

A Necessary and Sufficient Condition for Pseudo-symmetric Positive Solutions of Boundary Value Problems

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Abstract

We apply the monotone iterative technique to the second-order boundary value problems. We obtain a necessary and sufficient condition and discuss the uniqueness, a iterative sequence and an error estimation for pseudo-symmetric positive solutions. Moreover, an example is given to illustrate the applicability of our results.

Keywords: boundary value problems, pseudo-symmetric positive solutions, monotone iterative technique, necessary and sufficient condition

1. Introduction

Consider the pseudo-symmetric boundary value problem

$$\begin{cases} u''(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0, u(\eta) = u(1), \end{cases} \quad (1.1)$$

where $\eta \in (0, 1)$. For $\eta \in (0, 1)$, a function $u \in C[0, 1]$ is said to be pseudo-symmetric if u is symmetric over the interval $[\eta, 1]$. That is, $u(t) = u(1 + \eta - t)$, $t \in [\eta, 1]$. By a pseudo-symmetric positive solution of (1.1), we mean a pseudo-symmetric function $u \in C^2[0, 1]$ such that $u(t) > 0$ for $t \in (0, 1)$, and $u(t)$ satisfies (1.1).

Recently, many authors have focused on the question of symmetric positive solutions for ordinary differential equation boundary value problems, for example, see (Avery & Henderson, 2000; Çetin & Topal, 2012; Graef & Kong, 2008; Hamal & Yoruk, 2010; Jiang, Liu & Wu, 2013; Luo & Luo, 2010; Luo & Luo, 2012; Lin & Zhao, 2013; Tersenov, 2014) and the references therein. In (Avery & Henderson, 2003), Avery and Henderson gave the definition of pseudo-symmetric function. Since then, some papers have discussed the pseudo-symmetric question and established sufficient conditions for the existence of pseudo-symmetric positive solutions, see (Feng, Zhang & Ge, 2010; Guo, Han & Chen, 2010; Ji, 2008; Ma & Ge, 2007; Pang, 2009; Sun & Zhao, 2014). To the best of the authors' knowledge there is little known about necessary and sufficient conditions for second-order pseudo-symmetric nonlinear boundary value problem. Motivated by the works mentioned above, we aim to establish a necessary and sufficient condition for the existence of pseudo-symmetric positive solution of (1.1) by applying the monotone iterative technique.

The organization of the paper is as follows. Section 2 contains some preliminary lemmas and the basic assumptions. In Section 3, by applying the monotone iterative technique, we obtain a sufficient and necessary condition for the existence of at least one pseudo-symmetric positive solution for problem (1.1), we also discuss the uniqueness, a iterative sequence and an error estimation for the pseudo-symmetric positive solution to (1.1). In Section 4, an example will be presented to illustrate the applicability of our results.

2. Preliminaries

By routine calculations we have the following result.

Lemma 2.1. Let $v \in C[0, 1]$, then the boundary value problem

$$\begin{cases} u''(t) + v(t) = 0, & t \in (0, 1), \\ u(0) = 0, & u(\eta) = u(1) \end{cases}$$

has a unique solution

$$u(t) = \begin{cases} \int_0^t \int_s^{\frac{1+\eta}{2}} v(r) dr ds, & 0 \leq t \leq \frac{1+\eta}{2}, \\ \int_0^\eta \int_s^{\frac{1+\eta}{2}} v(r) dr ds + \int_t^1 \int_{\frac{1+\eta}{2}}^s v(r) dr ds, & \frac{1+\eta}{2} \leq t \leq 1, \end{cases}$$

where $\eta \in (0, 1)$.

Lemma 2.2. Assume that $u(t)$ is a pseudo-symmetric positive solution of (1.1). Then there exist constants c_1, c_2 with $0 < c_1 < 1 < c_2$ such that

$$c_1 \omega(t) \leq u(t) \leq c_2 \omega(t), t \in [0, 1]. \quad (2.1)$$

Proof. To prove (2.1) holds, first note that $u(t) > 0$ and $u''(t) \leq 0$ for $t \in (0, 1)$. From the pseudo-symmetry of $u(t)$, we have $u(\frac{1+\eta}{2}) = \max_{t \in [0, 1]} u(t)$. If $t \in (0, \frac{1+\eta}{2})$, then $u(t) \geq \frac{2u(\frac{1+\eta}{2})}{1+\eta} t$, and if $t \in (\frac{1+\eta}{2}, 1)$, then $u(t) \geq \frac{2u(\frac{1+\eta}{2})}{1+\eta} (1 + \eta - t)$. Hence, for $t \in (0, 1)$, $u(t) \geq \frac{2u(\frac{1+\eta}{2})}{1+\eta} \omega(t)$. Thus, we take c_1 with $0 < c_1 < \min\{1, \frac{2u(\frac{1+\eta}{2})}{1+\eta}\}$. The proof of the other half of (2.1) is similar, we have that $u(t) \leq u'(0)\omega(t)$, thus, choosing $c_2 > \max\{1, u'(0)\}$. The proof is complete.

Throughout this paper, we assume the following conditions hold without further mention.

(H1) $f : (0, 1) \times [0, \infty) \rightarrow [0, \infty)$ is continuous. For $(t, u) \in (0, 1) \times [0, \infty)$, $\eta \in (0, 1)$, $f(t, u)$ is pseudo-symmetric in t , i.e., f satisfies

$$f(1 + \eta - t, u) = f(t, u), t \in [\eta, 1].$$

(H2) For $(t, u) \in (0, 1) \times [0, \infty)$, $f(t, u)$ is nondecreasing in u and there exists a constant $\lambda \in (0, 1)$ such that if $\sigma \in (0, 1]$, then

$$\sigma^\lambda f(t, u) \leq f(t, \sigma u). \quad (2.2)$$

Example 2.1. Study the equation

$$f(t, u) = |2t - \eta - 1|u^{\frac{1}{4}}, (t, u) \in (0, 1) \times [0, \infty).$$

It is easy to see that the function f satisfies assumptions (H1) and (H2). In fact, if $\sigma \in (0, 1]$, there exists constant λ with $\frac{1}{4} \leq \lambda < 1$ such that $f(t, \sigma u) \geq \sigma^\lambda f(t, u)$.

Remark 2.1. Expression (2.2) implies that if $\sigma \in [1, \infty)$, we have

$$f(t, \sigma u) \leq \sigma^\lambda f(t, u). \quad (2.3)$$

For convenience, let

$$\omega(t) = \min\{t, 1 + \eta - t\}, t \in [0, 1], \quad (2.4)$$

and E be the Banach space $C^2[0, 1]$, and define

$$P = \left\{ u \in E : u(0) = 0, u(t) > 0 \text{ for } t \in (0, 1), u(t) = u(1 + \eta - t) \text{ for } t \in [\eta, 1] \text{ and there exist constants } l_u, L_u \text{ with } 0 < l_u < 1 < L_u \text{ such that } l_u \omega(t) \leq u(t) \leq L_u \omega(t) \text{ for } t \in [0, 1] \right\}. \quad (2.5)$$

Remark 2.2. The set P is not a cone as it is not closed.

3. Main Results

Theorem 3.1. Assume (H1) and (H2) hold. Then the boundary value problem (1.1) has at least one pseudo-symmetric positive solution if and only if

$$0 < \int_0^1 f(t, \omega(t))dt < \infty. \quad (3.1)$$

Proof. *Necessity.* Suppose first that $u(t)$ is a pseudo-symmetric positive solution of (1.1), we will show that (3.1) holds.

Let c_1 and c_2 be given as in Lemma 2.2 for this $u(t)$. By Lemma 2.2, $u(t)$ satisfies (2.1). Hence, by (H2), Remark (2.1) and (2.1), we have

$$\begin{aligned} \int_0^1 f(t, \omega(t))dt &\leq \int_0^1 f(t, c_1^{-1}u(t))dt \\ &\leq c_1^{-\lambda} \int_0^1 f(t, u(t))dt \\ &= c_1^{-\lambda}[u'(0) - u'(1)] < \infty \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \int_0^1 f(t, \omega(t))dt &\geq \int_0^1 f(t, c_2^{-1}u(t))dt \\ &\geq c_2^{-\lambda} \int_0^1 f(t, u(t))dt \\ &= c_2^{-\lambda}[u'(0) - u'(1)] > 0. \end{aligned} \quad (3.3)$$

Now, (3.1) follows from (3.2) and (3.3).

Sufficiency. Now assume that (3.1) holds, we will show that (1.1) has at least one pseudo-symmetric positive solution.

Define the operator $T : E \rightarrow E$ by

$$Tu(t) = \begin{cases} \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds, & 0 \leq t \leq \frac{1+\eta}{2}, \\ \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds + \int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, u(r))drds, & \frac{1+\eta}{2} \leq t \leq 1. \end{cases} \quad (3.4)$$

It is clear that u is a solution of (1.1) if and only if u is a fixed point of T .

Claim 1. The operator $T : P \rightarrow P$ is completely continuous and nondecreasing.

We first note that for $u \in P$ we have $Tu(0) = 0$, $Tu(t) > 0$ for $t \in (0, 1)$. We now prove that Tu is pseudo-symmetric about $\eta \in (0, 1)$.

In fact, for $t \in [\eta, \frac{1+\eta}{2}]$, then $1 + \eta - t \in [\frac{1+\eta}{2}, 1]$. From (H1), we have

$$\begin{aligned} Tu(1 + \eta - t) &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds + \int_{1+\eta-t}^1 \int_{\frac{1+\eta}{2}}^s f(r, u(r))drds \\ &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds + \int_\eta^t \int_{\frac{1+\eta}{2}}^{1+\eta-s} f(r, u(r))drds \\ &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds + \int_\eta^t \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds \\ &= \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds = Tu(t) \end{aligned}$$

and for $t \in [\frac{1+\eta}{2}, 1]$, we note that $1 + \eta - t \in [\eta, \frac{1+\eta}{2}]$. Thus,

$$\begin{aligned} Tu(1 + \eta - t) &= \int_0^{1+\eta-t} \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds + \int_\eta^{1+\eta-t} \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds + \int_t^1 \int_{1+\eta-s}^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds + \int_t^1 \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds = Tu(t). \end{aligned}$$

For $t \in [0, \frac{1+\eta}{2}]$, we have $\omega(t) = \min\{t, 1 + \eta - t\} = t$, so for any $u \in P$, from (2.2), (2.3) and (3.1),

$$\begin{aligned} Tu(t) &= \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &\leq \int_0^t \int_0^1 f(r, L_u \omega(r)) dr ds \\ &\leq t L_u^\lambda \int_0^1 f(r, \omega(r)) dr \leq L_{Tu} \omega(t) \end{aligned} \quad (3.5)$$

and for $t \in [\frac{1+\eta}{2}, 1]$, we have $\omega(t) = 1 + \eta - t$,

$$\begin{aligned} Tu(t) &= \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds + \int_t^1 \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &\leq \int_0^\eta \int_0^1 f(r, L_u \omega(r)) dr ds + \int_t^1 \int_0^1 f(r, L_u \omega(r)) dr ds \\ &\leq (1 + \eta - t) L_u^\lambda \int_0^1 f(r, \omega(r)) dr \leq L_{Tu} \omega(t), \end{aligned} \quad (3.6)$$

where $L_{Tu} > \max\{1, L_u^\lambda \int_0^1 f(r, \omega(r)) dr\}$.

On the other hand, for $t \in [0, \frac{1+\eta}{2}]$, there exists constant $\xi \in (0, \frac{1+\eta}{2})$ such that

$$\begin{aligned} Tu(t) &= \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &\geq \int_0^t \int_\xi^{\frac{1+\eta}{2}} f(r, u(r)) dr ds \\ &\geq t \int_\xi^{\frac{1+\eta}{2}} f(r, l_u \omega(r)) dr ds \\ &\geq t l_u^\lambda \int_\xi^{\frac{1+\eta}{2}} f(r, \omega(r)) dr \geq l_{Tu} \omega(t) \end{aligned} \quad (3.7)$$

and for $t \in [\frac{1+\eta}{2}, 1]$, there exist constants $\zeta \in (\frac{1+\eta}{2}, 1)$ such that

$$\int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, u(r)) dr ds \geq (1 - t) \int_{\frac{1+\eta}{2}}^\zeta f(r, u(r)) dr,$$

and $M > 0$ such that $\min\{\int_{\eta}^{\frac{1+\eta}{2}} f(r, \omega(r))dr, \int_{\frac{1+\eta}{2}}^{\zeta} f(r, \omega(r))dr\} > M$. Thus,

$$\begin{aligned} Tu(t) &= \int_0^{\eta} \int_s^{\frac{1+\eta}{2}} f(r, u(r))drds + \int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, u(r))drds \\ &\geq \eta \int_{\eta}^{\frac{1+\eta}{2}} f(r, l_u \omega(r))dr + (1-t) \int_{\frac{1+\eta}{2}}^{\zeta} f(r, l_u \omega(r))dr \\ &\geq l_u^{\lambda} \left[\eta \int_{\eta}^{\frac{1+\eta}{2}} f(r, \omega(r))dr + (1-t) \int_{\frac{1+\eta}{2}}^{\zeta} f(r, \omega(r))dr \right] \\ &\geq l_u^{\lambda} M(1+\eta-t) \geq l_{Tu} \omega(t), \end{aligned} \quad (3.8)$$

where $0 < l_{Tu} < \min\{1, l_u^{\lambda} \int_{\xi}^{\frac{1+\eta}{2}} f(r, \omega(r))dr, l_u^{\lambda} M\}$.

Hence, it follows from (3.5)-(3.8) that there exist constants l_{Tu} and L_{Tu} with $0 < l_{Tu} < 1 < L_{Tu}$ such that

$$l_{Tu} \omega(t) \leq Tu(t) \leq L_{Tu} \omega(t) \text{ for } t \in [0, 1].$$

Consequently, $Tu \in P$, and so $T : P \rightarrow P$. A standard argument can be used to show that T is completely continuous. From (H2), it is easy to see that T is nondecreasing for u . Hence, Claim 1 holds.

Claim 2. Let δ and γ be fixed numbers satisfying

$$0 < \delta \leq l_{T\omega}^{1/(1-\lambda)}, \quad \gamma \geq L_{T\omega}^{1/(1-\lambda)}, \quad (3.9)$$

and assume

$$u_0 = \delta \omega(t), \quad v_0 = \gamma \omega(t), \quad (3.10)$$

$$u_n = Tu_{n-1}, \quad v_n = Tv_{n-1}, \quad n = 1, 2, \dots \quad (3.11)$$

Then,

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0, \quad (3.12)$$

and there exists $u^* \in P$ such that

$$u_n(t) \rightarrow u^*(t), \quad v_n(t) \rightarrow u^*(t), \text{ uniformly on } [0, 1]. \quad (3.13)$$

In fact, $0 < l_{T\omega} < 1 < L_{T\omega}$ since $T\omega(t) \in P$. So, $0 < \delta < 1 < \gamma$. From (3.10), we have $u_0, v_0 \in P$ and $u_0 \leq v_0$.

On the other hand, for $t \in [0, \frac{1+\eta}{2}]$, from (2.2) and (2.3),

$$\begin{aligned} u_1 = Tu_0(t) &= \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, \delta \omega(r))drds \\ &\geq \delta^{\lambda} \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, \omega(r))drds \\ &= \delta^{\lambda} T\omega(t) \geq \delta^{\lambda} l_{T\omega} \omega(t) \\ &\geq \delta^{\lambda} \delta^{1-\lambda} \omega(t) = u_0 \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} v_1 = Tv_0(t) &= \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, \gamma \omega(r))drds \\ &\leq \gamma^{\lambda} T\omega(t) \leq \gamma^{\lambda} L_{T\omega} \omega(t) \\ &\leq \gamma^{\lambda} \gamma^{1-\lambda} \omega(t) = v_0. \end{aligned} \quad (3.15)$$

For $t \in [\frac{1+\eta}{2}, 1]$,

$$\begin{aligned} u_1 = Tu_0(t) &= \int_0^{\eta} \int_s^{\frac{1+\eta}{2}} f(r, \delta \omega(r))drds + \int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, \delta \omega(r))drds \\ &\geq \delta^{\lambda} T\omega(t) \geq \delta^{\lambda} l_{T\omega} \omega(t) \\ &\geq \delta^{\lambda} \delta^{1-\lambda} \omega(t) = u_0 \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} v_1 = Tv_0(t) &= \int_0^{\eta} \int_s^{\frac{1+\eta}{2}} f(r, \gamma\omega(r)) dr ds + \int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, \gamma\omega(r)) dr ds \\ &\leq \gamma^\lambda T\omega(t) \leq \gamma^\lambda L_{T\omega}(t) \\ &\leq \gamma^\lambda \gamma^{1-\lambda} \omega(t) = v_0. \end{aligned} \quad (3.17)$$

From (3.14)-(3.17), we have that $u_1 \geq u_0$ and $v_1 \leq v_0$ for $t \in [0, 1]$. Since $u_0 \leq v_0$ and T is nondecreasing, by induction, (3.12) holds.

Let $c_0 = \frac{\delta}{\gamma}$, then $0 < c_0 < 1$. It follows from

$$T(cu) \geq c^\lambda Tu, \text{ if } 0 < c < 1, u \in P$$

that for any natural number n

$$u_n = Tu_{n-1} = T^n u_0 = T^n(\delta\omega(t)) = T^n(c_0\gamma\omega(t)) \geq c_0^{\lambda^n} T^n(\gamma\omega(t)) = c_0^{\lambda^n} v_n.$$

Thus, for each natural number n and p^* , we have

$$0 \leq u_{n+p^*} - u_n \leq v_n - u_n \leq (1 - c_0^{\lambda^n})v_n \leq (1 - c_0^{\lambda^n})\gamma\omega(t), \quad (3.18)$$

which implies that there exists $u^* \in P$ such that (3.13) holds, and Claim 2 holds.

Let $n \rightarrow \infty$ in (3.11), we obtain $u^*(t) = Tu^*(t)$, which is a pseudo-symmetric positive solution of (1.1). The proof of the theorem is now complete.

Theorem 3.2. Assume (H1), (H2) and (3.1) hold. Then

(i) (1.1) has a unique pseudo-symmetric positive solution $u^*(t)$, and there exist constants $l, L \in R$ with $0 < l < 1 < L$ such that

$$l\omega(t) \leq u^*(t) \leq L\omega(t), \quad t \in [0, 1]. \quad (3.19)$$

(ii) For any initial value $x_0 \in P$, there exists a sequence $x_n(t)$ which uniformly converges to the unique pseudo-symmetric positive solution $u^*(t)$ for (1.1), and we have the error estimation

$$\max_{t \in [0, 1]} |x_n(t) - u^*(t)| = O(1 - k^{\lambda^n}), \quad (3.20)$$

where k is a constant with $0 < k < 1$ and determined by x_0 .

Proof. Let u_0, v_0, u_n, v_n be defined in (3.10) and (3.11).

(i) It follows from Theorem 3.1 that (1.1) has at least one pseudo-symmetric positive solution $u^*(t) \in P$, which implies that there exist constants l and L with $0 < l < 1 < L$ such that $u^*(t)$ satisfies (3.19). Let $v^*(t)$ be another pseudo-symmetric positive solution of (1.1), then from Lemma 2.2 we have that there exist constants c_1 and c_2 with $0 < c_1 < 1 < c_2$ such that

$$c_1\omega(t) \leq v^*(t) \leq c_2\omega(t) \text{ for } t \in [0, 1]. \quad (3.21)$$

Let δ defined in (3.9) small enough so that $\delta < c_1$ and γ defined in (3.9) large enough so that $\gamma > c_2$. Then from (3.10) and (3.21),

$$u_0(t) \leq v^*(t) \leq v_0(t).$$

Note that $Tv^* = v^*$ and T is nondecreasing, we have

$$u_n(t) \leq v^*(t) \leq v_n(t). \quad (3.22)$$

Let $n \rightarrow \infty$ in (3.22), from (3.13) we have that $v^* = u^*$. Hence, the pseudo-symmetric positive solution to (1.1) is unique.

(ii) From (i), we know that the pseudo-symmetric positive solution to (1.1) is unique. For any $x_0 \in P$, there exist constants l_{x_0} and L_{x_0} with $0 < l_{x_0} < 1 < L_{x_0}$ such that

$$l_{x_0}\omega(t) \leq x_0(t) \leq L_{x_0}\omega(t). \quad (3.23)$$

Like in (i), we can let δ and γ defined in (3.9) satisfy $\delta < l_{x_0}$ and $\gamma > L_{x_0}$. Then from (3.10) and (3.23)

$$u_0(t) \leq x_0(t) \leq v_0(t).$$

Let $x_n = T x_{n-1}$, then

$$x_n(t) = \begin{cases} \int_0^t \int_s^{\frac{1+\eta}{2}} f(r, x_{n-1}(r)) dr ds, & 0 \leq t \leq \frac{1+\eta}{2}, \\ \int_0^\eta \int_s^{\frac{1+\eta}{2}} f(r, x_{n-1}(r)) dr ds + \int_t^1 \int_{\frac{1+\eta}{2}}^s f(r, x_{n-1}(r)) dr ds, & \frac{1+\eta}{2} \leq t \leq 1, \end{cases} \quad (3.24)$$

where $n = 1, 2, \dots$. Note that T is nondecreasing and (3.11), we have

$$u_n(t) \leq x_n(t) \leq v_n(t). \quad (3.25)$$

Let $n \rightarrow \infty$ in (3.25), it follows from (3.13) that x_n uniformly converges to the unique pseudo-symmetric positive solution u^* for (1.1). At the same time, (3.20) follows from (3.18) and (3.25). Thus, the proof of the theorem is complete.

4. Example

Example 4.1. Study the boundary value problem

$$\begin{cases} -u''(t) = t^p(1+\eta-t)^p u^\alpha(t), & t \in (0, 1), \\ u(0) = 0, \quad u(\eta) = u(1), \end{cases} \quad (4.1)$$

where $p \in \mathbb{R}$, $0 < \alpha < 1$, $0 < \eta < 1$. Let

$$f(t, u) = t^p(1+\eta-t)^p u^\alpha, \quad (t, u) \in (0, 1) \times [0, \infty).$$

Note that the function f satisfies that $f(t, u) = f(1+\eta-t, u)$ for $t \in [\eta, 1]$, $f(t, u)$ is nondecreasing in u and if $\sigma \in (0, 1]$, there exists constant λ with $0 < \alpha \leq \lambda < 1$ such that $f(t, \sigma u) \geq \sigma^\lambda f(t, u)$ for all $(t, u) \in (0, 1) \times [0, \infty)$, which coincide with the assumptions (H1) and (H2). Thus, from Theorem 3.1 and Theorem 3.2, we have the following results.

Corollary 4.1. The boundary value problem (4.1) has at least one pseudo-symmetric positive solution if and only if $p > -\alpha$.

Corollary 4.2. Assume $p > -\alpha$. Then

(i) (4.1) has a unique pseudo-symmetric positive solution $u^*(t)$, and there exist constants l and L with $0 < l < 1 < L$ such that

$$l\omega(t) \leq u^*(t) \leq L\omega(t), \quad t \in [0, 1].$$

(ii) For any initial value $x_0 \in P$, there exists a sequence $x_n(t)$ which uniformly converges to the unique pseudo-symmetric positive solution $u^*(t)$ for (4.1), and we have the error estimation

$$\max_{t \in [0, 1]} |x_n(t) - u^*(t)| = O(1 - k^{\lambda^n}),$$

where k is a constant with $0 < k < 1$ and determined by x_0 .

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