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Common Fixed Point and Invariant Approximation Theorems for Mappings Satisfying Generalized Contraction Principle

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Abstract

We prove common fixed point theorems for weakly compatible mappings satisfying a generalized contraction principle by using a control function. As an application, we have established invariant approximation result. Our theorems generalize recent results existing in the literature.

Keywords: Common fixed point, Weakly compatible, Generalized weak contraction, Altering distance, Control function, Invariant approximation

1. Introduction

Generalizing Banach contraction principle in various ways has become a recent research interest and has been studied by many authors. For example, one may refer (Beg, I. & Abbas, M., 2006), (Dutta, P.N. & Choudhury, B.S., 2008), (Khan, M.S., Swaleh, M. & Sessa, S., 1984), (Rhoades, B.E., 2001), (Sastry, K.P.R. & Babu, G.V.R., 1999), (Suzuki, T., 2008). (Alber, Ya.I. & Guerre-Delabriere, S., 1997) has proved a generalization for weakly contractive mapping in Hilbert space which was proved by (Rhoades, B.E.,2001) in the setup of complete metric space.

On the other hand, (Park, S., (1980) and (Khan, M.S., Swaleh, M. & Sessa, S., 1984) proved fixed point theorem for a self mapping by altering distances between the points and using a control function, whereas (Sastry, K.P.R. & Babu, G.V.R., 1999) extended the concept for weakly commuting pairs of self mappings and proved common fixed point theorem in a complete metric space by using the control function.

More recently, (Dutta, P.N. & Choudhury, B.S., 2008) have obtained a fixed point result by generalizing the concept of control function and the weakly contractive mapping.

(Jungck, G., 1976) proved a common fixed point theorem for commuting mappings generalizing the Banach's contraction principle. (Sessa, S., 1982) introduced "Weakly commuting mappings" which was generalized by (Jungck, G., 1986) as "Compatible mappings". (Pant, R.P., 1994) coined the notion of "R-weakly commuting mappings", whereas (Jungck, G. & Rhoades, B.E., 1998) defined a term called "weakly compatible mappings".

(Meinardus, G., 1963) established the existence of invariant approximation using fixed point theorem which was generalized by (Brosowski, B., 1969). (Subrahmanyam, P.V., 1977) and (Singh, S.P., 1979) relaxed the linearity of the mapping and the convexity of the set of best approximants. Further generalizations may be seen in (Habiniak, L., 1989),(Hicks, T.L. & Humphries, M.D., 1982), (Jungck, G. & Sessa, S., 1995), (Khan, L.A. & Khan, A.R., 1995), (Sahab, S.A., Khan, M.S. & Sessa, S., 1988), (Shahzad, N., 2001).

In this paper, we give generalization of (Dutta, P.N. & Choudhury, B.S., 2008) and obtain common fixed point for weakly compatible mappings satisfying a more general weak contractive condition than the conditions given in (Alber, Ya.I. & Guerre-Delabriere, S.(1997), (Beg, I. & Abbas, M., 2006), (Dutta, P.N. & Choudhury, B.S., 2008), (Rhoades, B.E., 2001). As applications, we have also established best approximation results.

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2. Definitions and Preliminaries

Definition 2.1 *Two self mappings T and f of a metric space (X, d) are said to be weakly compatible, if $fTx = Tfx$ whenever $fx = Tx$ for all $x \in X$.*

Definition 2.2 *Let T and f be self mappings of a nonempty subset M of a metric space X. The mapping T is called f-contraction mapping, if there exists a real number $0 \leq k < 1$ such that*

$$d(Tx, Ty)) \leq d(fx, fy) \tag{1}$$

for all $x, y \in M$.

Definition 2.3 *(Khan, M.S., Swaleh, M. & Sessa, S., 1984), (Park, S., 1980) A control function ψ is defined as $\psi : R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\psi(t) = 0$ if and only if $t = 0$.*

Definition 2.4 *(Beg, I. & Abbas, M., 2006) A self mapping T of a metric space (X, d) is said to be weakly contractive with respect to a self mapping $f : X \to X$, if for each $x, y \in X$,*

$$d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)), \tag{2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$ and $lim_{t\to\infty}\phi(t) = \infty$.

If $f = I$, the identity mapping, then the Definition(2.4) reduces to the definition of weakly contractive mapping given by (Alber, Ya.I. & Guerre-Delabriere, S.(1997) and (Rhoades, B.E., 2001).

Combining the generalization of Banach contraction principle given by (Khan, M.S., Swaleh, M. & Sessa, S., 1984) and the generalization given by (Rhoades, B.E., 2001), (Dutta, P.N. & Choudhury, B.S., 2008) obtained the following result.

Theorem 2.1 *(Dutta, P.N. & Choudhury, B.S., 2008) Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \phi(d(fx, fy)), \tag{3}$$

where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Definition 2.5 *Let M be a nonempty subset of a metric space (X, d). The set of best M-approximants to $u \in X$, denoted as $P_M(u)$ is defined by*

$$P_M(u) = \{y \in M : d(y, u) = dist(u, M)\},$$

where $dist(u, M) = \inf\{d(x, u) : x \in M\}$.

3. Main Results

Theorem 3.1 *Let T and f be self mappings of a metric space (X, d) satisfying*

$$\psi(d(Tx, Ty)) \quad \leq \quad \psi(M(x, y)) - \phi(M(x, y)), \tag{4}$$

where

$$M(x, y) = \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fy, Tx) + d(fx, Ty)]\right\} \tag{5}$$

and $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous and monotone decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. If TX is a complete metric space and $TX \subset fX$, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then they have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary point. Construct the sequence $\{x_n\}$ such that $fx_n = Tx_{n-1}$ for each $n = 1, 2, 3, ...\infty$ which is possible since $TX \subset fX$. Now,

$$\psi(d(Tx_n, Tx_{n+1}) \leq \psi(M(x_n, x_{n+1})) - \phi(M(x_n, x_{n+1})), \tag{6}$$

where

$$\begin{aligned}
M(x_n, x_{n+1}) &= \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \\
&\quad \frac{1}{2}[d(fx_n, Tx_{n+1}) + d(Tx_n, fx_{n+1})]\} \\
&= \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \\
&\quad \frac{1}{2}[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)]\} \\
&\leq \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\}
\end{aligned}$$

Thus inequality (6) becomes

$$\begin{aligned}
\psi(d(Tx_n, Tx_{n+1}) &\leq \psi(\max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\}) \\
&\quad -\phi(\max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\}) \\
\psi(d(Tx_n, Tx_{n+1}) &\leq \psi(d(Tx_{n-1}, Tx_n)) - \phi(d(Tx_{n-1}, Tx_n)) \qquad (7) \\
\psi(d(Tx_n, Tx_{n+1})) &\leq \psi(d(Tx_{n-1}, Tx_n)).
\end{aligned}$$

By monotone property of ψ function, we have $d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n)$. Therefore, the sequence $\{d(Tx_n, Tx_{n+1})\}$ is monotone decreasing and continuous. Hence there exists a real number $r \geq 0$ such that,

$$lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = r \tag{8}$$

As $n \to \infty$ in (7), we have $\psi(r) \leq \psi(r) - \phi(r)$ which is possible only when $r = 0$. Thus

$$lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{9}$$

Next, we claim that $\{Tx_n\}$ is a Cauchy sequence. Assume the contrary. Then there exists an $\epsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$ along with

$$d(Tx_{m_i}, Tx_{n_i}) \geq \epsilon \text{ and } d(Tx_{m_i}, Tx_{n_i-1}) < \epsilon. \tag{10}$$

Then, it follows that

$$\epsilon \leq d(Tx_{m_i}, Tx_{n_i}) \leq d(Tx_{m_i}, Tx_{n_i-1}) + d(Tx_{n_i-1}, Tx_{n_i}) < \epsilon + d(Tx_{n_i-1}, Tx_{n_i})$$

By equation (9),

$$lim_{i\to\infty} d(Tx_{m_i}, Tx_{n_i}) = \epsilon. \tag{11}$$

Now,

$$\begin{aligned}
d(Tx_{m_i}, Tx_{n_i}) &\leq d(Tx_{m_i}, Tx_{m_i-1}) + d(Tx_{m_i-1}, Tx_{n_i-1}), d(Tx_{n_i-1}, Tx_{n_i}) \\
d(Tx_{m_i-1}, Tx_{n_i-1}) &\leq d(Tx_{m_i-1}, Tx_{m_i}) + d(Tx_{m_i}, Tx_{n_i}), d(Tx_{n_i}, Tx_{n_i-1})
\end{aligned}$$

Using inequalities (9) and (11), we have as $i \to \infty$

$$lim_{i\to\infty} d(Tx_{m_i-1}, Tx_{n_i-1}) = \epsilon. \tag{12}$$

Now using inequality (4) and (10), we have

$$\psi(\epsilon) \leq \psi(d(Tx_{m_i}, Tx_{n_i})) \leq \psi(M(x_{m_i}, x_{n_i})) - \phi(M(x_{m_i}, x_{n_i})), \tag{13}$$

where

$$\begin{aligned}
M(x_{m_i}, x_{n_i}) &= \max\{d(fx_{m_i}, fx_{n_i}), d(fx_{m_i}, Tx_{m_i}), d(fx_{n_i}, Tx_{n_i}), \\
&\quad \frac{1}{2}[d(fx_{m_i}, Tx_{n_i}) + d(fx_{n_i}, Tx_{m_i})]\} \\
&= \max\{d(Tx_{m_i-1}, Tx_{n_i-1}), d(Tx_{m_i-1}, Tx_{m_i}), d(Tx_{n_i-1}, Tx_{n_i}), \\
&\quad \frac{1}{2}[d(Tx_{m_i-1}, Tx_{n_i}) + d(Tx_{n_i-1}, Tx_{m_i})]\} \\
&= \max\{d(Tx_{m_i-1}, Tx_{n_i-1}), d(Tx_{m_i-1}, Tx_{m_i}), d(Tx_{n_i-1}, Tx_{n_i}), \\
&\quad \frac{1}{2}[d(Tx_{m_i-1}, Tx_{n_i-1}) + d(Tx_{n_i-1}, Tx_{n_i}) + d(Tx_{n_i-1}, Tx_{m_i})]\}.
\end{aligned}$$

Using inequalities (9), (10) and (12), we have

$$\begin{aligned}
lim_{i\to\infty} M(x_{m_i}, x_{n_i}) &= \max\{\epsilon, 0, 0, \frac{1}{2}[\epsilon + \epsilon]\} \\
lim_{i\to\infty} M(x_{m_i}, x_{n_i}) &= \epsilon.
\end{aligned}$$

As $i \to \infty$ and using (11), inequality (13) becomes,

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$$

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a contradiction, as $\epsilon > 0$. Thus $\{Tx_n\}$ is a Cauchy sequence in TX which in turn implies that $\{fx_n\}$ is also a Cauchy sequence in X. Since TX is complete, $\{Tx_n\}$ converges to some $v \in TX$. Since $TX \subset fX$ and $v = fu$ for some $u \in X$. Thus $\{fx_n\}$ converges to fu. Now,

$$lim_{n\to\infty}\psi(d(Tx_n, Tu)) \leq lim_{n\to\infty}[\psi(M(x_n, u)) - \phi(M(x_n, u))],$$

where

$$\begin{aligned} lim_{n\to\infty}M(x_n, u) &= lim_{n\to\infty}[\max\{d(fx_n, fu), d(fx_n, Tx_n), d(fu, Tu), \\ &\quad \frac{1}{2}[d(fx_n, Tu) + d(fu, Tx_n)]\}] \\ &= \max\{0, 0, d(v, Tu), \frac{1}{2}d(v, Tu)\} = d(v, Tu). \end{aligned}$$

By monotone increasing property of ψ and ϕ, we have

$$\begin{aligned} \psi[d(v, Tu)] &\leq \psi(d(v, Tu)) - \phi(d(v, Tu)) \\ \Rightarrow \phi(d(v, Tv)) &\leq 0 \end{aligned}$$

which is possible only when $d(v, Tu) = 0$. Thus $v = Tu = fu$ and u is the coincidence point of T and f. Since T and f are weakly compatible, they commute at their coincidence point. Hence $Tfu = fTu$ which implies $Tv = fv$. Now,

$$\psi(d(Tu, Tv)) \leq \psi(M(u, v)) - \phi(M(u, v)),$$

where

$$\begin{aligned} M(u, v) &= \max\{d(fu, fv), d(fu, Tu), d(fv, Tv), \frac{1}{2}[d(fu, Tv) + d(fv, Tu)]\} \\ &= d(v, Tv). \end{aligned}$$

Hence

$$\begin{aligned} \psi(d(v, Tv)) &= \psi(d(Tu, Tv)) \leq \psi(d(v, Tv)) - \phi(d(v, Tv)) \\ \Rightarrow \phi(d(v, Tv)) &\leq 0 \end{aligned}$$

which is possible only when $v = Tv$. Thus $v = Tv = fv$. Hence v is the common fixed point of T and f.

Uniqueness: Let v and w be two common fixed points of T and f. (i.e) $v = Tv = fv$ and $w = Tw = fw$. Using inequality (4), we have

$$\psi(d(Tv, Tw)) \leq \psi(M(v, w)) - \phi(M(v, w))$$

where,

$$\begin{aligned} M(v, w) &= \max\{d(fv, fw), d(fv, Tv), d(fw, Tw), \frac{1}{2}[d(fv, Tw) + d(fw, Tv)]\}. \\ &= d(v, w) \end{aligned}$$

Therefore,

$$\psi(d(v, w)) = \psi(d(Tv, Tw)) \leq \psi(d(v, w)) - \phi(d(v, w))$$

which is possible only when $v = w$. Hence v is the unique common fixed point of T and f.

Example 3.1 *Let $X = [0, 1]$ with the usual metric. Define two self mappings T and f of X by $Tx = \frac{x}{2}$ and $fx = x$ for all $x \in X$. Let $\psi : [0, \infty) \to [0, \infty)$ be defined by*

$$\psi(t) = \begin{cases} t + \frac{t^2}{2} & \text{if} \quad 0 \leq t \leq 1 \\ 0 \quad \text{if} & t > 1 \end{cases}$$

and $\phi : [0, \infty) \to [0, \infty)$ be defined by

$$\phi(t) = \begin{cases} \frac{3t^2}{8} & \text{if} \quad 0 \leq t \leq 1 \\ 0 \quad \text{if} & t > 1 \end{cases}$$

Now to verify inequality (4), LHS of (4)= $\psi(d(Tx,Ty)) = \psi(\frac{|x-y|}{2}) = \frac{|x-y|}{2} + \frac{|x-y|^2}{8}$.

RHS of (4)= $\psi(M(x,y)) - \phi(M(x,y))$,
where $M(x,y) = max\ \{|x-y|, \frac{|x|}{2}, \frac{|y|}{2}, |x-\frac{y}{2}|, |y-\frac{x}{2}|\} = |x-y|$.

Then RHS of (4) becomes $\psi(|x-y|) - \phi(|x-y|) = [|x-y| + \frac{|x-y|^2}{2}] - [\frac{3|x-y|^2}{8}] = |x-y| + \frac{|x-y|^2}{8}$. *Thus LHS* $\leq$ *RHS and inequality (4) is verified. Now, it is easy to see that* $TX = [0,\frac{1}{2}] \subset fX = [0,1]$. *Moreover, T and f are weakly compatible in X. Hence all the conditions of Theorem 3.1 are satisfied. It may be noted that '0' is the unique fixed point of T and f.*

Remark 3.1 *If in Theorem 3.1,* $\phi(t) = (1-k)\psi(t)$, *then we obtain Theorem 2.1 of (Pant, R.P., Jha, K. & Lohani, A.B., 2003).*

Remark 3.2 *By taking* $\psi(t) = t$ *and* $M(x,y) = d(fx, fy)$ *in Theorem 3.1, we obtain Theorem 2.1 and Theorem 2.5 of (Beg, I. & Abbas, M., 2006). If in addition,* $f = I$, *the identity mapping, then we obtain Theorem 1 of (Rhoades, B.E., 2001).*

4. Application to best approximation

Theorem 4.1 *Let* T *and* f *be self mappings of a metric space* (X,d). *Suppose that* $u \in X$, T *and* f *satisfy inequality(3.1),* T *leaves* f*-invariant compact subset* M *of closed subspace* fX *as invariant. For each* $b \in P_M(u)$, *let* $d(x,Tb) < d(x,fb)$ *and* $fb \in P_M(u)$. *If* T *and* f *are weakly compatible, then* u *has a best approximation in* M *which is also a common fixed point of* T *and* f.

Proof: Let $u \in F(T) \bigcap F(f)$. Since M is a compact subset of fX, $P_M(u) \neq \emptyset$. To prove that $T(P_M(u)) \subseteq f(P_M(u))$, assume the contrary. Then there exists $b \in P_M(u)$ with $Tb \notin f(P_M(u))$. Now,

$$d(u, fb) = dist(u, M) \leq d(u, Tb)) < d(u, fb)$$

which is a contradiction. Hence $T(P_M(u)) \subseteq f(P_M(u))$. Now, $f(P_M(u))$ being a closed subset of a complete space is complete. Hence $P_M(u) \bigcap F(T) \bigcap F(f)$ is singleton.

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