On Cartesian Products of Cyclic Orthogonal Double Covers of Circulants

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Abstract

A collection $\mathcal{G}$ of isomorphic copies of a given subgraph $G$ of $T$ is said to be orthogonal double cover (ODC) of a graph $T$ by $G$, if every edge of $T$ belongs to exactly two members of $\mathcal{G}$ and any two different elements from $\mathcal{G}$ share at most one edge. An ODC $\mathcal{G}$ of $T$ is cyclic (CODC) if the cyclic group of order $|V(T)|$ is a subgroup of the automorphism group of $\mathcal{G}$. In this paper, the CODCs of infinite regular circulant graphs by certain infinite graph classes are considered, where the circulant graphs are labelled by the Cartesian product of two abelian groups.

Keywords: orthogonal double cover, orthogonal labelling, circulant graph

1. Introduction

A generalization of notion of an orthogonal double cover (ODC) to arbitrary underlying graphs is as follows. Let $T$ be an arbitrary graph with $n$ vertices and let $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}\}$ be a collection of $n$ spanning subgraphs of $T$. $\mathcal{G}$ is called an orthogonal double cover (ODC) of $T$ if there exists a bijective mapping $\phi: V(T) \rightarrow \mathcal{G}$ such that:

1. Every edge of $T$ is contained in exactly two of the graphs $G_0, G_1, \ldots, G_{n-1}$.
2. For every choice of different vertices $a, b$ of $T$, $|E(\phi(a)) \cap E(\phi(b))| = \begin{cases} 1 & \text{if } \{a, b\} \in T, \\ 0 & \text{otherwise.} \end{cases}$

Where $E(\phi(a))$ and $E(\phi(b))$ refer to the edge sets of the graphs $\phi(a)$ and $\phi(b)$ respectively, generally $E(G)$ refers to the edge set of the graph $G$.

An automorphism of an orthogonal double cover (ODC) $\mathcal{G} = \{G_0, G_1, G_2, \ldots, G_{n-1}\}$ of $T$ is a permutation $\pi: V(T) \rightarrow V(T)$ such that $[\pi(G_0), \pi(G_1), \ldots, \pi(G_{n-1})] = \mathcal{G}$, where for $i \in \{0, 1, 2, \ldots, n-1\}$, $\pi(G_i)$ is a subgraph of $T$ with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{[\pi(u), \pi(v)] : [u, v] \in E(G_i)\}$. An orthogonal double cover (ODC) $\mathcal{G}$ of $T$ is cyclic orthogonal double cover (CODC) if the cyclic group of order $|V(T)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$. Let $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order $n$. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write $v_i$ for the corresponding vertex and define $[w_i, u_j] \in E(K_{n,n})$ if and only if $i \neq j$, for all $w, u \in \Gamma$ and $i, j \in \mathbb{Z}_2$.

Let $G$ be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph $G$ with $E(G + a) = \{(u + a, v + a) : (u, v) \in E(G)\}$ is called the a-translate of $G$. The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$. $G$ is called a half starter with respect to $\Gamma$ if $|E(G)| = n$ and the lengths of all edges in $G$ are different, i.e. $d(e) : e \in E(G) \neq \Gamma$. The following three results were established in (El-Shanawany, 2002).

Theorem 1. (El-Shanawany, 2002) If $G$ is a half starter, then the union of all translates of $G$ forms an edge decomposition of $K_{n,n}$, i.e. $\bigcup_{a \in \Gamma} E(G + a) = E(K_{n,n})$. 

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Here, the half starter will be represented by the vector: \( v(G) = (v_{y_0}, v_{y_1}, \ldots, v_{y_n}) \), where \( v_{y_i} \in \Gamma \) and \((v_{y_i})_0\) is the unique vertex \((v_{y_i}, 0) \in \Gamma \times \{0\}\) that belongs to the unique edge of length \( y_i \).

Two half starter vectors \( v(G_0) \) and \( v(G_1) \) are said to be orthogonal if \( \{v_{y_0}(G_0) - v_{y_1}(G_1) : y \in \Gamma\} = \Gamma \).

**Theorem 2.** (El-Shanawany, 2002) If two half starters \( v(G_0) \) and \( v(G_1) \) are orthogonal, then \( G = [G_{a,i} : (a, i) \in \Gamma \times \mathbb{Z}_2] \) with \( G_{a,i} = G_i + a \) is an ODC of \( K_{n,n} \).

The subgraph \( G_s \) of \( K_{n,n} \) with \( E(G_s) = \{[u_0, v_1] : [v_0, u_1] \in E(G)\} \) is called the symmetric graph of \( G \). Note that if \( G \) is a half starter, then \( G_s \) is also a half starter.

A half starter \( G \) is called a symmetric starter with respect to \( \Gamma \) if \( v(G) \) and \( v(G_s) \) are orthogonal.

**Theorem 3.** (El-Shanawany, 2002) Let \( n \) be a positive integer and let \( G \) be a half starter represented by \( v(G) = (v_{y_0}, v_{y_1}, \ldots, v_{y_n}) \). Then \( G \) is symmetric starter if and only if \( v_{y_0} - v_{y_1} + y : y \in \Gamma \) = \( \Gamma \).

**Definition 4.** (El-Shanawany, 2002) Let \( G = (\Gamma \times \mathbb{Z}_2, E(G)) \) be a symmetric starter, let \([a, a] \) be the edge in \( G \) with length zero. The graph \( H = (\Gamma, E(H)) \) is called corresponding graph of \( G \), where \([a, b] \in E(H)\) if and only if \((a, b) \in E(G)\) with \( a \neq b \).

**Remark 5.** (El-Shanawany, 2002) Note that \(|E(G) - [a, a]| = n - 1 = |E(H)|\) the number of edges of the graph \( H \).

\(|\cup_{a \in \Gamma} E(G + a) - \cup_{a \in \Gamma} [a, a]| = n^2 - n = n(n - 1) = \cup_{a \in \Gamma} E(H + a) = |E(H)|\) the number of edges of an ODC of \( K_n \) group generated by \( H \).

**Theorem 6.** (El-Shanawany, 2002) Let \( n \) be a positive integer. Let \( G \) be a symmetric starter of \( K_{n,n} \) and let \( H \) be the corresponding graph of \( G \). Then \( H \) is an orthogonal double cover (ODC) – generating graph with respect to \( \Gamma \).

The author of (Gronau, Mullin, & Rosa, 1997) introduced the notion of an orthogonal labelling. Given a graph \( G = (V, E) \) with \( n - 1 \) edges, a \( 1 \times 1 \) mapping \( \psi : V(G) \rightarrow \mathbb{Z}_n \) is an orthogonal labelling of \( G \) if:

1. For every \( l \in \{1, 2, \ldots, \left[\frac{n-1}{2}\right]\} \), \( G \) contains exactly two edges of length \( l \), and exactly one edge of length \( n/2 \) if \( n \) is even, and
2. \( \{r(l) : l \in \{1, 2, \ldots, \left[\frac{n-1}{2}\right]\}\} = \{1, 2, \ldots, \left[\frac{n-1}{2}\right]\} \).

The following theorem of (Gronau et al., 1997) relates cyclic orthogonal double covers (CODCs) of \( K_n \) and the orthogonal labelling.

**Theorem 7.** (Gronau et al., 1997). A cyclic orthogonal double covers (CODC) of \( K_n \) by a graph \( G \) exists if and only if there exists an orthogonal \( [d_1, d_2, \ldots, d_k] \)-labelling of \( G \).

The following theorem of Sampathkumar and Srinivasan is a generalization of Theorem 7.

**Theorem 8.** (Sampathkumar, et al., 2011). A cyclic orthogonal double cover (CODC) of \( \text{Circ}(n; [d_1, d_2, \ldots, d_k]) \) by a graph \( G \) exists if and only if there exists an orthogonal \([d_1, d_2, \ldots, d_k]\)-labelling of \( G \).

For results on orthogonal double cover of circulant graphs, see (Sampathkumar & Srinivasan, 2011), (Higazy, 2013), and (El-Shanawany & Shabana, 2014). In (Gronau et al., 1997), (Gronau, Hartmann, Grünthmüller, U. Leck, & V. Leck, 2002), (Scapellato, El-Shanawany, & Higazy, 2009), and (El-Shanawany, Higazy, & El-Mesady, 2013), other results of ODCs by different graph classes can be found. In (Balakrishnan & Ranganathan, 2012), the other terminology not defined here can be found.

(El-Shanawany et al., 2013) were the first ones introduced the cartesian product method as a recursive constructing method for orthogonal double covers of complete bipartite graphs, and this idea is potentially promising for treating various graph lift, see (Shang, 2012), and (Linial & Puder, 2010).

2. On Cartesian Products of Cyclic Orthogonal Double Covers of Circulants

Hereafter, we will use the operation \( \star \) for the usual multiplication and \( \times \) for cartesian product and if there is no danger of ambiguity, if \((i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m\) we can write \((i, j)\) as \(ij\). The above two Theorems 7, 8 motivated us to the following:
Using the fact that there exists a bijective mapping \( \Phi : \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \rightarrow \mathbb{Z}_{n_1n_2} \) defined by \( \Phi(ij) = n_2i + j : \ i \in \mathbb{Z}_{n_1}, \ j \in \mathbb{Z}_{n_2} \), and hence we consider \( xy > pq \) if \( x > p \) or if \( x = p \) and \( y > q \) where \( xy, \ pq \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \) and \( x \star y, \ p \star q \in \mathbb{Z}_{n_1n_2} \). The circulant graph \( \text{Circ}(n_1 \star n_2; X) \) has a vertex set \( \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \), where \( \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \) and \( X \subset \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \). Two vertices \( ab \) and \( cd \) are adjacent if and only if \( ab - cd = \pm(\alpha \beta) \), where \( \alpha \beta \in X \), and \( a, \ c, \ b, \ d \) and \( \alpha \) and \( \beta \) are calculated inside \( \mathbb{Z}_{n_1} \) and \( \mathbb{Z}_{n_2} \). For an edge \( \{ab, cd\} \) in \( \text{Circ}(n_1 \star n_2; X) \), the length of \( \{ab, cd\} \) is \( \min\{ab - cd, n_1n_2 - ab - cd\} \). Given two edges \( e_1 = \{ab, cd\} \) and \( e_2 = \{fg, uv\} \) of the same length \( \alpha \beta \) in \( \text{Circ}(n_1 \star n_2; X) \), the rotation-distance \( r(\alpha \beta) \) between \( e_1 \) and \( e_2 \) is \( r(\alpha \beta) = \min\{wz, st\} : \{ab + wz, cd + wz\} = e_2, \{fg + st, uv + st\} = e_1 \}, \) where addition and difference for \( a, c, f, \) and \( u \) are calculated inside \( \mathbb{Z}_{n_1} \) and for \( b, d, g, \) and \( v \) are calculated inside \( \mathbb{Z}_{n_2} \). Note that if \( r(\alpha \beta) = \beta \), then the edges \( e_1 \) and \( e_2 \) are adjacent; if \( r(\alpha \beta) \neq \beta \), then the edges \( e_1 \) and \( e_2 \) are nonadjacent. Consider the Cayley graph \( \text{Cay}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, X) = \text{Circ}(n_1 \star n_2; X) \). Given a subgraph \( G \) of \( \text{Circ}(n_1 \star n_2; X) \), a \( 1-1 \) mapping \( \Psi : V(G) \rightarrow \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \) is an orthogonal X-labelling of \( G \) if one of the following cases in the proof of the following theorem is verified.

**Theorem 9.** A cyclic orthogonal double cover (CODC) of \( \text{Circ}(n_1 \star n_2; X) \) by a graph \( G \) exists if and only if there exists an orthogonal X-labelling of \( G \).

**Proof.** Case 1. Let \( n_1 \) be even and \( n_2 > 1 \) be odd.

**Subcase 1.1.** For \( n_1 > 2 \), we find that:

(a) For every \( \alpha \beta \in X_1 : \)

\[
X_1 = \begin{cases} 
0\beta : & 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor, \\
\alpha \beta : & 1 \leq \alpha \leq \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{n_2}, \\
\frac{n_1}{2} \beta : & 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor.
\end{cases}
\]

\( G \) contains exactly two edges of length \( \alpha \beta \), and exactly one edge of length \( X_2 = \{0, 0\} \).

(b) \( \{r(\alpha \beta) : \alpha \beta \in X_1\} = X_1 \), then \( X = X_1 \cup X_2 \).

**Subcase 1.2.** For \( n_1 = 2 \), we find that:

(a) For every \( \alpha \beta \in X_1 : \)

\[
X_1 = \{ \alpha \beta : \alpha \in \mathbb{Z}_2, 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor \}.
\]

\( G \) contains exactly two edges of length \( \alpha \beta \), and exactly one edge of length \( X_2 = \{10\} \).

(b) \( \{r(\alpha \beta) : \alpha \beta \in X_1\} = X_1 \), then \( X = X_1 \cup X_2 \).

Case 2. Let \( n_1 > 1 \) be odd and \( n_2 \) be even.

**Subcase 2.1.** For \( n_2 > 2 \), we find that:

(a) For every \( \alpha \beta \in X_1 : \)

\[
X_1 = \begin{cases} 
\alpha 0 : & 1 \leq \alpha \leq \lfloor \frac{n_2}{2} \rfloor, \\
\alpha \beta : & 0 \leq \alpha \leq \frac{1}{2}(n_1 - 1), 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor - 1, \\
n_1 n_2 - \alpha \beta : & \frac{1}{2}(n_1 - 1) < \alpha < n_1, 1 \leq \beta \leq \lfloor \frac{n_2}{2} \rfloor - 1, \\
\frac{\alpha}{n_2} : & 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor.
\end{cases}
\]

\( G \) contains exactly two edges of length \( \alpha \beta \), and exactly one edge of length \( X_2 = \{0\} \).

(b) \( \{r(\alpha \beta) : \alpha \beta \in X_1\} = X_1 \), then \( X = X_1 \cup X_2 \).

**Subcase 2.2.** For \( n_2 = 2 \), we find that:

(a) For every \( \alpha \beta \in X_1 : \)

\[
X_1 = \{ \alpha \beta : 1 \leq \alpha \leq \lfloor \frac{n_2}{2} \rfloor, \beta \in \mathbb{Z}_2 \}.
\]

\( G \) contains exactly two edges of length \( \alpha \beta \), and exactly one edge of length \( X_2 = \{01\} \).

(b) \( \{r(\alpha \beta) : \alpha \beta \in X_1\} = X_1 \), then \( X = X_1 \cup X_2 \).

Case 3. Let \( n_1 \) and \( n_2 \) be odd.

**Subcase 3.1.** For \( n_1 > 3 \) and \( n_2 > 3 \), we find that:

(a) For every \( \alpha \beta \in X_1 : \)

\[
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\]
Subcase 3.2. For $n_1 = 3$ and $n_2 > 3$, we find that:

(a) For every $\alpha \beta \in X_1$:

$$X_1 = \begin{cases} 
\alpha \beta & : \ 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor, \\
\alpha \left\lfloor \frac{n_1}{2} \right\rfloor \beta & : \ \beta \in \mathbb{Z}_{n_1},
\end{cases}$$

$G$ contains exactly two edges of length $\alpha \beta$.

(b) if $r(\alpha \beta) : \alpha \beta \in X_1$, then $X = X_1$.

Subcase 3.3. For $n_1 > 3$ and $n_2 = 3$, we find that:

(a) For every $\alpha \beta \in X_1$:

$$X_1 = \begin{cases} 
\alpha \beta & : \ 1 \leq \alpha \leq \lfloor \frac{n_1}{2} \rfloor, \\
\alpha \left\lfloor \frac{n_1}{2} \right\rfloor \beta & : \ \beta \in \mathbb{Z}_{n_1},
\end{cases}$$

$G$ contains exactly two edges of length $\alpha \beta$.

(b) if $r(\alpha \beta) : \alpha \beta \in X_1$, then $X = X_1$.

Subcase 3.4. For $n_1 = 3$ and $n_2 = 3$ is trivial case.

Case 4. Let $n_1$ and $n_2$ be even.

Subcase 4.1. For $n_1 > 2$ and $n_2 > 2$, we find that:

(a) For every $\alpha \beta \in X_1$:

$$X_1 = \begin{cases} 
\alpha \beta & : \ \alpha \in \{0, \frac{n_1}{2} \}, \ 1 \leq \beta \leq \frac{n_1}{2} - 1, \\
\alpha \left\lfloor \frac{n_1}{2} \right\rfloor \beta & : \ 1 \leq \alpha \leq \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{n_1},
\end{cases}$$

$G$ contains exactly two edges of length $\alpha \beta$, and exactly one edge of length

$$X_2 = \left\{ \frac{n_1}{2}, 0, \frac{n_1}{2}, \frac{n_1}{2} \right\},$$

(b) if $r(\alpha \beta) : \alpha \beta \in X_1$, then $X = X_1 \cup X_2$.

Subcase 4.2. For $n_1 = 2$ and $n_2 > 2$, we find that:

(a) For every $\alpha \beta \in X_1$:

$$X_1 = \{ \alpha \beta : \ \alpha \in \mathbb{Z}_{2}, \ 1 \leq \beta \leq \frac{n_1}{2} - 1 \}.$$

$G$ contains exactly two edges of length $\alpha \beta$, and exactly one edge of length

$$X_2 = \left\{ 10, 0, \frac{n_1}{2}, \frac{n_1}{2} \right\},$$

(b) if $r(\alpha \beta) : \alpha \beta \in X_1$, then $X = X_1 \cup X_2$.

Subcase 4.3. For $n_1 > 2$ and $n_2 = 2$, we find that:

(a) For every $\alpha \beta \in X_1$:

$$X_1 = \{ \alpha \beta : \ 1 \leq \alpha \leq \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{2} \}.$$

$G$ contains exactly two edges of length $\alpha \beta$, and exactly one edge of length

$$X_2 = \left\{ \frac{n_1}{2}, 0, \frac{n_1}{2} \right\},$$

(b) if $r(\alpha \beta) : \alpha \beta \in X_1$, then $X = X_1 \cup X_2$.

Subcase 4.4. For $n_1 = 2$ and $n_2 = 2$ is trivial case.

For all positive integers $m$ with $\gcd(m, 3) = 1$, $k = l + m$, and $l \in \mathbb{Z}_{m}$, then for the following Theorem let $H_{1}^{m}$ be a
graph with the following edges set: $E(H_1^m) = \{(0l, 0l + ij) : ij \in Y_1 \setminus \{00\}\} \cup \{(0k, 0k + ij) : ij \in Y_2\} \cup \{(l, l + ij) : ij \in Y_3\} \cup \{(1k, 1k + ij) : ij \in Y_4\}$, where $Y_1 = A_1 \times A_2$, $Y_2 = A_1 \times A_4$, $Y_3 = A_3 \times A_2$, $Y_4 = A_3 \times A_4$, $A_1 = \{0, 2\}$, $A_2 = \{l, l + 2m\}$, $A_3 = \{1, 3\}$, $A_4 = \{l + m, l + 3m\}$.

**Theorem 10.** For all positive integers $m$ with $\gcd(m, 3) = 1$, there exists a CODC of $\text{Circ}(16m; X)$ by $H_1^m$ with respect to $\mathbb{Z}_4 \times \mathbb{Z}_{4m}$.

**Proof.** Let us define $\Psi : V(H_1^m) \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_{4m}$ by

$$
\Psi(v_{ij}) = \begin{cases}
0\alpha & : 0 \leq \alpha \leq 2m - 1, \\
0\delta & : 2m \leq \alpha \leq 3m - 1, \delta = 2(\alpha - m), \\
1\gamma & : 3m \leq \alpha \leq 5m - 1, \gamma = \alpha - 3m, \\
2\omega & : 5m \leq \alpha \leq 7m - 1, \omega = 2(\alpha - 5m).
\end{cases}
$$

Hence, the edge of length $0l$ is $(\Psi(v_l), \Psi(v_{l+2m}))$, the edge of length $2l$ is $(\Psi(v_l), \Psi(v_{l+5m}))$, the edge of length $0(l + 2m)$ is $(\Psi(v_{l+2m}), \Psi(v_{l+5m}))$, the edge of length $2(l + 2m)$ is $(\Psi(v_{l+2m}), \Psi(v_{l+6m}))$, the edge of length $0(l + m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+2m}))$, the edge of length $0(l + 3m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+5m}))$, the edge of length $2(l + m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+6m}))$, the edge of length $2(l + 3m)$ is $(\Psi(v_{l+m}), \Psi(v_{l+5m}))$, the edge of length $1l$ is $(\Psi(v_{l+3m}), \Psi(v_{l+4m})), \Psi(v_{l+5m}))$, the edge of length $3l$ is $(\Psi(v_{l+3m}), \Psi(v_{l+4m})), \Psi(v_{l+5m}))$, the edge of length $3(l + m)$ is $(\Psi(v_{l+3m}), \Psi(v_{l+4m})), \Psi(v_{l+5m}))$, the edge of length $3(l + 3m)$ is $(\Psi(v_{l+3m}), \Psi(v_{l+4m})), \Psi(v_{l+5m}))$.

From the edges set of $H_1^m$, and according to Subcase 4.1 of Theorem 9, we can find that:

(a) For every $\alpha \beta \in X_1$:

$$
X_1 = \begin{cases}
\alpha \beta & : \alpha \in \{0, 2\}, 1 \leq \beta \leq 2m - 1, \\
\alpha \beta & : \alpha = 1, \beta \in \mathbb{Z}_{4m}.
\end{cases}
$$

$H_1^m$ contains exactly two edges of length $\alpha \beta$, and exactly one edge of length $X_2 = \{20, 0\gamma, 2\gamma : \gamma = 2m\}$.

(b) $\{ r(\alpha \beta) : \alpha \beta \in X_1 \} = X_1$, then $X = X_1 \cup X_2$.

For an illustration of Theorem 10, let $m = 2$, then there exists a CODC of $\text{Circ}(32; X)$ by $H_1^2$ with respect to $\mathbb{Z}_4 \times \mathbb{Z}_8$, where $X = \{01, 02, 03, 21, 22, 23, 10, 11, 12, 13, 14, 15, 16, 17, 20, 04, 24\}$, see Figure 1.

![Figure 1](image.png)

3. Conclusion

In conclusion, in the future we can get the orthogonal double covers of circulant graphs by new graph classes where the circulant graphs are labelled by the Cartesian product of two abelian groups.

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References


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