

Optimal Use of Fertilizer to Overcome the Effects of Toxicity in Plants

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Abstract

In this paper we have considered a two species competitive system where one species produces a substance, which is toxic to the other. Various equilibrium points and their stability are discussed. Optimal policy of using fertilizer is determined by means of control theory. Finally, some numerical simulations are given.

Keywords: Toxicity, Plants, Fertilizer, Optimal control

1. Introduction

Plant-plant chemical interference play an important role in nature. It is common experience that there are two types of such interference (i) direct plant - plant interference mediated by allelo chemicals, and (ii) the effect of secondary compounds released by plants on abiotic and biotic soil processes that effect other plants. The toxic effect between two plants becomes vulnerable when we do not maintain the normal distance between them. The most important effects of compounds released into the soil environment by plants on other plants occur through indirect effects. We know that if we try to grow tomatoes near black walnut (*Juglans nigra*), then we face the devastating effect the chemical juglone can have on certain plants. Both blackwalnut and butternut (*J.cinerea*) produce juglone in sufficient amount to cause wilting and yellowing of leaves, and sometimes the death of the entire plants.

Sometimes due to shortage of land, it becomes necessary to grow different plants even if they do not maintain the normal distance between them. In such case, the problem is to see whether proper dose of fertilizer could be ascertained which would ensure the optimal growth of the plants in the best possible way.

Fertilizers are available in several forms, the most commonly used forms for house plants being liquids, spikes, tablets, powders, pellets and granules. Liquids come in concentrated or dilute form, the dilute "ready-to-use" forms being convenient but expensive. Spikes, pallets and tablets are often slow-release or time-release. When we shall use some fertilizer we have to keep in mind the price of the fertilizer. The rate and timing of fertilizer applications are very important in maintaining a vigorous and healthy turf as well as keeping the nutrients of our garden. We should keep in mind that too much fertilizer can be toxic to plants, damaging roots and leaves. Periodic leaching washing excess nutrients out of the soil by deep, long watering and appropriate application of fertilizer will help to reduce soluble salt burn.

The application of optimal control theory in ecology appeared at the end of the 1960s. Shoemaker (1973a, 1973b, 1973c) investigated the application of optimization techniques, especially dynamic programming, to making decision in agricultural pest management. To talk about various mathematical models to which optimal control theory have been used, we prefer to mention (Kirk (1970), Clark (1990), Kar and Matsuda (2008), Shastri and Diwekar (2006)). Many studies have been developed to better understand the effect of toxicants on populations (Barrett (1983), Hallam and Luna (1984), Luna and Hallam (1987), Bhattacharya (2004), Hallam et.al. (1983), Chatterjee (1996), Tillmann and John (2002), Das et.al. (2009), Sole et.al (2005)).

2. Mathematical Models

The models before and after the use of fertilizer are given by (I) and (II)

$$(I) \quad \begin{aligned} \frac{dx_1}{dt} &= x_1[\alpha_1 - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2] \\ \frac{dx_2}{dt} &= x_2[\alpha_2 - \beta_2 x_2 - v_2 x_1] \end{aligned}$$

$$(II) \quad \begin{aligned} \frac{dx_1}{dt} &= x_1[\alpha_1 - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2] + qux_1 \\ \frac{dx_2}{dt} &= x_2[\alpha_2 - \beta_2 x_2 - v_2 x_1] \end{aligned}$$

where α_1 and α_2 are natural growth, β_1 and β_2 are coefficients of intra-specific competition, v_1 and v_2 are coefficients of inter-specific competition of two biomasses x_1 and x_2 respectively; γ is the coefficient of toxicity of the biomass x_1 ; q is the coefficient of additional growth effective to x_1 due to the application of fertilizer of amount u per unit biomass.

3. Existence & Stability Analysis of Equilibrium

For model (I), there exist three boundary equilibrium points, namely $E_{10}(0, 0)$, $E_{11}(\alpha_1/\beta_1, 0)$ and $E_{12}(0, \alpha_2/\beta_2)$. The interior equilibrium $E^*(x_1^*, x_2^*)$ is given by

$$\alpha_1 - \beta_1 x_1^* - v_1 x_2^* - \gamma x_1^* (x_2^*)^2 = 0, \quad (1)$$

$$\alpha_2 - \beta_2 x_2^* - v_2 x_1^* = 0. \quad (2)$$

$$\text{From (2), } x_1^* = (\alpha_2 - \beta_2 x_2^*)/v_2, \quad (3)$$

and x_2^* is given by $\alpha_1 - \{\beta_1 + \gamma(x_2^*)^2\}x_1^* - v_1 x_2^* = 0$,

$$\text{or, } \gamma\beta_2(x_2^*)^3 - \gamma\alpha_2(x_2^*)^2 - (v_1 v_2 - \beta_1 \beta_2)x_2^* - (\alpha_2 \beta_1 - \alpha_1 v_2) = 0. \quad (4)$$

So, by Descartes' rule of signs there exists at least one positive root of (4) if

$$\alpha_2 \beta_1 - \alpha_1 v_2 > 0, \text{ i.e. if } (\alpha_2/\alpha_1) > (v_2/\beta_1). \quad (5)$$

again $x_1^* > 0$ if $x_2^* < (\alpha_2/\beta_2)$. For model (II), there exist three boundary equilibriums $E_{20}(0, 0)$, $E_{21}((\alpha_1 + qu)/\beta_1, 0)$, $E_{22}(0, \alpha_2/\beta_2)$, and the interior equilibrium $\bar{E}(\bar{x}_1, \bar{x}_2)$ is given by

$$(\alpha_1 + qu) - \beta_1 \bar{x}_1 - v_1 \bar{x}_2 - \gamma \bar{x}_1 (\bar{x}_2)^2 = 0, \quad (6)$$

$$\text{and } \alpha_2 - \beta_2 \bar{x}_2 - v_2 \bar{x}_1 = 0. \quad (7)$$

$$\text{Therefore, } \bar{x}_1 = (\alpha_2 - \beta_2 \bar{x}_2)/v_2, \quad (8)$$

where \bar{x}_2 is obtained from the equation

$$\gamma\beta_2(\bar{x}_2)^3 - \gamma\alpha_2(\bar{x}_2)^2 - (v_1 v_2 - \beta_1 \beta_2)\bar{x}_2 - \{\alpha_2 \beta_1 - v_2(\alpha_1 + qu)\} = 0. \quad (9)$$

$$\text{So, there exists at least one positive solution of (9) if } u < (\alpha_2 \beta_1 - \alpha_1 v_2)/v_2 q. \quad (10)$$

Again $\bar{x}_1 > 0$, if $\bar{x}_2 < (\alpha_2/\beta_2)$.

To investigate the stability of all the above equilibriums we consider the characteristic equation $|A - \lambda I| = 0$. For the model (I), $A = (a_{ij})_{3 \times 3}$ where

$$a_{11} = (\alpha_1 - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2) + x_1(-\beta_1 - \gamma x_2^2),$$

$$a_{12} = x_1(-v_1 - 2\gamma x_1 x_2), a_{21} = -v_2 x_2,$$

$$a_{22} = (\alpha_2 - \beta_2 x_2 - v_2 x_1) + x_2(-\beta_2).$$

Now for $E_{10}(0, 0)$, characteristic equation becomes $(\alpha_1 - \lambda)(\alpha_2 - \lambda) = 0$ and it is always unstable.

For $E_{11}(\alpha_1/\beta_1, 0)$, characteristic equation becomes $(\lambda + \alpha_1)(\lambda - \alpha_2 + \alpha_1 v_2/\beta_1) = 0$.

Hence, $E_{11}(\alpha_1/\beta_1, 0)$ is asymptotically stable if $(\alpha_2 \beta_1 - \alpha_1 v_2) < 0$ and is unstable if $(\alpha_2 \beta_1 - \alpha_1 v_2) > 0$.

Similarly, $E_{12}(0, \alpha_2/\beta_2)$ is asymptotically stable if $(\alpha_1 \beta_2 - \alpha_2 v_1) < 0$ and is unstable if $(\alpha_1 \beta_2 - \alpha_2 v_1) > 0$.

Now for the interior equilibrium, characteristic equation becomes,

$$\lambda^2 + A\lambda + B = 0, \quad (11)$$

where

$$A = \beta_1 x_1^* + \beta_2 x_2^* + \gamma x_1^* (x_2^*)^2, \tag{12}$$

$$B = \{(\beta_1 \beta_2 - v_1 v_2) + (\beta_2 x_2^* - 2v_2 x_1^*) \gamma x_2^*\} x_1^* x_2^*. \tag{13}$$

Therefore (x_1^*, x_2^*) is stable node if $B > 0$,

$$\text{i.e., if } (\beta_2/v_2) > \{(v_1 + 2\gamma x_1^* x_2^*)/(\beta_1 + \gamma x_2^{*2})\}. \tag{14}$$

For the model (II), $E_{20}(0, 0)$ is always unstable. $E_{21}((\alpha_1 + qu)/\beta_1, 0)$ is asymptotically stable if $\{\alpha_2 \beta_1 - (\alpha_1 + qu)v_2\} < 0$ and is unstable if $\{\alpha_2 \beta_1 - (\alpha_1 + qu)v_2\} > 0$.

$$\text{Also, } (\bar{x}_1, \bar{x}_2) \text{ is asymptotically stable if } (\beta_2/v_2) > \{(v_1 + 2\gamma \bar{x}_1 \bar{x}_2)/(\beta_1 + \gamma \bar{x}_2^2)\}. \tag{15}$$

So, we reach the Theorem 3.1.

Theorem 3.1. (i) The trivial equilibrium $(0,0)$ is always unstable for both the model (I) and (II).

(ii) The common equilibrium $(0, \alpha_2/\beta_2)$ of the model (I) and (II) is asymptotically stable for $(\alpha_2/\alpha_1) > (\beta_2/v_1)$.

(iii) The equilibrium point $(\alpha_1/\beta_1, 0)$ of the model (I) is asymptotically stable if $(\alpha_2/\alpha_1) < (v_2/\beta_1)$ and $((\alpha_1 + qu)/\beta_1, 0)$ of the model (II) is asymptotically stable if $\{\alpha_2/(\alpha_1 + qu)\} < v_2/\beta_1$.

(iv) The interior equilibrium (x_1^*, x_2^*) of the model (I) is asymptotically stable if $(\beta_2/v_2) > \{(v_1 + 2\gamma x_1^* x_2^*)/(\beta_1 + \gamma x_2^{*2})\}$ and same for the model (II) is asymptotically stable if $(\beta_2/v_2) > \{(v_1 + 2\gamma \bar{x}_1 \bar{x}_2)/(\beta_1 + \gamma \bar{x}_2^2)\}$.

Again if $v_t = \{u : 0 < u < u_{max}\}$ denote the control set, then $u_{max} = \{\alpha_2 \beta_1 - \alpha_1 v_2\}/v_2 q$. From (3) and (4) we see that, there exists a unique positive equilibrium (x_1^*, x_2^*) of the

$$\text{model (I) if } (\alpha_2/\alpha_1) > (v_2/\beta_1), G^2 + 4H^3 > 0, \text{ and } x_2^* < (\alpha_2/\beta_2), \tag{16}$$

$$\text{where } G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, H = a_0 a_2 - a_1^2, \tag{17}$$

$$a_0 = \gamma \beta_2, a_1 = -(\gamma \alpha_2)/3, a_2 = (\beta_1 \beta_2 - v_1 v_2)/3 \ \& \ a_3 = (\alpha_1 v_2 - \alpha_2 \beta_1). \tag{18}$$

Similarly, from (8) and (9), we see that there exists unique positive equilibrium (\bar{x}_1, \bar{x}_2)

$$\text{for the model (II) if, } (\alpha_2/(\alpha_1 + qu)) > (v_2/\beta_1), G_1^2 + 4H_1^3 > 0, \text{ and } \bar{x}_2 < (\alpha_2/\beta_2), \tag{19}$$

$$\text{where, } G_1 = (a'_0)^2 a'_3 - 3a'_0 a'_1 a'_2 + 2(a'_1)^3, H_1 = a'_0 a'_2 - (a'_1)^2, \tag{20}$$

$$a'_0 = a_0 = \gamma \beta_2, a'_1 = a_1 = -(\gamma \alpha_2)/3, a'_2 = a_2 = (\beta_1 \beta_2 - v_1 v_2)/3, \text{ and } a'_3 = (\alpha_1 + qu)v_2 - \alpha_2 \beta_1. \tag{21}$$

To establish the non-existence of periodic orbit encircling (x_1^*, x_2^*) of model (I) and (\bar{x}_1, \bar{x}_2) of model (II), we use Bendixon-Dulac criterion. Now the model (I) can be written as

$$\frac{dx_1}{dt} = f_1(x_1, x_2), \frac{dx_2}{dt} = f_2(x_1, x_2), \text{ where } f_1(x_1, x_2) = x_1[\alpha_1 - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2],$$

$f_2(x_1, x_2) = x_2[\alpha_2 - \beta_2 x_2 - v_2 x_1]$. we consider Dulac function $D(x_1, x_2) = 1/(x_1 x_2)$. Obviously, $D(x_1, x_2) > 0$ for all $x_1, x_2 > 0$.

Then, $\frac{\partial}{\partial x_1}(Df_1) + \frac{\partial}{\partial x_2}(Df_2) = \{(-\beta_1 - \gamma x_2^2)/x_2\} + (-\beta_2/x_1) < 0$, for all $x_1, x_2 > 0$, since all other parameters are strictly positive.

For model (II), let $\frac{dx_1}{dt} = F_1(x_1, x_2), \frac{dx_2}{dt} = F_2(x_1, x_2)$,
where $F_1(x_1, x_2) = x_1\{(\alpha_1 + qu) - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2\}$,
 $F_2(x_1, x_2) = x_2\{\alpha_2 - \beta_2 x_2 - v_2 x_1\}$.

Then, $\frac{\partial}{\partial x_1}(DF_1) + \frac{\partial}{\partial x_2}(DF_2) = \{(-\beta_1 - \gamma x_2^2)/x_2\} + (-\beta_2/x_1) < 0$.

Therefore, there exists no limit cycle for the model (I) and (II). So, we reach the theorem 3.2.

Theorem 3.2 Whenever the conditions (16) to (21) hold, both the interior equilibrium of the model (I) and (II) are globally stable.

4. Bionomic Aspect of the Model

Let π be the profit function defined by $\pi = (pqx_1 - c_0) u(t)$, where p is the market value per unit biomass of x_1 , c_0 is the cost per unit amount of fertilizer. Then bionomic equilibrium $(x_{1\infty}, x_{2\infty})$ of the model (II) is defined as the point of intersection of $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ and $\pi = 0$ provided the corresponding $u \in v_t$.

$$\text{Now, } \pi = 0 \implies x_1 = \frac{c_0}{pq} > 0, \quad (22)$$

$$x_2 = \frac{\alpha_2 - v_2 x_1}{\beta_2} = \frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq}, \quad (23)$$

$$u = \frac{1}{q} \left[\frac{c_0 \beta_1}{pq} + \frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq} \left\{ v_1 + \frac{c_0 \gamma}{pq} \left(\frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq} \right) \right\} - \alpha_1 \right].$$

$$\text{So, } x_{1\infty} = \frac{c_0}{pq}, \quad x_{2\infty} = \frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq} \text{ and}$$

$$u_\infty = \frac{1}{q} \left[\frac{c_0 \beta_1}{pq} + \frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq} \left\{ v_1 + \frac{c_0 \gamma}{pq} \left(\frac{\alpha_2 pq - c_0 v_2}{\beta_2 pq} \right) \right\} - \alpha_1 \right].$$

Now $x_{2\infty} > 0$ provided $c_0 < \frac{\alpha_2 pq}{v_2}$. Thus the system has a bionomic equilibrium.

Theorem 4.1 Let us consider the model (II) with the restrictions given by (10) and (15). Let the objective function be $J = \int_0^T \pi(x_1, u, t) dt$, where $\pi = (pqx_1 - c_0)u(t)$ and T is the total time of application of the fertilizer $u(t)$. Our goal is to maximize J over $u(t)$, $u(t) \in v_t$.

The optimal biomass (x_1, x_2) are determined by the point of intersection of $(pqx_1 - c_0)[(\beta_1 \beta_2 - v_1 v_2) + \gamma x_2 (\beta_2 x_2 - 2v_2 x_1)] + pq^2 \beta_2 u = 0$, $\alpha_2 - \beta_2 x_2 - v_2 x_1 = 0$ and $(\alpha_1 + qu) - \beta_1 x_1 - v_1 x_2 - \gamma x_1 x_2^2 = 0$. In that case corresponding $u = u^*$ is the optimal value of u .

Proof : Let $P(x_1, x_2) = \alpha_1 x_1 - \beta_1 x_1^2 - v_1 x_1 x_2 - \gamma x_1^2 x_2^2$,
 $Q(x_1, x_2) = \alpha_2 x_2 - \beta_2 x_2^2 - v_2 x_1 x_2$. Then the Hamiltonian corresponding to (II) is given by $H = (pqx_1 - c_0)u(t) + \lambda_1(t)[P(x_1, x_2) + qux_1] + \lambda_2(t)Q(x_1, x_2)$, where $\lambda_1(t)$ and $\lambda_2(t)$ are adjoint variables.

Now the adjoint equations for $\lambda_1(t)$ and $\lambda_2(t)$ are

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x_1} = -pqu(t) - \lambda_1(t)[P_{x_1}(x_1, x_2) + qu] - \lambda_2(t)[Q_{x_1}(x_1, x_2)] \\ &= -pqu(t) - \lambda_1(t)[- \beta_1 x_1 - \gamma x_1^2 x_2^2] + \lambda_2(t)v_2 x_2 \end{aligned} \quad (24)$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -\lambda_1(t)[-v_1 x_1 - 2\gamma x_1^2 x_2] - \lambda_2(t)[- \beta_2 x_2]. \quad (25)$$

Here we use the steady state solution as we are concerned with optimal equilibrium and we consider x_1 & x_2 as constant in the subsequent steps.

So, at the optimal point $\frac{d\lambda_1}{dt} = 0 = \frac{d\lambda_2}{dt}$ imply

$$\lambda_1(t)(\beta_1 x_1 + \gamma x_1^2 x_2) + \lambda_2(t)v_2 x_2 = pqu, \quad (26)$$

$$\lambda_1(t)(v_1 x_1 + 2\gamma x_1^2 x_2) + \lambda_2(t)\beta_2 x_2 = 0. \quad (27)$$

$$\text{Therefore we get } \lambda_1(t) = \frac{pq\beta_1 x_2 u}{[(\beta_1 \beta_2 - v_1 v_2) + \gamma x_2 (\beta_2 x_2 - 2v_2 x_1)] x_1 x_2}. \quad (28)$$

Let us now assume that H is maximum for $u^* \in v_t$ i.e. $\left. \frac{\partial H}{\partial u} \right|_{u=u^*} = 0$.

Then $(pqx_1 - c_0) + \lambda_1(t)qx_1 = 0$

$$\text{or, } (pqx_1 - c_0)[(\beta_1 \beta_2 - v_1 v_2) + \gamma x_2 (\beta_2 x_2 - 2v_2 x_1)] + pq^2 \beta_2 u = 0 \quad (29)$$

which represents the optimal path.

Optimal biomass (x_1, x_2) are determined by the point of intersection of (29), $\alpha_2 - \beta_2 x_2 - v_2 x_1 = 0$, and $(\alpha_1 + qu) - \beta_2 x_1 v_1 x_2 - \gamma x_1 x_2^2 = 0$, corresponding value of u is the optimal value of u .

5. Numerical Simulation & Discussion

If we choose the values of parameters as $\alpha_1 = 9$, $\beta_1 = 0.4$, $\alpha_2 = 10$, $\beta_2 = 0.2$, $v_1 = 0.01$, $v_2 = 0.02$, $\gamma = 0.002$ and $q = 1$, we find that $u = 58.50314$, $u_{max} = 191$.

Here $\alpha_2/\alpha_1 = 1.111$, $v_2/\beta_1 = 0.05$, therefore $(\alpha_2/\alpha_1) > (v_2/\beta_1)$.

Now, $a'_0 = \gamma\beta_2 = 0.0004$, $a'_1 = -(\gamma\alpha_2)/3 = -0.0067$, $a'_2 = (\beta_1\beta_2 - v_1v_2)/3 = 0.0266$, $a'_3 = (\alpha_1v_2 - \alpha_2\beta_1) + quv_2 = -2.65$.

Now $H_1 = a'_0a'_2 - (a'_1)^2 = -0.3380444 \times (10)^{-4}$, $G_1 = -0.8037825 \times (10)^{-6}$, $G_1^2 + 4H_1^3 = 0.4915476 \times (10)^{-12} > 0$. Since $G_1^2 + 4H_1^3 > 0$ there exists a pair of complex roots of the equation (9). Again $\alpha_2/(\alpha_1 + qu) = 0.1481413$, $v_2/\beta_1 = 0.05$. So $\alpha_2/(\alpha_1 + qu) > v_2/\beta_1$ and hence (9) has at least one positive root. Combining the above two results we can claim that (9) has a unique positive root. Also for the above choice of parameters $\bar{x}_2 = 48.69689 < \alpha_2/\beta_2$. So, for the above choice of parameters, there exists a unique positive equilibrium of the model (II) (see figure 1).

Since there exists no limit cycle and the positive equilibrium is unique it will be globally stable (see figures 2 & 3).

For bionomic equilibrium we have the above set of parameters as it is with $p=3.5$ and $c_0 = 75$. Then we get the bionomic equilibrium point $(x_{1\infty}, x_{2\infty})$, where $x_{1\infty} = 21.42857$ and $x_{2\infty} = 47.85714$ with $u^* = 98.20598 < u_{max}(191)$. Here we see that existence condition for the bionomic equilibrium point holds, since $c_0 < \frac{\alpha_2 pq}{v_2}$ (see figure 4).

To get the optimal equilibrium point we solve the equations

$$10 - 0.2x_2 - 0.02x_1 = 0,$$

$$59.0799 - 0.4x_1 - 0.01x_2 - 0.002x_1x_2^2 = 0,$$

$$\text{and } (3.5x_1 - 75)[0.0798 + 0.002x_2(0.2x_2)(0.4x_1)] + 35.05593 = 0.$$

We get the optimal equilibrium point (x_1, x_2) , where $x_1 = 11.31945$, $x_2 = 48.86805$, and the corresponding $u^* = 50.0799$, which is the optimal value of u (see figure 5).

Now, if we choose the set of parameters as follows, $\alpha_1 = 9$, $\alpha_2 = 10$, $\beta_1 = 0.4$, $\beta_2 = 0.2$, $v_1 = 0.01$, $v_2 = 0.07$, $p = 1$, $q = 1$, $c_0 = 25$, we get the optimal point $(12.33669, 45.68216)$ and the corresponding optimal value of u is 47.8821 , which is less than the maximum value of $u(u_{max} = 48.14286)$.

From (5.5), we get the value of adjoint variable $\lambda_1(t) = 1.0026455$. Next we draw the bang-bang diagram (see figure 6).

It is observed that optimal paths always take less time than suboptimal path to reach the optimal steady state.

References

- Barrett, J. A. (1983). Estimating relative fitness in plant parasites: some general problems. *phytopathology*, 73, 510 -512.
- Bhattacharya, D. K. (2004). Toxicity in planets and optimal growth under fertilizer. *J. Appl. Mathematical and Computing*, Vol. 16, 355 -369.
- Chatterjee, J. (1996). Effect of toxic substance on a two species competitive system. *Ecol. Model*, 84, 287-289.
- Clark, C. W. (1990). *Mathematical Bio-economics. The optimal management of renewable resources*, Wiley Eastern, New York.
- Das, T., Mukherjee, R.N. and Chaudhuri, K.S. (2009). Harvesting of a prey-predator fishery in the presence of toxicity. *Appl. Math. Model*, 33, 2282-2292.
- De Luna, T. J. and Hallam, T. G. (1987). Effects of toxicants on populations; a qualitative approach IV. Resource-consumer toxicant models. *Ecol. Model*, 35, 249-273.
- Hallam, T. G., Clark, C. E., and Lassiter, R. R. (1983). Effects of toxicants on populations, a qualitative approach I. Equilibrium Environmental exposure. *Ecol. Model*, 18, 291 -304.
- Hallam, T. G. and De Luna, T. J. (1984). Effects of toxicants on populations a qualitative approach III. Environmental and food chain pathways. *J. Theo. Biol.*, 109, 411-429.
- Kar, T.K. and Matsuda, H. (2008). A bionomic model of a single - species fishery with marine reserve. *J. Environmental Management*, 86, 171-180.
- Kirk, D. (1970). *Optimal control theory-An introduction* (Prentice Hall).
- Shastri, Y., Diwekar, U. (2006). Sustainable ecosystem management using optimal control theory, 241, 522-532.

Shoemaker, C. (1973a). Optimization of agricultural pest management. III. Results and extensions of a model. *Math. Biosci.*, 18, 1-22.

Shoemaker, C. (1973b). Optimization of agricultural pest management. I. Biological and Mathematical back ground. *Math. Biosci.*, 16,143-175.

Shoemaker, C. (1973c). Optimization of agricultural pest management. II. Formulation of a control model. *Math. Biosci.*, 17, 357-365.

Sole, J., Garcia-Ladona, E., Ruardij, P., Estrada, M. (2005). Modelling allelopathy among marine algae. *Ecol. Model.*, 183, 373-384.

Tillmann, U., John, U. (2002). Toxic effect of alexandrium spp. on heterotrophic dinoflagellates: an allelochemical defence mechanism independent of psp-toxin content. *Mar. Ecol. Prog.*, 230, 47-58.

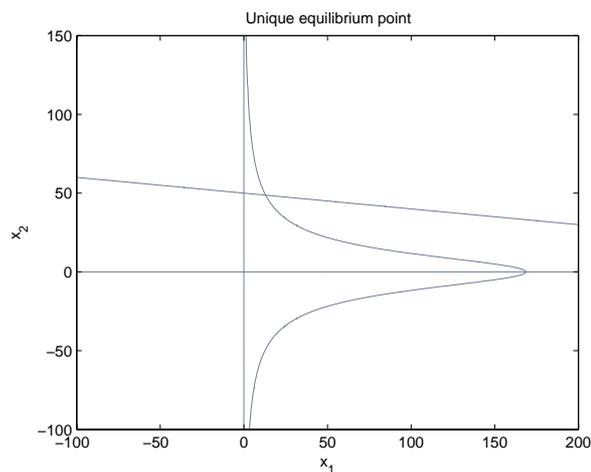


Figure 1. Showing the unique equilibrium point

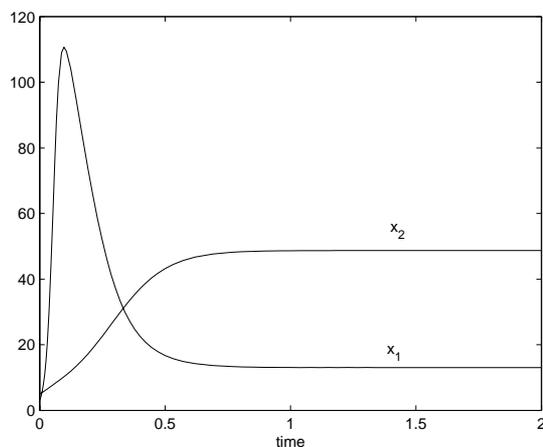


Figure 2. Solution curves

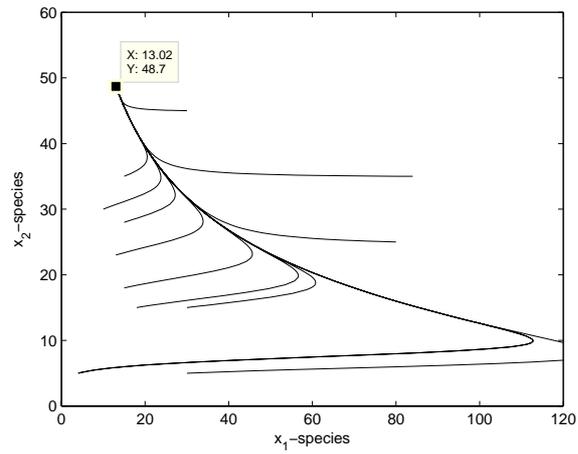


Figure 3. Phase-plane diagram with initial conditions

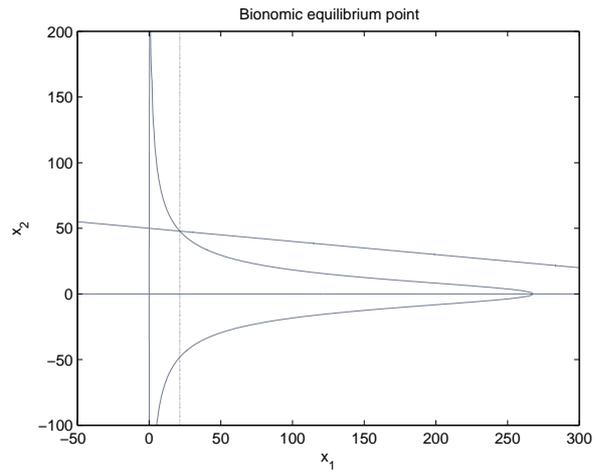


Figure 4. Showing the unique bionomic equilibrium point

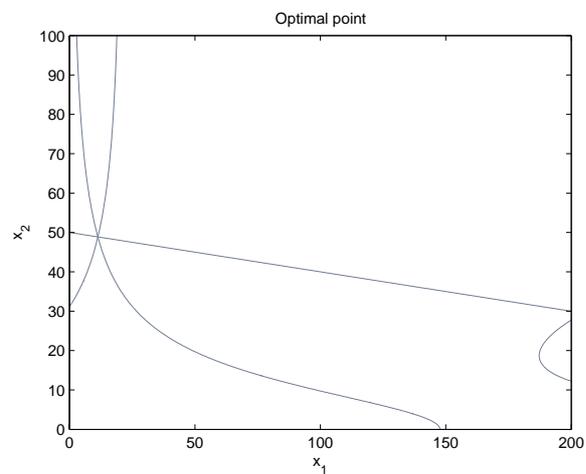


Figure 5. Showing the optimal equilibrium point

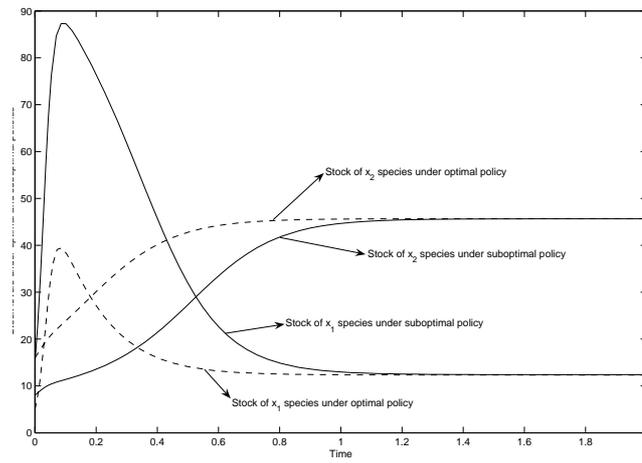


Figure 6. Optimal and suboptimal approach paths