \tag{25}$$

and

$$\Box^k_{c_1} v(x) = \delta. \tag{26}$$

Where $\Box^k$ *and* $\Box^k_{c_1}$ *defined by (3) and (13) respectively,* $x \in \mathbb{R}^n$ *and* δ *is the Dirac-delta distribution. Then we obtain*

$$u(x) = R^H_{2k}(x) \quad , \quad v(x) = S^H_{2k}(x)$$

are an elementary solution of (15) and (16) respectively where $R^H_{2k}(x)$ *and* $S^H_{2k}(x)$ *are defined by (15) and (16) with* $\alpha = \beta = 2k$.

Proof. (S.E. Trione, 1987, p.11).

Lemma 2 *Given the equation*

$$\triangle^k u(x) = \delta \tag{27}$$

and

$$\triangle^k_{c_1} v(x) = \delta. \tag{28}$$

Where $\triangle^k$ *and* $\triangle^k_{c_1}$ *defined by (2) and (24) respectively,* $x \in \mathbb{R}^n$ *and* δ *is the Dirac-delta distribution. Then we obtain*

$$u(x) = (-1)^k R^e_{2k}(x) \quad , \quad v(x) = (-1)^k L^e_{2k}(x)$$

are an elementary solution of (17) and (18) respectively, $R^e_{2k}(x)$ *and* $L^e_{2k}(x)$ *are defined by (19) and (20) with* $\gamma = \rho = 2k$

Proof. (W.F. Donoghue, 1969, p.118).

Lemma 3 *Let* $S_\alpha(x)$*and* $R_\beta(x)$ *be the function defined by (13) and (14) respectively. Then*

$$S_\alpha(x) * S_\beta(x) = S_{\alpha+\beta}(x)$$

and

$$R_\beta(x) * R_\alpha(x) = R_{\beta+\alpha}(x)$$

where α *and* β *are a positive even number.*

Proof. *(Aguirre Manuel A., 2008, pp.171-190).* □

Lemma 4 *The function* $R_{-2k}(x)$ *and* $(-1)^k S_{-2k}(x)$ *are the inverse in the convolution algebra of* $R_{2k}(x)$ *and* $(-1)^k S_{2k}(x)$ *, respectively.That is*

$$R_{-2k}(x) * R_{2k}(x) = R_{-2k+2k}(x) = R_0(x) = \delta(x)$$

and

$$(-1)^k S_{-2k}(x) * (-1)^k S_{2k}(x) = (-1)^{2k} S_{-2k+2k}(x) = S_0(x) = \delta(x)$$

Proof. (S.E. Trione, 1987, p.123), (W.F. Donoghue, 1969, p.118, p.158), and (A. Kananthai, 2000, p.10).

Lemma 5 *Given P is a hyper-function then*

$$P\delta^k(p) + k\delta^{(k-1)}(p) = 0$$

where $\delta^{(k)}$ *is the Dirac-delta distribution with k derivatives.*

Proof. (I.M.Gelfand, 1964, p.233).

Lemma 6 *Given the equation*

$$\Box^k u(x) = 0. \tag{29}$$

Where $\Box^k$ *is defined by (3) and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *then* $u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}$ *is a solution of (29) where* $(R^H_{2(k-1)}(\upsilon))^{(m)}$ *is defined by (15) with m - derivatives and* $\alpha = 2(k-1)$, $m = \frac{n-4}{2}$, $n \geq 4$ *and n is even dimension.*

Proof. We first to show that the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \ldots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2$, $p + q = n$ is a solution of the equation

$$\Box u(x) = 0. \tag{30}$$

Where $\Box$ is defined by (3) with $k = 1$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$\frac{\partial}{\partial x_i}\delta^{(m)}(r^2 - s^2) = 2x_i\delta^{(m+1)}(r^2 - s^2)$$

$$\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) = 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2\delta^{(m+2)}(r^2 - s^2)$$

$$\begin{aligned}
\Box\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)\\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2\delta^{(m+2)}(r^2 - s^2)\\
&= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2)\\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2)\\
&= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m + 2)\delta^{(m+1)}(r^2 - s^2)\\
&+ 4s^2\delta^{(m+2)}(r^2 - s^2)\\
&= (2p - 4(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4s^2\delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

By Lemma 1 with $P = r^2 - s^2$. Similarly,

$$\begin{aligned}
\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(r^2 - s^2) &= (-2q + 4(m + 2))\delta^{(m+1)}(r^2 - s^2)\\
&\quad + 4r^2\delta^{(m+2)}(r^2 - s^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Box\delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(r^2 - s^2)\\
&= (2(p + q) - 8(m + 2))\delta^{(m+1)}(r^2 - s^2) - 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2)\\
&= (2n - 8(m + 2))\delta^{(m+1)}(r^2 - s^2) + 4(m + 2)\delta^{(m+1)}(r^2 - s^2)\\
&= (2n - 4(m + 2))\delta^{(m+1)}(r^2 - s^2).
\end{aligned}$$

If $2n - 4(m + 2) = 0$, we have $\Box\delta^{(m)}(r^2 - s^2) = 0$. That is $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (29) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. We write

$$\Box^k u(x) = \Box(\Box^{k-1}u(x)) = 0$$

From the above proof we have $\Box^{k-1}u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(x)$, we obtain

$$R^H_{2(k-1)}(x) * \Box^{k-1}u(x) = R^H_{2(k-1)}(x) * \delta^{(m)}(r^2 - s^2)$$

$$\Box^{k-1}(R^H_{2(k-1)}(x)) * u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}, \quad \textit{where} \ \ \upsilon = (r^2 - s^2)$$

$$\delta * u(x) = u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}.$$

Thus $u(x) = (R^H_{2(k-1)}(\upsilon))^{(m)}$ is a solution of (29) with $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Lemma 7 *Given the equation*

$$\Box^k_{c_1} u(x) = 0 \tag{31}$$

where $\Box^k_{c_1}$ is defined by (23). Then we obtain $u(x) = \left(S^H_{2(k-1)}(x)\right)^{(m)}$ as a solution of (31) where $\left(S^H_{2(k-1)}(x)\right)^{(m)}$ is defined by (16) with m derivative and $\beta = 2(k - 1)$, $m = \frac{n-4}{2}, n \geq 4$ and n is even dimension.

Proof. The proof of Lemma 7 is similar to the proof of Lemma 6.

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Lemma 8 *Given the equation*

$$L^k G(x) = \delta(x) \tag{32}$$

where $L^k = (\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2)^k$, $\triangle$ and $\Box$ is defined by (2) and (3) with $k = 1$ respectively. Then we obtain $G(x)$ is an elementary solution of (32) where

$$G(x) = \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} \tag{33}$$

and

$$C(x) = \frac{3}{4} R^H_4(x) + \frac{1}{4}(-1)^2 R^e_4(x). \tag{34}$$

$C^{*k}(x)$ denotes the convolution of S it self $k-$times , $\left(C^{*k}(x)\right)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $G(x)$ is a tempered distribution.

Proof. From (32), we have

$$L^k = \left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^k G(x) = \delta(x)$$

or we can write

$$\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R^H_4(x) * (-1)^2 R^e_4(x)$,

$$\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right) * \left(R^H_4(x) * (-1)^2 R^e_4(x)\right)\left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} G(x) = \delta(x) * R^H_4(x) * (-1)^2 R^e_4(x)$$

or

$$\left(\frac{3}{4}\triangle^2(R^H_4(x) * (-1)^2 R^e_4(x)) + \frac{1}{4}\Box^2(R^H_4(x) * (-1)^2 R^e_4(x))\right) * \left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} G(x) = \delta(x) * R^H_4(x) * (-1)^2 R^e_4(x).$$

By properties of convolution, we obtain

$$\left(\frac{3}{4}\triangle^2((-1)^2 R^e_4(x)) * R^H_4(x)) + \frac{1}{4}\Box^2(R^H_4(x)) * (-1)^2 R^e_4(x)\right) * \left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} G(x) = \delta(x) * R^H_4(x) * (-1)^2 R^e_4(x).$$

By Lemma 1 and Lemma 2, we obtain

$$\left(\frac{3}{4}R^H_4(x) + \frac{1}{4}(-1)^2 R^e_4(x)\right) * \left(\frac{3}{4}\triangle^2 + \frac{1}{4}\Box^2\right)^{k-1} G(x) = R^H_4(x) * (-1)^2 R^e_4(x)$$

keeping on convolving both sides of the above equation by $R^H_4(x) * (-1)^2 R^e_4(x)$ up to $k-1$ times, we obtain

$$C^{*k}(x) * G(x) = \left(R^H_4(x) * (-1)^2 R^e_4(x)\right)^{*k}$$

the symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(x)$, we have

$$\left(R^H_4(x) * (-1)^2 R^e_4(x)\right)^{*k} = R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x).$$

$$C^{*k}(x) * G(x) = \left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\right)$$

$$G(x) = \left(R^H_{4k}(x) * (-1)^2 R^e_{4k}(x)\right) * (C^{*k}(x))^{*-1}$$

is an elementary solution of (32) where $R^H_{4k}(x)$ and $R^e_{4k}(x)$ are deined by (15), (20) with $\alpha = \gamma = 4k$ respectively.

Lemma 9 *Given the equation*

$$\Box u(x) = f(x, u(x)) \tag{35}$$

where f is defined and has continuous first deivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ is the boundary of Ω. Assume that f is bounded, that is $|f(x, u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (35).

Proof. We can prove the existence of the solution $u(x)$ of (35) by the method of iterations and the Schuder's estimates. The details of the proof are given by Courant and Hilbert, (R.Courant, 1966, pp.369-372).

3. Main Results

Theorem

Consider the nonlinear equation

$$\otimes^k \diamond_{c_1}^k u(x) = f(x, \square^{k-1} L^k \diamond_{c_1}^k u(x)) \tag{36}$$

Where $\square^k, \diamond_{c_1}^k$ are defined by (3) , (11) respectively and the operator L^k is defined by (6). Let f be defined and having continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Suppose f is bounded function, that is

$$|f(x, \square^{k-1} L^k \diamond_{c_1}^k u(x))| \leq N \tag{37}$$

for all $x \in \Omega$ and the boundary condition

$$\square^{k-1} L^k \diamond_{c_1}^k u(x) = 0 \tag{38}$$

for all $x \in \partial\Omega$. Then we obtain

$$u(x) = R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x) * W(x) \tag{39}$$

as a solution of (36) with the boundary condition

$$u(x) = G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x) * (R^H_{2(k-2)}(\upsilon))^{(m)} \tag{40}$$

for all $x \in \partial\Omega$, $m = (n-4)/2$, $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, and $G(x)$ defined by (33). The function $L^e_{2k}(x), S^H_{2k}(x)$ are defined by (22), (16) with $\rho = 2k, \beta = 2k$ respectively and $(R^H_{2(k-2)}(\upsilon))^{(m)}$ is defined by (15) with $\alpha = 2(k-2)$. Moreover, for $k = 1$ we obtain

$$M(x) = \left(R^H_{-2}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(C^{*1}(x)\right) * (-1)^k S_{-2}(x) * u(x)$$

as a solution of the inhomogeneous equation

$$\square \square_{c_1} M(x) \ = \ W(x).$$

Where $\square$ and $\square_{c_1}$ are defined by (3), (23) with $k = 1$ respectively and $u(x)$ is obtained from (48). Furthermore, if we put $p = k = 1$ then the operator $\square^k$ and $\square^k_{c_1}$ reduces to

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \ \ and \ \ \frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

respectively and the solution $M(x) = I^H_2(x) * N^H_2(x) * W(x)$ which is the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot \left(\frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) M(x) = W(x).$$

Where $I^H_2(x)$ is defined by (18) with $\alpha = 2$ and $N^H_2(x)$ is defined by (19) with $\beta = 2$. **Proof.** We have

$$\begin{aligned} \otimes^k \diamond_{c_1}^k u(x) &= \square \square^{k-1} L^k \diamond_{c_1}^k u(x) \\ &= f(x, \square^{k-1} L^k \diamond_{c_1}^k u(x)). \end{aligned} \tag{41}$$

Since $u(x)$ has continuous derivative up to order $6k$ for $k = 1, 2, 3, \ldots$ and $\square^{k-1} L^k \diamond_{c_1}^k u(x)$ exists as the generalized function. Thus we can assume

$$\square^{k-1} L^k \diamond_{c_1}^k u(x) = W(x) \ \ , \ \ \forall x \in \Omega. \tag{42}$$

Then (41) can be written in the form

$$\otimes^k \diamond_{c_1}^k u(x) = \square W(x) = f(x, W(x)) \tag{43}$$

by (37)

$$|f(x, W(x))| \leq \ N \ \ , \ \ x \in \Omega \tag{44}$$

and by (38), $W(x) = 0$, $x \in \partial\Omega$ or

$$\square^{k-1} L^k \diamond_{c_1}^k u(x) = 0 \ \ , \ \ \forall x \in \partial\Omega. \tag{45}$$

We obtain a unique solution of (43) which satisfies (37) by Lemma 9. Since

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$R^H_{2(k-1)}(x)$, $(-1)^k L^e_{2k}(x)$ and $S^H_{2k}(x)$ are the elementary solution of the operators $\Box^{k-1}$, $\triangle^k_{c_1}$ and $\Box^k_{c_1}$ respectively, and by Lemma 8, we have $G(x)$ is an elementary of the operator L^k where $G(x)$ defined by (33), i.e.

$$\Box^{k-1} R^H_{2(k-1)}(x) \;=\; \delta \quad , \quad \triangle^k_{c_1}(-1)^k L^e_{2k}(x) \;=\; \delta \tag{46}$$

and

$$\Box^{k-1}_{c_1} S^H_{2k}(x) \;=\; \delta \quad , \quad L^k G(x) \;=\; \delta \tag{47}$$

From (42)

$$\Box^{k-1} L^k \diamond^k_{c_1} u(x) \;=\; W(x).$$

Convolving the above equation by $R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)$, we obtain

$$\begin{aligned}&\left(\Box^{k-1} L^k \diamond^k_{c_1} u(x)\right) * \left(R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right)\\ &= \left(R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right) * W(x).\end{aligned}$$

By the properties of convolution, we obtain

$$\begin{aligned}&(\Box^{k-1} R^H_{2(k-1)}(x)) * (L^k G(x))(\diamond^k_{c_1} * (-1)^k L^e_{2k} * S^H_{2k}) * u(x)\\ &= \left(R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right) * W(x)\end{aligned}$$

or

$$\delta * \delta * \delta * u(x) = \left(R^H_{2(k-1)}(u) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right) * W(x)$$

Thus

$$u(x) = \left(R^H_{2(k-1)}(x) * G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right) * W(x) \tag{48}$$

as a solution of (36).

Next, consider the condition (45). From

$$\Box^{k-1} L^k \diamond^k_{c_1} u(x) = 0. \tag{49}$$

By Lemma 6, we have

$$L^k \diamond^k_{c_1} u(x) = (R^H_{2(k-2)}(\upsilon))^{(m)}. \tag{50}$$

Where $m = \frac{n-4}{2}$, $n \geq 4$ and n is even dimension. Convolving both sides of (50) by $G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)$. We obtain

$$\left(G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)\right) * L^k \Box^k_{c_1} u(x) = G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x) * (R^H_{2(k-2)}(\upsilon))^{(m)}$$

By the properties of convolution, we obtain

$$(L^k G(x)) * (\diamond^k_{c_1}(-1)^k L^e_{2k}(x) * S^H_{2k}(x)) * u(x) = G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x) * (R^H_{2(k-2)}(\upsilon))^{(m)}$$

By Lemma 8 and Lemma 1, Lemma 2, we obtain

$$\delta * \delta * u(x) = (G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x)) * (R^H_{2(k-2)}(x))^{(m)}$$

Thus for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \ldots ...$

$$u(x) = G(x) * (-1)^k L^e_{2k}(x) * S^H_{2k}(x) * (R^H_{2(k-2)}(\upsilon)^{(m)} \tag{51}$$

as required. Now, for $k = 1$ in (48), we have

$$u(x) = \delta(x) * G(x) * (-1) L^e_2(x) * S_2(x) * W(x). \tag{52}$$

By Lemma 8, we have

$$G(x) = \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(C^{*1}(x)\right)^{*-1}.$$

Taking into account (52), we obtain

$$u(x) = \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(C^{*1}(x)\right)^{*-1} * (-1)^1 L^e_2(x) * S^H_2(x) * W(x) \tag{53}$$

as a solution of (36) for $k = 1$.

Convolving both sides of (53) by

$$\left(R^H_{-2}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(C^{*1}(x)\right) * (-1) L^e_{-2}(x).$$

By Lemma 4, we obtain

$$\left(R^H_{-2}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(C^{*1}(x)\right) * (-1)L^e_{-2}(x) * u(x) = R^H_2(x) * S^H_2(x) * W(x).$$

By Lemma 1, we obtain

$$M(x) = \left(R^H_{-2}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(C^{*1}(x)\right) * (-1)L^e_{-2}(x) * u(x) \tag{54}$$

as a solution of the inhomogeneous equation

$$\Box\Box_{c_1} M(x) \ = \ W(x). \tag{55}$$

Now, consider the boundary condition for $k = 1$ in (38), we have

$$L\diamond_{c_1} u(x) = 0 \ \ or \ \ \Box_{c_1}\triangle_{c_1} Lu(x) = 0$$

for $x \in \partial\Omega$. Thus by Lemma 6, for $k = 1$ we obtain

$$L\triangle_{c_1} u(x) = \delta^{(m)}(x) \tag{56}$$

for $x \in \partial\Omega$ where $\delta^{(m)}(x) = S^H_0(x)$. We convolved the above equation by $G(x) * (-1)L^e_2(x)$ where $G(x)$ is defined by (33) with $k = 1$ and $(-1)L^e_2(x)$ is defined by (22) with $\rho = 2$, we obtain

$$G(x) * (-1)L^e_2(x) * \left(L\triangle_{c_1} u(x)\right) = \delta^{(m)}(x) * G(x) * (-1)L^e_2(x).$$

By properties of convolution

$$LG(x) * \triangle_B(-1)L^e_2(x) * u(x) = \delta^{(m)}(x) * G(x) * (-1)L^e_2(x).$$

By Lemma 8 and Lemma 2, we obtain,

$$\delta(x) * \delta(x) * u(x) = \delta^{(m)}(\upsilon) * G(x) * (-1)L^e_2(x).$$

It follows that

$$u(x) = \delta^{(m)}(\upsilon) * G(x) * (-1)L^e_2(x). \tag{57}$$

By (33) with $k = 1$, we have

$$G(x) = \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(C^{*1}(x)\right)^{*-1}.$$

Taking into account (57), we obtain

$$u(x) = \delta^{(m)}(\upsilon) * \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(C^{*1}(x)\right)^{*-1} * (-1)L^e_2(x) \ \ for \ \ x \in \partial\Omega. \tag{58}$$

Now consider the case $k = 1, \ p = 1$ and $q = n - 1$ that is from (56), $R^H_2(x)$ reduced to $I^H_2(x)$ where $I^H_2(x)$ is defined by (18) with $\alpha = 2$ and $S^H_2(x)$ reduced to $N^H_2(x)$ where $N^H_2(x)$ is defined by (19) with $\beta = 2$ and then the operator $\Box$ defined by (3) reduces to the wave operator

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

and $\Box_{c_1}$ defined by (26) reduces to the wave operator

$$\Box^*_{c_1} = \frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

and then the solution $M(x)$ reduced to

$$M(x) = I^H_2(x) * N^H_2(x) * W(x)$$

which is the solution of inhomogeneous wave equation

$$\Box^*\Box^*_{c_1} M(x) = W(x).$$

or

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot \left(\frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) M(x) = W(x).$$

Where c_1 is a positive constant. $\Box$

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