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Remarks on the Continuity of the Local Minimizer of Scalar Integral Functionals With Nonstandard General Growth Conditions

Dedicated to Caterina and Delia Granucci

Tiziano Granucci[1]

[1] Instituto P. Calamandrei, Italy

Correspondence: Tiziano Granucci, Instituto P. Calamandrei, Italy. E-mail: tizianogranucci@libero.it

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Abstract

In this paper we show a regularity theorem for local minima of scalar integral functionals of the Calculus of Variations with nonstandard general growth conditions. Let us consider functionals in the following form

$$\mathcal{F}[u,\Omega]=\int_{\Omega} f(x,u(x),\nabla u(x)),dx$$

where $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying the inequalities

$$\Phi(|z|)-c_1\le f(x,s,z)\le c_2\left[1+(\Phi^*(|z|))^\beta+(\Phi^*(|s|))^\beta\right]$$

for each $z\in\mathbb{R}^N$, $s\in\mathbb{R}$ and for $\mathcal{L}^N$-a. e. $x\in\Omega$, where c_1 and c_2 are two positive real constants, with $c_1<c_2$, Ω is an open subset of $\mathbb{R}^N$, $N\ge 2$, $\Phi\in\triangle_2^m\cap\nabla_2^r$ [Definition 6 and Definition 8], $1\le r<m<N$ and the function Φ^* is the Sobolev conjugate of Φ [Definition 12], β is a positive real number that we will opportunely fix [Hypothesis $H_{1,f}$].

Keywords: variational inequality, regularity, Hölder continuity, nonstandard growth conditions

AMS Classification: [2010] Primary 35B65; Secondary 49J45, 49J40

1. Introduction

In this paper we show a regularity theorem for local minima of scalar integral functionals of the Calculus of Variations with nonstandard general growth conditions.

Let us consider functionals in the following form

$$\mathcal{F}[u,\Omega]=\int_{\Omega} f(x,u(x),\nabla u(x))\,dx$$

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the Sobolev conjugate of Φ [Definition 12], β is a positive real number that we will opportunely fix [Hypothesis $H_{1,f}$]. The research of regularity results for weak solutions of elliptic and parabolic PDEs starts from the basic results of De Giorgi (1957) and Nash (1958). In 1990s a remarkable production of regularity results for functionals with general growths has been developed.

In 1996, Mascolo and Papi have determined an Harnack inequality for the minimizer of the functional (1.1) under the following conditions: $f(z) = \Phi(|z|)$ where Φ is a N-function and $\Phi \in \triangle_2 \cap \nabla_2$. We observe that $\Phi \in \triangle_2 \cap \nabla_2$ implies

$$c_3 t^p - c_4 < \Phi(t) < c_5 t^m + c_6 \qquad \text{for } t > 0 \tag{1.1}$$

with real positive constants c_3, c_4, c_5, c_6 and $1 < p \leq m$. A classical regularity theorem for functionals with standard growth conditions ($p = m$) has been proved in (De Giorgi, 1957) (we refer also to Giaquinta & Giusti, 1984; Giusti, 1994). In 1991, Moscariello and Nania have obtained a results of Hölder continuity for the local-minima of functional of the type (1.1) under the following hypothesis, $f(z) = \Phi(|z|)$ where Φ is a convex and increasing function, $\Phi \in \triangle_2$ and (1.2) holds with $1 < p \leq m < \frac{Np}{N-p}$. In 1991, Lieberman proved an Harnack inequality for the minimizer of the functional (1.1) with $\Phi \in C^2$ suth that

$$c_7 \leq \frac{t\ddot{\Phi}(t)}{\dot{\Phi}(t)} \leq c_8 \qquad \text{for } t > 0$$

with $0 < c_7 < c_8$. In (Granucci, 2006, 2014a, 2014b), the author has extended, partially, the precedents results. Moreover in 1994, Klimov studies this problem when Φ satisfies ∇_2 but not a $\triangle_2$ condition.

In this paper we proof a theorem on the regulaity of quasi-minima of the functional $\mathcal{F}[u, \Omega]$ with the following hypotheses.

[$\mathbf{H}_{0,f}$] Let $\Phi \in \triangle_2^m \cap \nabla_2^r$ be a N-function and $1 \leq r \leq m < N$.

[$\mathbf{H}_{1,f}$] $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the inequalities

$$\Phi(|z|) - c_1 \leq f(x, s, z) \leq c_2 \left[1 + (\Phi^*(|z|))^\beta + (\Phi^*(|s|))^\beta\right] \tag{1.2}$$

for each $z \in \mathbb{R}^N$, $s \in \mathbb{R}$ and for $\mathcal{L}^N$-a. e. $x \in \Omega$, where c_1 and c_2 are two positive real constants, with $c_1 < c_2$, Ω is an open subset of $\mathbb{R}^N$, $N \geq 2$, $\Phi \in \triangle_2^m \cap \nabla_2^r$ and the function Φ^* is the Sobolev conjugate of Φ. Let us define

$$\Gamma_1(N, m, r, \beta) = \alpha_1 N + \left[\alpha_1 N + (N - m^*\beta)(1 + \alpha_2)\right]\vartheta\alpha_1 - m^*\beta + (N - m^*\beta)\alpha_2$$

where $\alpha_1 = \frac{[N-(1-\beta)m]r - Nm\beta}{(N-m)r}$, $\alpha_2 = \frac{\beta m N}{(N-m)r} - 1$, $m^* = \frac{mN}{N-m}$ and ϑ is the positive solution of $\vartheta(\vartheta + 1) = \alpha_1$, i.e. $\vartheta = \frac{-1+\sqrt{1+4\alpha_1}}{2}$. Let us consider the system

$$\begin{cases} 1 \leq r \leq m < N \\ \frac{r(N-m)}{mN} < \beta < \min\left(\frac{r(N-m)}{m(N-r)}, \frac{N-m}{m}\right) \\ \alpha_1 N + (N - m^*\beta)(1 + \alpha_2) \leq N \\ \alpha_1 N + \left[\alpha_1 N + (N - m^*\beta)(1 + \alpha_2)\right]\vartheta\alpha_1 - m^*\beta + (N - m^*\beta)\alpha_2 \geq 0 \end{cases} \tag{1.3}$$

then for every r and m, with $1 \leq r \leq m < N$, there exists $\beta_{1,r,m,N}$, implicitly defined by $\Gamma_1(N, m, r, \beta) = 0$, such that if $\frac{r(N-m)}{mN} < \beta < \beta_{1,r,m,N}$, then β is a solution of the system (1.4). Our hypoteses on β will be

$$\frac{r(N-m)}{mN} < \beta < \beta_{1,r,m,N}. \tag{1.4}$$

$\mathbf{H}_{2,f}$ For every $t \geq 0$ we have

$$t^r \leq \Phi(t). \tag{1.5}$$

Remark 1 For the whole paper we will suppose that the system (1.4) has some solutions, in Appendix we will study in detail such system and we will find some conditions of existence, the relationship (1.5).

Remark 2 If $r = m$ then $\Phi(t) \sim t^r$ [Definition 4] and $\Phi^*(t) \sim t^{r^*}$ [Definition 12 and Lemma 3], this case has been studied by the author in (Granucci, submitted) and with more restrictive hypotheses in (Moscariello & Nania, 1991).

Remark 3 The hypothesis $\mathrm{H}_{2,f}$ is purely tecnical, it can be removed [Corollary 1].

Moreover we will assume that f satisfies on of the following hypotheses.

$\mathbf{H}_{3,1,f}$ For almost every $x \in \Omega$ $f(x, \cdot, \cdot)$ is a convex function.

$\mathbf{H}_{3,2,f}$ For almost every $x \in \Omega$ and for every $s \in \mathbb{R}$, $f(x, s, \cdot)$ is a convex function; moreover there exists a constant $c_3 > 0$ such that

$$f(x, s_1, \xi) \le c_3 f(x, s_2, \xi) \tag{1.6}$$

for almost every $x \in \Omega$, for every pair $s_1, s_2 \in \mathbb{R}$, with $|s_1| \le |s_2|$, and for every $\xi \in \mathbb{R}^N$.

Definition 1 If $u \in V = \{v \in W^{1,1}(\Omega) \colon \mathcal{F}[u, \Omega] < +\infty\}$ then u is a local minimizer of $\mathcal{F}[u, \Omega]$ if

$$\mathcal{F}[u, supp(\varphi)] \le \mathcal{F}[u + \varphi, supp(\varphi)] \tag{1.7}$$

for every $\varphi \in W^{1,1}(\Omega)$ with $\mathrm{supp}(\varphi) \subset\subset \Omega$.

Our principal result is the following theorem.

Theorem 1 (Main Theorem) *Let $u \in V$ be a local minimizer of $\mathcal{F}[u, \Omega]$, with f satisfying conditions $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,1,f}$ or $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,2,f}$; then u is locally Hölder continuous in Ω.*

The precedent Theorem 1 extends the results gotten in (Dall'Aglio, Mascolo and Papi, 1997, 1998).

Corollary 1 *Let $u \in V$ be a local minimizer of $\mathcal{F}[u, \Omega]$, with f satisfying conditions $H_{0,f}$, $H_{1,f}$, $H_{3,1,f}$ or $H_{0,f}$, $H_{1,f}$, $H_{3,2,f}$; then u is locally Hölder continuous in Ω.*

Proof. Using Lemma 2 (iii) there exists a N-function $\Phi_1 \sim \Phi$ such that Φ_1 satisfys $H_{2,f}$. Moreover, since $\Phi_1 \sim \Phi$, there exist some positive constants $\varkappa_1$, $\varkappa_2$ and t_0 such that

$$\Phi(\varkappa_1 t) \le \Phi_1(t) \le \Phi(\varkappa_2 t) \tag{1.8}$$

for all $t > t_0$. If $0 \le t \le t_0$, since Φ_1 and Φ are increasing and continuous functions, we have

$$\Phi(\varkappa_1 t) - c_{10} \le \Phi_1(t) \le \Phi(\varkappa_2 t) + c_{11} \tag{1.9}$$

where c_{10} and c_{11} are positive constants. Using (1-9) and (1.10) it follows

$$\Phi(\varkappa_1 t) - c_{12} \le \Phi_1(t) \le \Phi(\varkappa_2 t) + c_{13} \tag{1.10}$$

for all $t \ge 0$. Let us consider $\Phi_2(t) = \Phi(\varkappa_1 t)$, since Φ fulfils the hypotesis $H_{0,f}$ and $H_{3,1,,f}$, then Φ_2 fulfils the hypothesis $H_{0,f}$ and $H_{3,1,,f}$; using (1.2) and Lemma 1 (i) we get

$$\begin{aligned} \Phi_2(|z|) - c_{14} &\le f(x, s, z) \\ &\le c_{15}\left[\left(\tfrac{\varkappa_2}{\varkappa_1}\right)^{\beta m^*}(\Phi^*(|z|))^{\beta} + \left(\tfrac{\varkappa_2}{\varkappa_1}\right)^{\beta m^*}(\Phi^*(|s|))^{\beta} + 1\right] \\ &\le c_{15}\left(\tfrac{\varkappa_2}{\varkappa_1}\right)^{\beta m^*}\left[(\Phi^*(|z|))^{\beta} + (\Phi^*(|s|))^{\beta} + 1\right]. \end{aligned} \tag{1.11}$$

Corollary 1 follows using (1.12) and Theorem 1. □

Particulary Theorem1 and Corollary 1 can be applied in the following cases

$\Phi(t) = t^p$ with $p > 1$,

$\Phi(t) = t^p \ln^a(1 + t)$ with $p \ge 1$ and $a > 0$,

$$\Phi(t) = \begin{cases} t^p & \text{if } 0 < t < t_0 \\ e^{\left(\frac{p+q}{2} + \frac{q-p}{2}\sin(\ln(\ln(t)))\right)\ln(t)} & \text{if } t \ge t_0 \end{cases}$$ where $\sin(\ln(\ln(t_0))) = -1$ and $1 < p < q < N$.

2. Definitions

Definition 2 A continuous and convex function $\Phi\colon [0, +\infty) \to [0, +\infty)$ is called N-function if it satisfies

$$\begin{aligned} &\Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ if } t > 0; \\ &\lim_{t \to o^+} \tfrac{\Phi(t)}{t} = 0; \\ &\lim_{t \to +\infty} \tfrac{\Phi(t)}{t} = +\infty. \end{aligned} \tag{2.1}$$

For exemple the function $\Phi_{p,\beta}(t) = t^p \ln^\beta(1+t)$ for $p > 1$ and $\beta \geq 0$ or $p = 1$ and $\beta > 0$ is a N-function. Actually, only the growth at infinity really matters in the definition of N-function. Indeed, given a continuous and convex function $A{:}[0,+\infty) \to [0,+\infty)$ satisfying

$$\lim_{t\to+\infty} \frac{A(t)}{t} = +\infty$$

there exist a N-function Φ and $t_0 > 0$ such that for every $t > t_0$ there holds

$$A(t) = \Phi(t).$$

The function A is called principal part of the N-function Φ. For exemple there exists a N-function Φ such that $\Phi(t) = t^{\ln(t)}$ near infinity or there exists a N-function Φ such that $\Phi(t) = t\ln(t)$ near infinity.

Definition 3 If Φ_1 and Φ_2 are two N-functions we say that Φ_1 dominates Φ_2 near infinity if there exist two positive constants $\varkappa$ and t_0 such that

$$\Phi_2(t) \leq \Phi_1(\varkappa t)$$

for all $t \geq t_0$.

Definition 4 If Φ_1 and Φ_2 are two N-functions we say that Φ_1 and Φ_2 are equivalent near infinity ($\Phi_1 \sim \Phi_2$) if and only if there exist some positive constants $\varkappa_1$, $\varkappa_2$ and t_0 such that

$$\Phi_1(\varkappa_1 t) \leq \Phi_2(t) \leq \Phi_1(\varkappa_2 t)$$

for all $t \geq t_0$.

Remark 4 If $0 < \lim\limits_{t\to+\infty} \frac{\Phi_1(t)}{\Phi_2(t)} < +\infty$ then Φ_1 and Φ_2 are equivalent near infinity. Let us introduce two important classes of N-functions.

Definition 5 A N-function Φ is of class $\triangle_2$ globally in $(0,+\infty)$ if exists $k > 1$ such that

$$\Phi(2t) \leq k\Phi(t) \qquad \forall t \in (0,+\infty). \tag{2.2}$$

Definition 6 A convex function Φ is of class $\triangle_2^m$ globally in $(0,+\infty)$, with $m \geq 1$, if for every $\lambda > 1$

$$\Phi(\lambda t) \leq \lambda^m \Phi(t) \qquad \forall t \in (0,+\infty). \tag{2.3}$$

Remark 5 The class $\triangle_2^1$ contains only linear functions.

Definition 7 A N-function Φ is of class ∇_2 globally in $(0,+\infty)$ if exists $l > 1$ such that

$$\Phi(t) \leq \frac{\Phi(lt)}{2l} \qquad \forall t \in (0,+\infty). \tag{2.4}$$

Definition 8 A convex function Φ is of class ∇_2^r globally in $(0,+\infty)$, with $r \geq 1$, if for every $\lambda > 1$

$$\lambda^r \Phi(t) \leq \Phi(\lambda t) \qquad \forall t \in (0,+\infty). \tag{2.5}$$

Remark 6 Every N-functions belongs to∇_2^1 .

Remark 7 We observe that

$$\Delta_2 = \bigcup_{m>1} \triangle_2^m$$

and

$$\nabla_2 = \bigcup_{r>1} \nabla_2^r.$$

The N-functions $\Phi \in \triangle_2^m$ are characterized by the following result.

Lemma 1 *Let Φ be a N-function and let $\dot{\Phi}_-$ be its left derivative. For $m \geq 1$ the following properties are equivalent:*

(i) $\Phi(\lambda t) \leq \lambda^m \Phi(t)$, for every $t \geq 0$, for every $\lambda > 1$;

(ii) $t\dot{\Phi}_-(t) \leq m\Phi(t)$, for every $t \geq 0$;

(iii) the function $\frac{\Phi(t)}{t^m}$ is nonincreasing on $(0,+\infty)$.

The N-functions $\Phi \in \nabla_2^r$ are characterized by the following result.

Lemma 2 *Let Φ be a N-function and let $\dot{\Phi}_-$ be its left derivative. For $r \geq 1$ the following properties are equivalent:*

(i') $\Phi(\lambda t) \geq \lambda^r \Phi(t)$, for every $t \geq 0$, for every $\lambda > 1$

(ii') $t\dot{\Phi}_-(t) \geq r\Phi(t)$, for every $t \geq 0$;

(iii') the function $\frac{\Phi(t)}{t^r}$ is nondecreasing on $(0, +\infty)$.

Remark 8 If $\Phi \in \triangle_2^m$ using Lemma 1 (iii) it follows

$$\Phi(t) \leq c_{16} t^m$$

for every $t > 1$.

Remark 9 If $\Phi \in \nabla_2^r$ using Lemma 2 (iii) it follows

$$\Phi(t) \geq c_{17} t^r$$

for every $t > 1$.

Remark 10 Moreover, if $\Phi \in \triangle_2^m \cap \nabla_2^r$ then

$$c_{17} t^r \leq \Phi(t) \leq c_{16} t^m$$

for every $t > 1$.

Remark 11 If Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$, then we have $\Phi(\lambda t) \leq \lambda^m \Phi(t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^m} \leq \Phi\left(\frac{s}{\lambda}\right)$ and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^m}\right) \leq \frac{s}{\lambda}$ for every $s \in (0, +\infty)$ and $\lambda > 1$. Let us put $s = \Phi^{-1}(w)$ then we have $\Phi^{-1}\left(\frac{w}{\lambda^m}\right) \leq \frac{\Phi^{-1}(w)}{\lambda}$ for every $w \in (0, +\infty)$ and $\lambda > 1$. Let us put $\frac{1}{\lambda^m} = a$ then we have $\Phi^{-1}(aw) \leq a^{\frac{1}{m}} \Phi^{-1}(w)$ for every $w \in (0, +\infty)$ and $a \in (0, 1)$.

Definition 9 We say that the N-function Φ satisfies the $\triangle'$- condition if there exist positive constants c_9 and t_0 such that

$$\Phi(ts) \leq c_9 \Phi(t) \Phi(s) \tag{2.6}$$

for every $t, s \geq t_0$. If $t_0 = 0$ we say that Φ satisfies globally the $\triangle'$- condition ($\Phi \in \triangle'$ in $(0, +\infty)$).

Let us consider the N-functions

$$\begin{aligned}
&\Phi_1(t) = t^p && \text{with } p > 1; \\
&\Phi_2(t) = t^p(|\ln(t)| + 1) && \text{with } p > 1; \\
&\Phi_3(t) = (1+t)\ln(1+t) - t; && \\
&\Phi_4(t) = \frac{t^2}{1+\ln(1+t)}; && \\
&\Phi_5(t) = e^t - t - 1. &&
\end{aligned} \tag{2.7}$$

We observe that Φ_1 and Φ_2 satisfy the $\triangle'$-condition globally in $[0, +\infty)$; moreover Φ_1 and Φ_2 belong to the class $\triangle_2 \cap \nabla_2$ globally in $[0, +\infty)$. The function Φ_3 satisfy $\triangle'$-condition for all $t \geq t_0$ but $\Phi_3 \notin \nabla_2$. The function $\Phi_5 \in \nabla_2$ but $\Phi_5 \notin \triangle_2$. Finally $\Phi_4 \in \triangle_2 \cap \nabla_2$ but $\Phi_4 \notin \triangle'$. For further details refer to (Adams, 1975; Dall'Aglio, Mascolo, & Papi, 1997, 1998; Klimov, 2000; Krasnosel'skij, 1961; Mascolo & Papi, 1996; Moscariello & Nania, 1991; Rao & Ren, 1991). Now we can introduce Orlicz spaces and Orlicz Sobolev Spaces, L^Φ and W^1L^Φ. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and open set, the Orlicz class $K^\Phi(\Omega)$ is the set of all measurable functions u: $\Omega \to \mathbb{R}$ (equivalence classes modulo equality $\mathcal{L}^N$ a.e. in Ω) satisfying $\int_\Omega \Phi(|u|)\, d\mathcal{L}^N < +\infty$. The Orlicz space $L^\Phi(\Omega)$ is defined to be the linear hull of $K^\Phi(\Omega)$, thus it consists of all measurable functions u such that $\lambda u \in K^\Phi(\Omega)$ for some $\lambda > 0$. Moreover, the equality $K^\Phi(\Omega) \equiv L^\Phi(\Omega)$ holds if and only if $\Phi \in \triangle_2$.

Definition 10 If $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\Phi \in \triangle_2$ then we define

$$W^1L^\Phi(\Omega) = \left\{u \in L^\Phi(\Omega) : \partial_i u \in L^\Phi(\Omega) \text{ for } i = 1, ..., N\right\}$$

where $\partial_i u$ are the weak derivatives of u for $i = 1, ..., N$.

Theorem 2 *Let $\Phi \in \triangle_2$, then $L^{\Phi}(\Omega)$ and $W^1L^{\Phi}(\Omega)$ are Banach spaces with the following norms*

$$\|u\|_{\Phi,\Omega} = \inf\left(k > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{k}\right) d\mathcal{L}^N \leq 1\right)$$

and

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \sum_{i=1}^{N} \|\partial_i u\|_{\Phi,\Omega} .$$

For greater details we refer to (Adams, 1975; Dall'Aglio, Mascolo, & Papi, 1997, 1998; Klimov, 2000; Krasnosel'skij, 1961; Mascolo & Papi, 1996; Moscariello & Nania, 1991; Rao & Ren, 1991).

Let Φ be a N-function then there exists a real valued function p defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right continuous on $(0, +\infty)$ such that

$$\Phi(t) = \int_0^t p(s)\, ds \qquad \text{for every } t \in (0, +\infty)$$

and

$$\dot{\Phi}_+(t) = p(t) \qquad \text{a.e. on } (0, +\infty).$$

Definition 11 Let p be a real valued function defined on $[0, +\infty)$ and having the following properties: $p(0) = 0$, $p(t) > 0$ if $t > 0$, p is increasing and right continuous on $(0, +\infty)$. We define

$$\widetilde{p}(s) = \sup_{p(t) \leq s} (t)$$

and

$$\widetilde{\Phi}(t) = \int_0^t \widetilde{p}(s)\, ds.$$

The N-functions Φ and $\widetilde{\Phi}$ are complementary N-functions.

Particularly from the relationship (2.1) of the Definition 2 we get the following Young inequality

$$ab \leq \widetilde{\Phi}(a) + \Phi(b). \tag{2.8}$$

We now recall the notion of Sobolev conjugate of a N-function. For a sake of simplicity, we will only consider the case of a function in $\triangle_2^m$.

Definition 12 Assume that $\Phi \in \triangle_2^m$, with $1 \leq m < N$. We define the Sobolev conjugate of Φ as the function Φ^* whose inverse is defined by

$$(\Phi^*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds. \tag{2.9}$$

Remark 12 Using condition (ii) of Lemma 1 we have $\Phi^{-1}(s) < cs^{\frac{1}{m}}$ for $0 < s < 1$, then the integral in (2.9) is finite and it is easy to check that Φ^* is a N-function. If $\Phi(t) = t^m$, with $1 \leq m < N$, then $\Phi^*(t) = \left(\frac{t}{m^*}\right)^{m^*}$ where $m^* = \frac{mN}{N-m}$ is the Sobolev conjugate exponent of m. Moreover, if $\Phi(t)$ is equivalent near infinity to $t^m \ln^a(1 + t)$, with $1 \leq m < N$ and $a > 0$, then $\Phi^*(t)$ is equivalent near infinity to $t^{m^*}(\ln(1 + t))^{\frac{aN}{N-m}}$.

Lemma 3 *Let Φ be a N-function in $\triangle_2^m \cap \nabla_2^r$ with $1 \leq r \leq m < N$, then $\Phi^* \in \triangle_2^{m^*} \cap \nabla_2^{r^*}$.*

3. Local Boundedness

Let E be a $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$, then with $|E|$ we denote the $\mathcal{L}^N$-measure of E. If $u \in W^1_{loc}L^{\Phi}(\Omega)$, k is a real number and $Q_R(x_0)$ is a cube strictly contained in Ω we set

$$A(k, R) = \{x \in Q_R : u(x) > k\} = \{u > k\} \cap Q_R,$$

$$B(k, R) = \{x \in Q_R : u(x) < k\} = \{u < k\} \cap Q_R.$$

Remark 13 We have $|A(k,R)| = |Q_R| - |B(k,R)|$ for almost every $k \in \mathbb{R}$, so that when necessary we can assume without loss of generality that all the vcalues k under consideration will satisfy this relation.

Our proof is based on the following Caccioppoli inequalities.

Theorem 3 *Let $u \in V$ be a local minimizer of $\mathcal{F}[u,\Omega]$, with f satisfying conditions $H_{1,f}$, $H_{2,1,f}$ or $H_{1,f}$, $H_{2,2,f}$; then there exist two real positive constants $C_{1,Cac}$, $C_{2,Cac}$ depending only on c_1, c_2, N, p and β such that for every $x_0 \in \Omega$, for every cube $Q_\varrho(x_0) \subset\subset Q_R(x_0) \subset\subset \Omega$ and for every $k \in \mathbb{R}$ we have*

$$\int_{A(k,\varrho)} \Phi(|\nabla u|)\, d\mathcal{L}^N \le \frac{C_{1,Cac}}{(R-\varrho)^{\beta p^*}} \int_{A(k,R)} (\Phi^*(u-k))^\beta\, d\mathcal{L}^N + C_{2,Cac}(\Phi^*(k))^\beta |A(k,R)| \tag{3.1}$$

and

$$\int_{B(k,\varrho)} \Phi(|\nabla u|)\, d\mathcal{L}^N \le \frac{C_{1,Cac}}{(R-\varrho)^{\beta p^*}} \int_{B(k,R)} (\Phi^*(k-u))^\beta\, d\mathcal{L}^N + C_{2,Cac}(\Phi^*(k))^\beta |B(k,R)|. \tag{3.2}$$

Proof. It follows by Theorem 4.1 of Dall'Aglio, Mascolo, and Papi (1997). □

For more details see Dall'Aglio, Mascolo, and Papi (1997, 1998).

Remark 14 Let us take $v = u+1$ and $h = k+1$, then using (3.1) and (3.2) we get

$$\int_{A(h,\varrho)} \Phi(|\nabla v|)\, d\mathcal{L}^N \le \frac{H_1}{(R-\varrho)^{\beta p^*}} \int_{A(h,R)} (\Phi^*(v-h))^\beta\, d\mathcal{L}^N + H_2(\Phi^*(h))^\beta |A(h,R)|$$

and

$$\int_{B(h,\varrho)} \Phi(|\nabla v|)\, d\mathcal{L}^N \le \frac{H_1}{(R-\varrho)^{\beta p^*}} \int_{B(h,R)} (\Phi^*(h-v))^\beta\, d\mathcal{L}^N + H_2(\Phi^*(h))^\beta |B(h,R)|.$$

Using an abuse of notation we will always identify u with v and h with k.

We can now introduce the adequate De Giorgi classes related to the functional (1.1).

Definition 13 Let $u \in W^{1,1}_{loc}(\Omega)$; we say that $u \in DG^+_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0)$ if for all couple of concentric cubes $Q_\varrho \subset Q_R \subset Q_{R_0} \Subset \Omega$, with $\varrho < R < R_0$, and for all $k \ge k_0 \ge 0$ we have

$$\int_{A(k,\varrho)} \Phi(|\nabla u|)\, d\mathcal{L}^N \le \frac{H_1}{(R-\varrho)^{\beta p^*}} \int_{A(k,R)} (\Phi^*(u-k))^\beta\, d\mathcal{L}^N + H_2(\Phi^*(k))^\beta |A(k,R)|. \tag{3.3}$$

Definition 14 Let $u \in W^{1,1}_{loc}(\Omega)$; we say that $u \in DG^-_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0)$ if for all couple of concentric cubes $Q_\varrho \subset Q_R \subset Q_{R_0} \Subset \Omega$, with $\varrho < R < R_0$, and for all $k \le -k_0 \le 0$ we have

$$\int_{B(k,\varrho)} \Phi(|\nabla u|)\, d\mathcal{L}^N \le \frac{H_1}{(R-\varrho)^{\beta p^*}} \int_{B(k,R)} (\Phi^*(k-u))^\beta\, d\mathcal{L}^N + H_2(\Phi^*(k))^\beta |B(k,R)|. \tag{3.4}$$

Definition 15 Let $u \in W^{1,1}_{loc}(\Omega)$; we say that $u \in DG_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0)$ if $u \in DG^\pm_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0)$, that is

$$DG_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0) = DG^+_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0) \cap DG^-_{\Phi,\Phi^{*\beta}}(\Omega, H_1, H_2, R_0, k_0).$$

Lemma 4 *Let $\theta > 0$ and let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of real positive numbers such that*

$$x_{i+1} \le CB^i x_i^{1+\theta} \tag{3.5}$$

with $C > 0$ and $B > 1$. If $x_0 \le C^{-\frac{1}{\theta}} B^{-\frac{1}{\theta^2}}$ then we get

$$x_i \le C^{-\frac{1}{\theta}} x_0 \tag{3.6}$$

and

$$\lim_{i\to+\infty} x_i = 0. \tag{3.7}$$

Theorem 4 *Let $u \in V$ be a local minimizer of $\mathcal{F}[u,\Omega]$, with f satisfying conditions $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,1,f}$ or $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,2,f}$; then u is locally bounded in Ω and there exists a $R_0 > 0$ such that for every $x_0 \in \Omega$ and $0 < R \leq \min(R_0, dist(x_0, \partial\Omega))$ we have*

$$\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(|u|)\right) \leq 1 + C(R)\left[\int_{Q_R} \Phi^*(|u|)^\beta \, d\mathcal{L}^N\right]^{\frac{r^*-m}{m(1-\beta)}} \tag{3.8}$$

where $C(R)$ is a real positive constant depending on R, N, m, r and β.

Proof. It follows by Theorem 3.1 of Dall'Aglio, Mascolo, and Papi (1997). □

For more details see Dall'Aglio, Mascolo, and Papi (1997, 1998).

Now we introduce our first result that improves the precedent theorem shown in general in (Dall'Aglio, Mascolo, & Papi, 1997, 1998).

Theorem 5 *Let $u \in V$ be a local minimizer of $\mathcal{F}[u,\Omega]$, with f satisfying conditions $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,1,f}$ or $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,2,f}$; then if $|k_0| + \sup(u) \leq M$ we have*

$$\left(\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(u)\right)\right)^\beta \leq \frac{C_{22}\,|A(k_0,R)|^\vartheta}{R^{\frac{m^*\beta-\gamma\alpha_2}{\alpha_1}}} \int_{A(k_0,R)} (\Phi^*(u-k_0))^\beta \, d\mathcal{L}^N + \Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right) \tag{3.9}$$

where $\alpha_1 = \frac{[N-(1-\beta)m]r - Nm\beta}{(N-m)r}$, $\alpha_2 = \frac{\beta mN}{(N-m)r} - 1$, $\gamma = N - m^*\beta$, ϑ is the positive solution of $\vartheta(\vartheta+1) = \alpha_1$, i.e. $\vartheta = \frac{-1+\sqrt{1+4\alpha_1}}{2}$, and C_{22} is a real positive number depending on N, β, α_1, α_2 and γ.

Proof. For $\frac{R}{2} \leq \sigma < \tau \leq R$ let $\eta(x)$ be a function of class $C_c^\infty\left(Q_{\frac{\tau+\sigma}{2}}\right)$ with $\eta = 1$ on Q_σ and $|\nabla\eta| \leq \frac{c}{\tau-\sigma}$. Setting $k > h > k_0$, $\zeta = \eta(u-k)_+$ we have

$$\int_{A(k,\sigma)} (\Phi^*(u-k))^\beta \, d\mathcal{L}^N \leq \int_{E_\zeta} (\Phi^*(\zeta))^\beta \, d\mathcal{L}^N \tag{3.10}$$

where E_ζ =Supp(ζ) $\subset A\left(k, \frac{\tau+\sigma}{2}\right)$. Since

$$\int_{E_\zeta} (\Phi^*(\zeta))^\beta \, d\mathcal{L}^N \leq \int_{E_\zeta} (\Phi^*(\zeta+t_0))^\beta \, d\mathcal{L}^N \leq c\int_{E_\zeta} (\zeta+t_0)^{m^*\beta} \, d\mathcal{L}^N$$

using the Sobolev Inequality we get

$$\begin{aligned} \int_{E_\zeta} (\zeta+t_0)^{m^*\beta} \, d\mathcal{L}^N &= \int_{Q_{\frac{\tau+\sigma}{2}}} (\zeta+t_0)^{m^*\beta} \, d\mathcal{L}^N \\ &\leq c_{N,m,\beta}\left[\int_{Q_{\frac{\tau+\sigma}{2}}} |\nabla\zeta|^{\frac{\beta mN}{N-(1-\beta)m}} \, d\mathcal{L}^N\right]^{\frac{N-(1-\beta)m}{N-m}} \\ &= c_{N,m,\beta}\left[\int_{E_\zeta} |\nabla\zeta|^{\frac{\beta mN}{N-(1-\beta)m}} \, d\mathcal{L}^N\right]^{\frac{N-(1-\beta)m}{N-m}}. \end{aligned} \tag{3.11}$$

Since $\beta < \frac{r(N-m)}{m(N-r)}$ using the Hölder Inequality it follows

$$\int_{E_\zeta} |\nabla\zeta|^{\frac{\beta mN}{N-(1-\beta)m}} \, d\mathcal{L}^N \leq \left|E_\zeta\right|^{1-\frac{Nm\beta}{(N-(1-\beta)m)r}}\left[\int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N\right]^{\frac{Nm\beta}{(N-(1-\beta)m)r}} \tag{3.12}$$

and

$$\int_{E_\zeta} (g^*(\zeta))^\beta \, d\mathcal{L}^N \leq c_{N,p,\beta}\left|E_\zeta\right|^{\alpha_1}\left[\int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N\right]^{\frac{\beta mN}{(N-m)r}} \tag{3.13}$$

where $\alpha_1 = \frac{[N-(1-\beta)m]r-Nm\beta}{(N-m)r}$. Since $\beta > \frac{(N-m)r}{mN}$ then $\frac{\beta mN}{(N-m)r} > 1$ and

$$\int_{E_\zeta} (g^*(\zeta))^\beta \, d\mathcal{L}^N \leq c_{N,p,\beta} \left|E_\zeta\right|^{\alpha_1} \int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N \left[\int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N\right]^{\alpha_2} \tag{3.14}$$

where $\alpha_2 = \frac{\beta mN}{(N-m)r} - 1$. Since $t^r \leq \Phi(t)$ [Refer $H_{2,f}$] we have

$$\int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N \leq \int_{A\left(k,\frac{\tau+\sigma}{2}\right)} \Phi(|\nabla u|) \, d\mathcal{L}^N + c\left[\int_{A\left(k,\frac{\tau+\sigma}{2}\right)} \left(\Phi^*\left(\frac{u-k}{\tau-\sigma}\right)\right)^\beta d\mathcal{L}^N + \left|A\left(k,\frac{\tau+\sigma}{2}\right)\right|\right].$$

Using the Caccioppoli inequality we obtain

$$\int_{E_\zeta} |\nabla\zeta|^r \, d\mathcal{L}^N \leq c\Upsilon(u) \tag{3.15}$$

where

$$\Upsilon(u) = C_{1,Cac} \int_{A(k,\tau)} \left(\Phi^*\left(\frac{u-k}{\tau-\sigma}\right)\right)^\beta d\mathcal{L}^N + C_{2,Cac}\left((\Phi^*(k))^\beta + 1\right)|A(k,\tau)|.$$

From (3.14) and (3.15) it follows

$$\int_{E_\zeta} (\Phi^*(\zeta))^\beta \, d\mathcal{L}^N \leq c_{N,p,\beta} \left|E_\zeta\right|^{\alpha_1} [\Upsilon(u)] [\Upsilon(u)]^{\alpha_2}. \tag{3.16}$$

Let $M > d > (\Phi^*)^{-1}\left(R^{m^*}\right)$ be a constant that we shall fix later, and define

$$\begin{aligned} k_0 &= d \\ k_{i+1} &= k_i + \Phi_{*,\beta}^{-1}\left(\frac{\Phi_{*,\beta}(d)}{2^{im}}\right) \quad \text{for } i \geq 1; \\ r_i &= \frac{R}{2}\left(1 + 2^{-i}\right); \\ u_i &= U(k_i, r_i); \end{aligned} \tag{3.17}$$

where $\Phi_{*,\beta}(t) = (\Phi^*(t))^\beta$, then, using (3.16), we get

$$\int_{A_{i+1}} (\Phi^*(u-k_{i+1}))^\beta \, d\mathcal{L}^N \leq c_{N,p,\beta} |A_i|^{\alpha_1} [U(k_i, r_i)] \cdot [U(k_i, r_i)]^{\alpha_2}. \tag{3.18}$$

where

$$\begin{cases} A_i = A(k_i, r_i) \\ U(k_i, r_i) = C_{1,Cac} \int_{A(k_i,r_i)} \left(\Phi^*\left(\frac{u-k_i}{r_i - r_{i+1}}\right)\right)^\beta d\mathcal{L}^N + C_{2,Cac}\left((\Phi^*(k_i))^\beta + 1\right)|A(k_i, r_i)| \end{cases}$$

Now we give two alternative estimates of

$$U(k_i, r_i) = C_{1,Cac} \int_{A(k_i,r_i)} \left(\Phi^*\left(\frac{u-k_i}{r_i - r_{i+1}}\right)\right)^\beta d\mathcal{L}^N + C_{2,Cac}\left((\Phi^*(k_i))^\beta + 1\right)|A(k_i, r_i)|.$$

Since u is bounded we get

$$U(k_i, r_i) \leq \frac{C_{1,Cac} 2^{m^+\beta} 2^{m^+\beta i}}{R^{m^+\beta}} R^N (\Phi^*(M))^\beta + C_{2,Cac}\left((\Phi^*(M))^\beta + 1\right) R^N$$

then

$$U(k_i, r_i) \leq G_M 2^{m^+\beta i} R^\gamma \tag{3.19}$$

where $G_M = 2^{m^+\beta}\left(C_{1,Cac}+C_{2,Cac}\right)\left((\Phi^*(M))^\beta+1\right)$, $C_{Cac}=\left(C_{1,Cac}+C_{2,Cac}\right)$ and $\gamma = N-m^*\beta$, since $\frac{N-m}{m}>\beta$ then $\gamma>0$.

Moreover, since

$$|A(k_i,r_i)| \le \frac{1}{(\Phi^*(k_{i+1}-k_i))^\beta}\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N$$

we get

$$U(k_i,r_i)\le\left[\frac{C_{1,Cac}}{(r_i-r_{i+1})^{m^*\beta}}+\frac{C_{2,Cac}\left((g^*(k_i))^\beta+1\right)}{(\Phi^*(k_i-k_i))^\beta}\right]\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N;$$

since $k_i = d+\sum_{j=1}^{i}\Phi^{-1}_{*,\beta}\left(\frac{\Phi_{*,\beta}(d)}{2^{m^+\beta j}}\right)\le d+\sum_{j=1}^{i}\frac{d}{2^{m^+\beta j}}\le 2^{m^+\beta}d$ and $(\Phi^*(d))^\beta > R^{m^*\beta}$, it follows

$$\begin{aligned}U(k_i,r_i) &\le \left[\frac{C_{1,Cac}2^{m^*\beta}2^{m^*\beta i}}{R^{m^*\beta}}+\frac{C_{2,Cac}2\left(2^{m^+\beta}\right)^{m^+\beta}2^{m^*\beta i}}{R^{m^*\beta}}\right]\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N\\ &\le C_{Cac}C_{2,m,N}\frac{2^{m^*\beta i}}{R^{m^*\beta}}\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N.\end{aligned}\tag{3.20}$$

Using (3.18), (3.19) and (3.20) it follows

$$\begin{aligned}\int_{A_{i+1}}(\Phi^*(u-k_{i+1}))^\beta\, d\mathcal{L}^N &\le c_{N,p,\beta}|A_i|^{\alpha_1}\cdot\left[G_M2^{m^+\beta i}R^\gamma\right]^{\alpha_2}\cdot\\ &\quad\cdot\left[C_{Cac}C_{2,m,N}\frac{2^{m^*\beta i}}{R^{m^*\beta}}\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N\right].\end{aligned}\tag{3.21}$$

Since $|A(k_i,r_i)|\le\frac{1}{(\Phi^*(k_{i+1}-k_i))^\beta}\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N$ we have

$$\int_{A_{i+1}}(\Phi^*(u-k_{i+1}))^\beta\, d\mathcal{L}^N\le\frac{C_{2,m,N,M,\beta}}{\left((\Phi^*(d))^\beta\right)^{\alpha_1}}\frac{2^{m^*\beta(1+\alpha_1+\alpha_2)i}}{R^{m^*\beta-\gamma\alpha_2}}\cdot\left[\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N\right]^{1+\alpha_1}.\tag{3.22}$$

and

$$\omega_{i+1}\le\frac{C_{2,m,N,M,\beta}}{R^{m^*\beta-\gamma\alpha_2}\left((g^*(d))^\beta\right)^{\alpha_1}}2^{m^*\beta(1+\alpha_1+\alpha_2)i}\omega_i^{1+\alpha_1}\tag{3.23}$$

where $\omega_i = |A(k_i,r_i)|^\vartheta\int_{A(k_i,r_i)}(\Phi^*(u-k_i))^\beta\, d\mathcal{L}^N$ and ϑ is the positive solution of $\vartheta(\vartheta+1)=\alpha_1$.

Choosing d such that

$$\varpi_0\le\left(\frac{C_{2,m,N,M,\beta}}{R^{m^*\beta-\gamma\alpha_2}\left((\Phi^*(d))^\beta\right)^{\alpha_1}}\right)^{-\frac{1}{\alpha_1}}2^{-\frac{m^*\beta(1+\alpha_1+\alpha_2)}{\alpha_1^2}}\tag{3.24}$$

we can apply Lemma 4; then

$$\lim_{i\to+\infty}\varpi_i=0\tag{3.25}$$

and

$$U\left(d,\frac{R}{2}\right)=0.\tag{3.26}$$

The condition imposed on d will be satisfied taking

$$(\Phi^*(d))^\beta=\frac{\left(C_{2,m,N,M,\beta}\right)^{\frac{1}{\alpha_1}}2^{\frac{m^*\beta(1+\alpha_1+\alpha_2)}{\alpha_1^2}}}{R^{\frac{m^*\beta-\gamma\alpha_2}{\alpha_1}}}\varpi_0+\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right)$$

hence we have

$$\left(\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(u)\right)\right)^\beta \leq \frac{C\,|A(k_0,R)|^\vartheta}{R^{\frac{m^*\beta-\gamma\alpha_2}{\alpha_1}}} \int_{A(k_0,R)} (\Phi^*(u-k_0))^\beta \, d\mathcal{L}^N + \Phi_{*,\beta}^{-1}\left(R^{m^*\beta}\right) \tag{3.27}$$

where $C = \left(C_{2,m,N,M,\beta}\right)^{\frac{1}{\alpha_1}} 2^{\frac{m^*\beta(1+\alpha_1+\alpha_2)}{\alpha_1^2}}$ and ϑ is the positive solution of $\vartheta(\vartheta+1)=\alpha_1$, i.e $\vartheta = \frac{-1+\sqrt{1+4\alpha_1}}{2}$. □

4. Proof of the Main Theorem

Lemma 5 *Let $u \in V$ be a local minimizer of $\mathcal{F}[u,\Omega]$, with f satisfying conditions $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,1,f}$ or $H_{0,f}$, $H_{1,f}$, $H_{2,f}$, $H_{3,2,f}$, and let $2k_0 = M(2R) - m(2R)$. Assume that $|A(k_0,R)| \leq \gamma\,|Q_R|$ for some $\gamma < 1$. If for an integer ν, it holds that*

$$osc(u,2R) \geq 2^{\nu+1}\Phi_{*,\beta}^{-1}\left(R^{m^*\beta}\right) \tag{4.1}$$

where $\Phi_{*,\beta}^{-1}(t)$ is the inverse of $\Phi_{*,\beta}(t) = (\Phi^*(t))^\beta$, then, setting $k_\nu = M(2R) - 2^{-\nu-1}osc(u,2R)$ we have

$$|A(k_\nu,R)| \leq c_{N,m,\beta,M}\frac{R^{N\alpha_1+(N-m^*\beta)(1+\alpha_2)}}{\nu^{\alpha_1}} \tag{4.2}$$

where $\alpha_1 = \frac{[N-(1-\beta)m]r-Nm\beta}{(N-m)r}$, $\alpha_2 = \frac{\beta mN}{(N-m)r} - 1$ are the constants of Theorem 5.

Proof. For $k_0 < h < k$ let us define

$$v(x) = \begin{cases} k-h & \text{if } u \geq k \\ u-h & \text{if } h < u < k \\ 0 & \text{if } u \leq h. \end{cases} \tag{4.3}$$

We have $v = 0$ in $Q_R \backslash A(k_0,R)$ and since $|Q_R \backslash A(k_0,R)| \geq (1-\gamma)\,|Q_R|$ by Sobolev inequality we get

$$\begin{aligned} \int_{A(k,R)} (\Phi^*(v))^\beta \, d\mathcal{L}^N &\leq \int_{A(k,R)} (\Phi^*(v+t_0))^\beta \, d\mathcal{L}^N \\ &\leq \int_{A(k,R)} (v+t_0)^{m^*\beta} \, d\mathcal{L}^N \\ &\leq c_{N,m,\beta}\left[\int_{\Delta_R} |\nabla u|^{\frac{\beta mN}{N-(1-\beta)m}} \, d\mathcal{L}^N\right]^{\frac{N-(1-\beta)m}{N-m}} c_{N,p,\beta}\,|\Delta_R|^{\alpha_1}\left[\int_{\Delta_R} |\nabla u|^r \, d\mathcal{L}^N\right]^{\frac{\beta mN}{(N-m)r}} \end{aligned} \tag{4.4}$$

where $\Delta_{R,k,h} = A(k,R) - A(h,R)$ and $\alpha_1 = \frac{[N-(1-\beta)m]r-Nm\beta}{(N-m)r}$. Since $t^r \leq g(t)$ and $\beta > \frac{(N-m)r}{mN}$ then $\frac{\beta mN}{(N-m)r} > 1$ and we get

$$\int_{A(k,R)} (\Phi^*(v))^\beta \, d\mathcal{L}^N \leq c_{N,p,\beta}\,|\Delta_R|^{\alpha_1}\left[\int_{\Delta_R} \Phi(|\nabla u|)\, d\mathcal{L}^N\right]\left[\int_{\Delta_R} \Phi(|\nabla u|)\, d\mathcal{L}^N\right]^{\alpha_2}. \tag{4.5}$$

where $\alpha_2 = \frac{\beta mN}{(N-m)r} - 1$. Since for $h \leq k_\nu$ we have $M(2R) - h \geq M(2R) - k_\nu \geq 2^{-\nu-1}osc(u,2R) \geq \Phi_{*,\beta}^{-1}\left(R^{m^*\beta}\right)$ then, using Caccioppoli inequality (3.1) for the levels $k = k_i = M(2R) - 2^{-i-1}osc(u,2R)$ and $h = k_{i-1}$, proceeding as in the previous Theorem 5 we get

$$|A(k_\nu,R)|^{\frac{1}{\alpha_1}} \leq \left(c_{N,m,\beta,M}\right)^{\frac{1}{\alpha_1}} \left|\Delta_{R,i}\right| R^{\frac{(N-m^*\beta)(1+\alpha_2)}{\alpha_1}}. \tag{4.6}$$

Summing over i from 1 to ν we have

$$\nu\,|A(k_\nu,R)|^{\frac{1}{\alpha_1}} \leq \left(c_{N,m,\beta,M}\right)^{\frac{1}{\alpha_1}} R^N R^{\frac{(N-m^*\beta)(1+\alpha_2)}{\alpha_1}} \tag{4.7}$$

and

$$|A(k_\nu,R)| \leq c_{N,m,\beta,M}\frac{R^{N\alpha_1+(N-m^*\beta)(1+\alpha_2)}}{\nu^{\alpha_1}}. \tag{4.8}$$

□

Lemma 6 *Let φ be a positive function, and assume that there exist a constant q and a number τ, $0 < \tau < 1$ such that for every $R < R_0$*

$$\varphi(\tau R) \leq \tau^\delta \varphi(R) + BR^\varkappa \tag{4.9}$$

with $0 < \varkappa < \delta$ *and*

$$\varphi(t) \leq q\varphi\left(\tau^k R\right) \tag{4.10}$$

for every $t \in \left(\tau^{k+1}R, \tau^k R\right)$. *Then for every* $\varrho < R \leq R_0$ *we have*

$$\varphi(\varrho) \leq C\left\{\left(\frac{\varrho}{R}\right)^{\varkappa}\varphi(R) + B\varrho^{\varkappa}\right\} \tag{4.11}$$

where C is a constant depending only on q, τ*,* $\varkappa$ *and* δ*.*

Now we can prove our main Theorem.

Proof. (Proof of Theorem 1) Let $k_0 = \frac{M(2R)+m(2R)}{2}$ we can assume $|A(k_0, R)| \leq \frac{1}{2}|Q_R|$ since otherwise we can assume $|B(k_0, R)| = |Q_R| - |A(k_0, R)| \leq \frac{1}{2}|Q_R|$ and it will be sufficient to write $-u$ instead of u. Setting $k_\nu = M(2R) - 2^{-\nu-1}osc(u, 2R)$ we have $k_\nu > k_0$ and

$$\begin{aligned}\left(\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(u-k_\nu)\right)\right)^\beta &\leq \frac{C}{R^{\frac{m^*\beta-\gamma\alpha_2}{\alpha_1}}}|A(k_\nu, R)|^\vartheta \int_{A(k_0,R)} (\Phi^*(u-k_\nu))^\beta \, d\mathcal{L}^N + +\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right) \\ &\leq \frac{C}{R^{\frac{m^*\beta-\gamma\alpha_2}{\alpha_1}}}|A(k_\nu, R)|^\vartheta R^N\left(\Phi^*\left(ess-\sup_{Q_R}(u-k_\nu)\right)\right)^\beta + +\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right).\end{aligned} \tag{4.12}$$

Let us choose an integer ν such that

$$c\left(\frac{c_{N,m,\beta,M}}{\nu^{\alpha_1}}\right)^\vartheta < \frac{1}{2^\varkappa} \tag{4.13}$$

If $osc(u, 2R) \geq 2^{\nu+1}\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right)$ then by Lemma 5 we get

$$\left(\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(u-k_\nu)\right)\right)^\beta \leq \frac{1}{2^\varkappa}R^\Lambda\left(\Phi^*\left(ess-\sup_{Q_R}(u-k_\nu)\right)\right)^\beta + \Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right) \tag{4.14}$$

where $\Lambda = N + (N\alpha_1 + (N - m^*\beta)(1+\alpha_2))\vartheta - \frac{m^*\beta-\gamma\alpha_2}{\alpha_1}$. Since $\Lambda \geq 0$ [refer $H_{1,f}$ and (1.3)] it follows

$$\left(\Phi^*\left(ess-\sup_{Q_{\frac{R}{2}}}(u-k_\nu)\right)\right)^\beta \leq \frac{1}{2^\varkappa}\left(\Phi^*\left(ess-\sup_{Q_R}(u-k_\nu)\right)\right)^\beta + \Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right) \tag{4.15}$$

and

$$osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right)osc(u, R) + \chi R. \tag{4.16}$$

In conclusion, either $osc(u, 2R) \leq 2^{\nu+1}\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right)$ or, if $osc(u, 2R) \geq 2^{\nu+1}\Phi^{-1}_{*,\beta}\left(R^{m^*\beta}\right)$, $osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right)$ $osc(u, R) + \chi R$, in any case we get

$$osc\left(u, \frac{R}{2}\right) \leq \left(1 - \frac{1}{2^{\nu+2}}\right)osc(u, R) + \chi R^\omega \tag{4.17}$$

with $0 < \omega < 1$. Apply Lemma 6 with $\tau = \frac{1}{4}$ and $\delta = \log_{\frac{1}{4}}\left(1 - \frac{1}{2^{\nu+2}}\right)$ and $\omega < \delta$ we have

$$osc(u, \varrho) \leq C\left\{\left(\frac{\varrho}{R}\right)^\omega osc(u, R) + B\varrho^\omega\right\} \tag{4.18}$$

for every $\varrho < R \leq \min(R_0, dist(x_0, \partial\Omega))$. □

5. Appendix: Hypotheses on β

Let us consider the system

$$\begin{cases} 1 \leq r \leq m < N \\ \frac{r(N-m)}{mN} < \beta < \min\left(\frac{r(N-m)}{m(N-r)}, \frac{N-m}{m}\right) \\ \alpha_1 N + (N - m^*\beta)(1+\alpha_2) \leq N \\ \alpha_1 N + [\alpha_1 N + (N - m^*\beta)(1+\alpha_2)]\vartheta\alpha_1 - m^*\beta + (N - m^*\beta)\alpha_2 \geq 0 \end{cases} \tag{5.1}$$

where $\alpha_1 = \frac{[N-(1-\beta)m]r-Nm\beta}{(N-m)r}$, $\alpha_2 = \frac{\beta mN}{(N-m)r} - 1$, $m^* = \frac{mN}{N-m}$ and ϑ is the positive solution of $\vartheta(\vartheta+1) = \alpha_1$, i.e. $\vartheta = \frac{-1+\sqrt{1+4\alpha_1}}{2}$. Moreover, since $\frac{r(N-m)}{m(N-r)} < 1$ then we get $\min\left(\frac{r(N-m)}{m(N-r)}, \frac{N-m}{m}\right) < 1$. Let us take

$$\begin{aligned}
\alpha_1 &= \frac{[N-(1-\beta)m]r-Nm\beta}{(N-m)r} \\
&= \frac{(N-m)r-m\beta(N-r)}{(N-m)r} \\
&= 1 - \frac{N-r}{r}\frac{m}{N-m}\beta \\
&= 1 - \frac{m^*}{r^*}\beta \\
\alpha_2 &= \frac{\beta mN}{(N-m)r} - 1 \\
&= \frac{m^*}{r}\beta - 1 \\
\vartheta &= \frac{-1+\sqrt{1+4\alpha_1}}{2} \\
&= \frac{-1+\sqrt{1+4\left(1-\frac{m^*}{r^*}\beta\right)}}{2} \\
&= \frac{-1+\sqrt{5-4\frac{m^*}{r^*}\beta}}{2}
\end{aligned}$$

and

$$\begin{aligned}
m^* &= N\frac{x}{1-x} && \text{where } x = \frac{m}{N} \\
r^* &= N\frac{y}{1-y} && \text{where } y = \frac{r}{N} \\
\frac{m^*}{r^*} &= \frac{x(1-y)}{y(1-x)} && \beta = z \\
\frac{m^*}{r} &= \frac{x}{y(1-x)}
\end{aligned}$$

then the system (5.1) can be written in this way

$$\begin{cases}
\frac{1}{N} \le y \le x \le 1 \\
\frac{y(1-x)}{x} < z < \min\left(\frac{y(1-x)}{x(1-y)}, \frac{1-x}{x}\right) \\
\left(1 - \frac{xz}{1-x}\right)\frac{xz}{y(1-x)} < \frac{x(1-y)}{y(1-x)}z \\
T(x,y,z) \ge 0
\end{cases}$$

where

$$\begin{aligned}
T(x,y,z) &= \left[1 - \frac{x(1-y)}{y(1-x)}z + \left(1 - \frac{xz}{1-x}\right)\frac{xz}{y(1-x)}\right]\left(\frac{-1+\sqrt{5-4\frac{x(1-y)}{y(1-x)}z}}{2}\right)\left(1 - \frac{x(1-y)}{y(1-x)}z\right) - \\
&\quad - \frac{xz}{1-x} + \left(1 - \frac{xz}{1-x}\right)\left(\frac{xz}{y(1-x)} - 1\right) + 1 - \frac{x(1-y)}{y(1-x)}z
\end{aligned}$$

If

$$w = w(x,z) = \frac{xz}{1-x}$$

we get

$$\begin{cases}
\frac{1}{N} \le y \le x \le 1 \\
y < w < \min\left(\frac{y}{(1-y)}, 1\right) \\
(1-w)\frac{w}{y} < \frac{(1-y)}{y}w \\
M(y,w) \ge 0
\end{cases}$$

where

$$\begin{aligned}
M(y,w) &= \left[1 - \frac{(1-y)}{y}w + (1-w)\frac{w}{y}\right]\left(\frac{-1+\sqrt{5-4\frac{(1-y)}{y}w}}{2}\right)\left(1 - \frac{(1-y)}{y}w\right) - \\
&\quad -w + (1-w)\left(\frac{w}{y} - 1\right) + 1 - \frac{(1-y)}{y}w
\end{aligned}$$

With simple algebraic calculations it follows that the system (5.1) is equivalent to the system

$$\begin{cases} \frac{1}{N} \leq y \leq x \leq 1 \\ y < w < \min\left(\frac{y}{(1-y)}, 1\right) \\ (1-w)\frac{w}{y} < \frac{(1-y)}{y}w \Longleftrightarrow y < w \\ E(y,w) \geq 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{N} \leq y \leq x \leq 1 \\ y < w < \min\left(\frac{y}{(1-y)}, 1\right) \\ \\ E(y,w) \geq 0 \end{cases}$$

where

$$\begin{aligned} E(y,w) &= [y-(1-y)w+(1-w)w]\left(\frac{-1+\sqrt{5-4\frac{(1-y)}{y}w}}{2}\right)\left(1-\frac{(1-y)}{y}w\right)- \\ &\quad -wy+(1-w)(w-y)+y-(1-y)w. \end{aligned}$$

Since

$$\begin{array}{ll} e_1(y) = E(y,y) = y^2\left(\frac{-1+\sqrt{1+4y}}{2}\right) \geq 0 & \text{for every } 0 \leq y \leq 1 \\ e_2(y) = E\left(y, \frac{y}{1-y}\right) = -\frac{2y^3}{1-y} \leq 0 & \text{for every } 0 \leq y < 1 \\ e_3(y) = E(y,1) = y-1+\frac{(2y-1)^2}{y}\left(\frac{-1+\sqrt{5-4\frac{(1-y)}{y}}}{2}\right) & \text{for every } \frac{1}{2} \leq y < 1 \end{array}.$$

and

$$\frac{\partial E}{\partial y} = w + G(y,w) \geq 0$$

for every $w, y \in (0,1)$, where

$$\begin{aligned} G(y,w) &= (1+w)\left(\frac{-1+\sqrt{5-4\frac{(1-y)}{y}w}}{2}\right)\left(1-\frac{(1-y)}{y}w\right)+ \\ &\quad +[y-(1-y)w+(1-w)w]\left(\frac{-1+\sqrt{5-4\frac{(1-y)}{y}w}}{2}\right)\left(\frac{w}{y^2}\right)+ \\ &\quad +[y-(1-y)w+(1-w)w]\left(1-\frac{(1-y)}{y}w\right)\frac{w}{y^2\sqrt{5-4\frac{(1-y)}{y}w}} \geq 0 \\ &\quad \text{for every } w, y \in (0,1). \end{aligned}$$

using the Bolzano-Weiestrass Teoremand the Implicit Theorem we get that the equation

$$E(y,w) = 0$$

implicitly defines a fuction

$$w = \tilde{w}(y)$$

for every $0 < y < 1$. Moreover the system

$$\begin{cases} \frac{1}{N} \leq y \leq x \leq 1 \\ y < w < \min\left(\frac{y}{(1-y)}, 1\right) \\ E(y,w) \geq 0 \end{cases}$$

has as solutions the region defined by

$$y < w < \bar{w}_y$$

where $\bar{w}_y$ is the function defined by

$$\bar{w}_y = \min(\tilde{w}(y), 1)$$

for every $0 < y < 1$. With a simple algebraic calculations we get the following relations

$$\begin{cases} 1 \leq r \leq m \leq N \\ \frac{r}{m^*} < \beta < \frac{N}{m^*}\bar{w}_{r,N}. \end{cases}$$

Let us take

$$\bar{\beta}_{r,m,N} = \frac{N}{m^*}\bar{w}_{r,N}$$

then we have the following hypoteses on β

$$\begin{cases} 1 \le r \le m \le N \\ \frac{r}{m^*} < \beta < \bar{\beta}_{r,m,N}. \end{cases}$$

In the following figures we draw the graphs of $E(y, w)$, $e_1(y)$, $e_2(y)$ and $e_3(y)$.

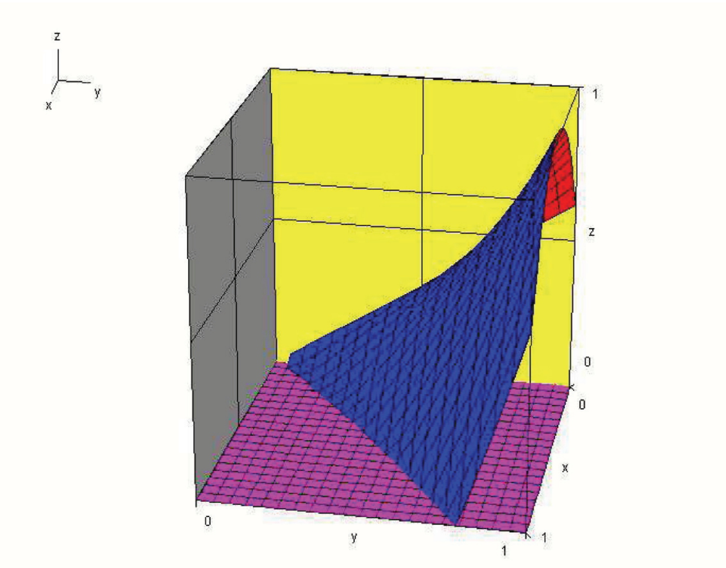

Figure 1. The graphs of $E(y, w)$

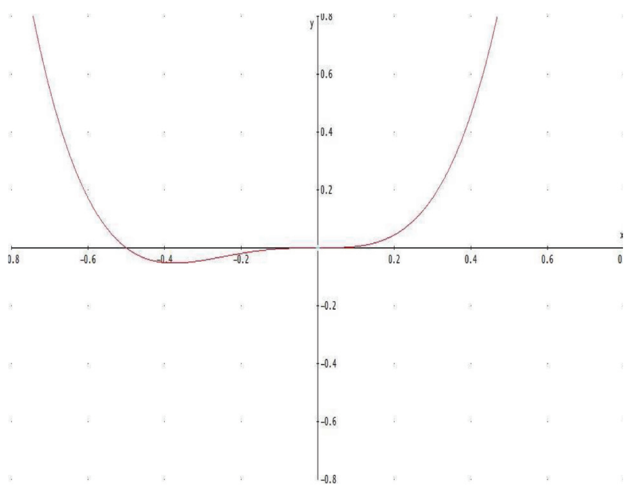

Figure 2. The graphs of $e_1(y)$

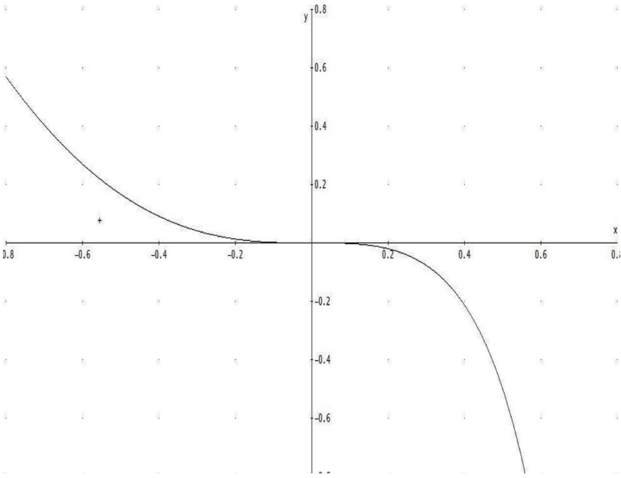

Figure 3. The graphs of $e_2(y)$

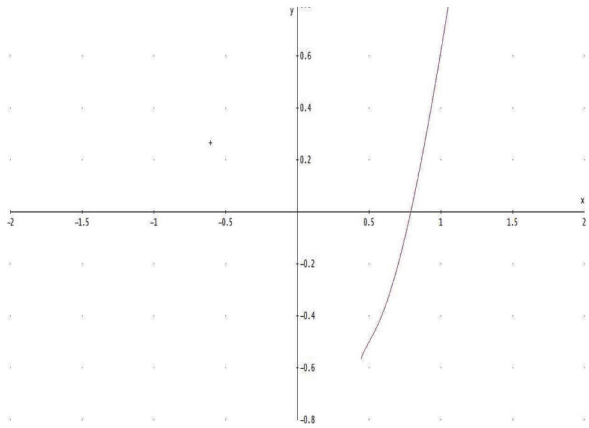

Figure 4. The graphs of $e_3(y)$

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