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Strongly Hopfian and Strongly Cohopfian Objects in the Category of Complexes of Left A-Modules

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Abstract

The object of this paper is the study of *strongly hopfian*, *strongly cohopfian*, *quasi-injective*, *quasi-projective*, *Fitting* objects of the category of complexes of A-modules.

In this paper we demonstrate the following results:

a)If C is a strongly hopfian chain complex (respectively strongly cohopfian chain complex) and E a subcomplex which is direct summand then E and C/E are both strongly Hopfian (respectively strongly coHopfian) then C is strongly Hopfian (respectively strongly coHopfian).

b)Given a chain complex C, if C is quasi-injective and strongly-hopfian then C is strongly cohopfian.

c)Given a chain complex C, if C is quasi-projective and strongly-cohopfian then C is strongly hopfian.

In conclusion the main result of this article is the following theorem:

Any *quasi-projective* and *strongly-hopfian* or *quasi-injective* and *strongly-cohofian* chain complex of A-modules is a *Fitting* chain complex.

Keywords: chain complex, strongly-hopfian, strongly-cohopfian, quasi-injective, quasi-projective, fitting chain complex

1. Introduction

In this paper, we will study *strongly hopfian*, *strongly cohopfian*, *quasi-injective*, *quasi-projective* objects of the category of complexes of A-modules denoted by $COMP$.

We also study Fitting objects of the category of complexes of A-modules.

The objects of $COMP$ are chain complexes and the morphisms are maps of chains.

A chain complex (C, d): $\ldots \rightarrow C_{n+1} \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \ldots$ is a sequence (d_n) of homomorphisms of left A-modules verifying $d_n \circ d_{n+1} = 0$, for all $n \in \mathbf{Z}$.

A chain map f of (C, d) into (C', d') is defined by:

$$\begin{tikzcd}
(C,d) : \ldots \arrow[r] \arrow[d, "f"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
(C',d') : \ldots \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

with $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}, \forall n \in \mathbb{Z}$.

Given (C, d) an object of $COMP$ and f an endomorphism of (C, d).

C is said to be hopfian (respectively cohopfian) if any epimorphism (respectively monomorphism) is an isomorphism.

A chain complex (C, d): $\ldots \rightarrow C_{n+1} \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \ldots$ is said to be fully invariant if and only if each $A-$ module C_n is fully invariant, $\forall n \in \mathbb{Z}$.

A chain complex (C, d): $\ldots \rightarrow C_{n+1} \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \ldots$ is said essential if and only if each $A-$ module C_n is essential, $\forall n \in \mathbb{Z}$.

A chain complex (C, d) is said $Fitting$ if for any endomorphism f of (C, d), there exists an integer n such $C = Imf^n \oplus Kerf^n$.

After (C, d) is noted by C.

In this paper we demonstrate the following results:

a) If C is a strongly hopfian chain complex (respectively strongly cohopfian chain complex) and E a subcomplex is direct summand then E and C/E are both strongly Hopfian (respectively strongly coHopfian) then C is strongly Hopfian (respectively strongly coHopfian).

b) If E is a fully invariant chain complex such that E and C/E are strongly Hopfian (respectively strongly coHopfian) then C is strongly Hopfian (respectively strongly coHopfian).

c) Given $C = \oplus C^i$ such C^i is fully invariant, if C^i is a strongly hopfian (respectively strongly cohopfian) chain complex then C is a strongly hopfian (respectively strongly cohopfian) chain complex.

d) Given a chain complex C: $\ldots \rightarrow C_{n+1} \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \ldots$, C is $quasi-injective$, if and only if, C_n is $quasi-injective$, for all $n \in \mathbb{Z}$.

e) A chain complex C: $\ldots \rightarrow C_{n+1} \overset{d_{n+1}}{\rightarrow} C_n \overset{d_n}{\rightarrow} C_{n-1} \overset{d_{n-1}}{\rightarrow} \ldots$, is $quasi-projective$, if and only if, C_n is $quasi-projective$, for all $n \in \mathbb{Z}$.

f) Given a chain complex C, if C is quasi-injective and strongly- hopfian then C is strongly cohopfian.

g) Given a chain complex C, if C is quasi-projective and strongly- cohopfian then C is strongly hopfian.

h) Any *quasi-projective* and *strongly-hopfian* or *quasi-injective* and *strongly-cohofian* chain complex of A-modules is a $Fitting$ chain complex.

In this paper A denotes a not inevitably commutative, unitarian associative ring and M a left unifere module.

2. Definitions and Preliminary Results on the Category $COMP$

Definition 1 Given (C, d) a chain complex of left A modules and f an endomorphism of (C, d) such that:

$$\begin{tikzcd}
(C,d): \ldots \arrow[r] \arrow[d, "f"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
(C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We call $f \circ f$ the chain map compounded of f by itself denoted f^2.

We also define $(f^2)_n = f_n \circ f_n$, for all $n \in \mathbb{Z}$:

$$\begin{tikzcd}
C \ldots \arrow[r] \arrow[d, "f"] \arrow[dd, bend right=60, "f^2"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
C \ldots \arrow[r] \arrow[d, "f"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
C \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r, "d_{n-1}"] & \cdots
\end{tikzcd}$$

We also define f^k such $f_n^k = f_n \circ f_n \circ \ldots \circ f_n$, with k factors, for all $n \in \mathbb{Z}$.

Proposition 1 *Considering (C, d) a chain complex of $A-$ modules and f an endomorphism of (C, d):*

$$\begin{tikzcd}
(C,d): \ldots \arrow[r] \arrow[d, "f"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
(C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Given

$$\begin{array}{rccc} \Delta^k_{n+1}: & Kerf^k_{n+1} & \to & Kerf^k_n \\ & x & \mapsto & d_{n+1}(x) \end{array}$$

where $f^k_{n+1} = f_{n+1} \circ f_{n+1} \circ \ldots \circ f_{n+1}$ with k factors, then Δ^k_{n+1} is the induced morphism by Δ^{k+1}_{n+1}.

Proof. Considering

$$\begin{array}{rccc} \Delta^k_{n+1}: & Kerf^k_{n+1} & \to & Kerf^k_n \\ & x & \mapsto & d_{n+1}(x) \end{array}$$

and

$$\begin{array}{rccc} \Delta^{k+1}_{n+1}: & Kerf^{k+1}_{n+1} & \to & Kerf^{k+1}_n \\ & x & \mapsto & d_{n+1}(x). \end{array}$$

We obtain $Kerf^k_{n+1} \subseteq Kerf^{k+1}_{n+1}$ et $Kerf^k_n \subseteq Kerf^{k+1}_n$ therefore Δ^k_{n+1} is the induced by Δ^{k+1}_{n+1}. □

Proposition 2 *Let be (C, d) a chain complex of A-modules and f an endomorphism of (C, d) such that:*

$$\begin{tikzcd}
(C,d): \ldots \arrow[r] \arrow[d, "f"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\
(C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Given

$$\begin{array}{rccc} \delta^k_{n+1}: & Imf^k_{n+1} & \to & Imf^k_n \\ & x & \mapsto & d_{n+1}(x) \end{array}$$

Then δ^{k+1}_{n+1} is the induced morphism by δ^k_{n+1}.

Proof. Given

$$\begin{array}{rccc} \delta^k_{n+1}: & Imf^k_{n+1} & \to & Imf^k_n \\ & x & \mapsto & d_{n+1}(x) \end{array}$$

and

$$\begin{array}{rccc} \delta^k_{n+1}: & Imf^{k+1}_{n+1} & \to & Imf^{k+1}_n \\ & x & \mapsto & d_{n+1}(x) \end{array}$$

then $Imf^{k+1}_{n+1} \subseteq Imf^k_{n+1}$ et $Imf^{k+1}_n \subseteq Imf^k_n$ d'o δ^{k+1}_{n+1} is the morphism induced by δ^k_{n+1}. □

Definition 2 Let C: $\ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \ldots$ a chain complex and E: $\ldots \to E_{n+1} \overset{u_{n+1}}{\to} E_n \overset{u_n}{\to} E_{n-1} \overset{u_{n-1}}{\to} \ldots$ a subcomplex of C. Then E is direct summand, if and only if, E_n is direct summand.

Definition 3 Let N a sub-module of a $A-$ module M. Then N is said to be fully invariant, if for any endomorphism f of M we have $f(N) \subseteq N$.

Definition 4 Given $C : \ldots C_{n+1} \to C_n \to C_{n-1} \to \ldots$ a chain complexes of $A-$ modules and E: $\ldots E_{n+1} \to E_n \to E_{n-1} \to \ldots$ a subcomplex of C. Then E is said to be subcomplex fully invariant in C, if for all $n \in \mathbb{Z}$, E_n is fully invariant.

3. Strongly Hopfian Objects and Strongly CoHopfian Objects of the Category of Complexes of Left A-Modules

3.1 Strongly Hopfian and Strongly CoHopfian

Definition 5 Let be f an endomorphism of chain complex (C, d):

$$\begin{tikzcd} (C,d): \ldots \arrow[r] \arrow[d, "f"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\ (C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Then:

a) the chain complex

$$(Kerf^k) : \ldots \to\ Kerf^k_{n+1} \overset{\Delta^k_{n+1}}{\to} Kerf^k_n \overset{\Delta^k_n}{\to} Kerf^k_{n-1} \overset{\Delta^k_{n-1}}{\to} \ldots$$

stabilizes, if and only if, it exists $k_0 \in \mathbb{N}^\star$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$, for all $s \in \mathbb{N}$.

b) the chain complex

$$(Imf^k) : \ldots \to\ Imf^k_{n+1} \overset{\delta^k_{n+1}}{\to} Imf^k_n \overset{\delta^k_n}{\to} Imf^k_{n-1} \overset{\delta^k_{n-1}}{\to} \ldots$$

stabilizes if it exists $k_0 \in \mathbb{N}^\star$ such that $(Imf^{k_0}) = Imf^{k_0+s}$, for all $s \in \mathbb{N}$.

Proposition 3 *Given f an endomorphism of a chain complex (C, d):*

$$\begin{tikzcd} (C,d): \ldots \arrow[r] \arrow[d, "f"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\ (C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Then $(Kerf^k)$ stabilizes if and only if $(Kerf^k_n)$ stabilizes for all $n \in \mathbb{Z}$.

Proof. If $(Kerf^k)$ stabilizes then it exists $k_0 \in \mathbb{N}^\star$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$, for all $s \in \mathbb{N}$ hence $Kerf^{k_0}_n = Kerf^{k_0+s}_n$ for all $n \in \mathbb{Z}$. This implies that que $(Kerf^k_n)$ stabilizes for all $n \in \mathbb{Z}$.

Reciprocally suppose that $(Kerf^k_n)$ stabilizes for all $n \in \mathbb{Z}$ then it exists $k_0 \in \mathbb{N}^\star$ such that for all $s \in \mathbb{N}$ then $Kerf^{k_0}_n = Kerf^{k_0+s}_n$ for all $n \in \mathbb{Z}$.

So $(Kerf^k)$ stabilizes for all positive integer k. □

Proposition 4 *Given f an endomorphism of a chain complex (C, d):*

$$\begin{tikzcd} (C,d): \ldots \arrow[r] \arrow[d, "f"'] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \\ (C,d): \ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Then (Imf^k) stabilizes if and only if (Imf^k_n) stabilizes for all $n \in \mathbb{Z}$.

Proof. Suppose (Imf^k) stabilizes then it exists $k_0 \in \mathbb{N}^\star$ such that $(Imf^{k_0}) = (Imf^{k_0+s})$, for all $s \in \mathbb{N}$ hence $Imf^{k_0}_n = Imf^{k_0+s}_n$ for all $n \in \mathbb{Z}$. Hence (Imf^k_n) stabilizes for all $n \in \mathbb{Z}$.

Reciprocally suppose that (Imf^k_n) stabilizes for all $n \in \mathbb{Z}$ then there is $k_0 \in \mathbb{N}^\star$ such that for all $s \in \mathbb{N}$ so $Imf^{k_0}_n = Imf^{k_0+s}_n$ for all $n \in \mathbb{Z}$. That proove (Imf^k) stabilizes for all k. □

Definition 6 An $A-$ module M is said to be strongly hopfian (respectively strongly cohopfian) if for any endomorphism f of M the sequence $(Kerf^n)$ stabilizes: $kerf^2 \subseteq kerf^3 \ldots \subseteq Kerf^n$ (respectively $Imf \supseteq Imf^2 \ldots \supseteq Imf^n \ldots$).

Definition 7 A chain complex C of A-modules is said to be strongly hopfian(respectively strongly cohopfian), if for any endomorphism f of (C, d), $(Kerf^k)$ (respectively (Imf^k)) stabilizes.

Proposition 5 *A chain complex C is strongly hopfian if it exists $k \in \mathbb{N}$ such that $Kerf_n^k \cap Imf_n^k = 0$,for all $n \in \mathbb{Z}$.*

Proof. Using the Proposition 2.5 (see Hmaimou, 2007) and let be $k \in \mathbb{N}$ such that $Kerf_n^k \cap Imf_n^k = 0$ then $(Kerf_n^k)$ stabilizes for all $n \in \mathbb{Z}$. That prove $(Kerf^k)$ stabilizes, so C is a strongly hopfian chain complex. □

Proposition 6 *A chain complex C:*

$$\begin{tikzcd} (C,d):\ldots \arrow[r] \arrow[d, "f"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ (C,d):\ldots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

is strongly cohopfian, if it exists a positive integer k such that:

$$Imf_n^k + Kerf_n^k = C_n, \text{for all } n \in \mathbb{Z}.$$

Proof. Using the Proposition 2.6 (see Hmaimou, 2007), we can suppose it exists a positive integer k such that: $Imf_n^k + Kerf_n^k = C_n$, for all $n \in \mathbb{Z}$ then (Imf_n^k) stabilizes for all $n \in \mathbb{Z}$ hence (Imf^k) stabilizes so C is a strongly cohopfian complex. □

Theorem 1 *Considering C a chain complex and E a subcomplex of C. If C is strongly hopfian (respectively is strongly cohopfian) and E is direct summand then E et C/E both are strongly hopfian (respectively strongly cohopfian).*

Proof. Let us demonstrate at first that E is strongly hopfian (respectively strongly cohopfian).

Suppose $C = E \oplus K$ and let be g a chain map of E in itself which can be extended to C such as $f = g \oplus 0$, with 0 the zero morphism of K.

Given C is strongly hopfian (respectively strongly cohopfian), then $(Kerf^k)$ (respectively (Imf^k)) stabilizes then it exists $k_0 \in \mathbb{N}^\star$ such that $(Kerf^{k_0}) = (Kerf^{k_0+s})$ (respectively $(Imf^{k_0}) = (Imf^{k_0+s})$), for all $s \in \mathbb{N}$, then $Kerf_n^{k_0} = Kerf_n^{k_0+s}$ (respectively $Imf_n^{k_0} = Imf_n^{k_0+s}$ pour tout $n \in \mathbb{Z}$), therefore C_n is strongly hopfian (respectively strongly cohopfian) and E_n is direct summand.

Hence C_n and C_n/E_n for all $n \in \mathbb{Z}$ are both strongly hopfian (respectively strongly cohopfian).

That prove E and C/E are strongly hopfian (respectively strongly cohopfian). □

Theorem 2 *Considering C a chain complex and E a subcomplexe of C, if E is fully invariant with E and C/E both strongly hopfian (respectively strongly cohopfian) then C is strongly hopfian (respectively strongly cohopfian).*

Proof. Let be (C,d) a chain complex of A-modules and f an endomorphism of (C,d) verifying:

$$\begin{tikzcd} (C,d):\ldots \arrow[r] \arrow[d, "f"] & C_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "f_{m+1}"] & C_m \arrow[r, "d_m"] \arrow[d, "f_m"] & C_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & \cdots \\ (C,d):\ldots \arrow[r] & C_{m+1} \arrow[r, "d_{m+1}"] & C_m \arrow[r, "d_m"] & C_{m-1} \arrow[r] & \cdots \end{tikzcd}$$

Given E is fully invariant in C then f induces a chain map h of E in itself such that:

$$\begin{tikzcd} (E,d):\ldots \arrow[r] \arrow[d, "h"] & E_{m+1} \arrow[r, "d_{m+1}"] \arrow[d, "h_{m+1}"] & E_m \arrow[r, "d_m"] \arrow[d, "h_m"] & E_{m-1} \arrow[r] \arrow[d, "h_{m-1}"] & \cdots \\ (E,d):\ldots \arrow[r] & E_{m+1} \arrow[r, "d_{m+1}"] & E_m \arrow[r, "d_m"] & E_{m-1} \arrow[r] & \cdots \end{tikzcd}$$

f induces also a chain map g of C/E in itself such that:

$$\begin{tikzcd} (C/E,u):\ldots \arrow[r] \arrow[d, "g"] & C_{m+1}/E_{m+1} \arrow[r, "u_{m+1}"] \arrow[d, "g_{m+1}"] & C_m/E_m \arrow[r, "u_m"] \arrow[d, "g_m"] & C_{m-1}/E_{m-1} \arrow[r] \arrow[d, "g_{m-1}"] & \cdots . \\ (C/E,u):\ldots \arrow[r] & C_{m+1}/E_{m+1} \arrow[r, "u_{m+1}"] & C_m/E_m \arrow[r, "u_m"] & C_{m-1}/E_{m-1} \arrow[r] & \cdots \end{tikzcd}$$

Given E and C/E are both strongly cohopfian, hence $Imh^n = Imh^{n+k}$ and $Img^n = Img^{n+k}$, therefore for all $m \in \mathbb{Z}$, we have $Imh^n_m = Imh^{n+k}_m$ and $Img^n_m = Img^{n+k}_m$.

If $p = 2n$, for $x \in C_m$, we obtain $g^n_m(x + E_m) = g^{n+1}_m(y + E_m)$ for some $y \in E_m$. Then $t = f^n_m(x) - f^n_m(y) \in E_m$, hence $f^n_m(t) = f^{p+1}_m(z)$ for some $z \in E_m$, therefore $f^p_m(x) = f^{p+1}_m(y + z)$, so C is strongly cohopfian.

Suppose that E and C/E are both hopfian, then $Kerh^n = Kerh^{n+k}$ and $Kerg^n = Kerg^{n+k}$, hence for all $m \in \mathbb{Z}$, we have $Kerh^n_m = Kerh^{n+k}_m$ and $Kerg^n_m = Kerg^{n+k}_m$.

Let be $x \in Kerf^{2n}_m$, then $g^{2n+1}(x + E_m) = 0$, so $y = f^n_m(x) \in E_m$ and $f^{n+1}_m(y) = 0$. Therefore $y \in Kerh^{n+1}_m = Kerh^n_m$, then $x \in Kerf^{2n}_m$, so C is strongly hopfian. □

Corollary 1 *Let be $C = \oplus C^i$ where C^i is a subcomplex fully invariant of C for all $i \in I$. If C^i is strongly hopfian (respectively strongly cohopfian), then C is strongly hopfian (respectively strongly cohopfian).*

Proof. Let be f an endomorphism of chain complex C. Then it exists a unique family (f^i) such that $i \in I$ where $f^i = f\ |_{C^i}$ and $m = \sum n_i$. f^i is denoted by $[f^i]$. C^i is strongly hopfian then $Ker[f^i]^{n_i} = Ker[f^i]^{n_i+1}$, therefore $\oplus Ker[f^i]^m = \oplus Ker[f^i]^{n_i+1}$. So $Ker(\oplus[f^i])^m = Ker(\oplus[f^i])^{m+1}$, then C is strongly hopfian.

Let be f an endomorphism of C. Then it exists a unique family (f^i) where $i \in I$ $f^i = f\ |_{C^i}$ and $m = \sum n_i$. C^i is strongly cohopfian, then $Im[f^i]^{n_i} = Im[f^i]^{n_i+1}$, therefore $\oplus Im[f^i]^m = \oplus Im[f^i]^{n_i+1}$. So $Im(\oplus[f^i])^m = Im(\oplus[f^i])^{m+1}$, then C is strongly cohopfian. □

3.2 Quasi-Injective Chain Complex

Definition 8 An A-module M is *quasi-injective*, if for any monomorphism g of A-module N into M and for any morphism f of N into M, there exists an endomorphism h of M such: $f = h \circ g$.

$$\begin{tikzcd}
M & \\
N \arrow[u, "f"] \arrow[r, "g"'] & M \arrow[ul, dotted, "h"']
\end{tikzcd}$$

Definition 9 Let us considering two chains complexes of A-modules: C and E. C is said to be quasi-injective, if for any monomorphism g of E into C and for any chain map f of E into C, there exists a chain map h of C into C verifying $f = h \circ g$.

Theorem 3 *Given C: $\ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \to \ldots$ a chain complex of A-modules. C is quasi-injective, if and only if for all $n \in \mathbb{Z}$, C_n a quasi-injective A-module.*

Proof. Suppose that R is quasi-injective chain complex and f: $M \to N$ a monomorphism of $A-$ modules and ϕ_n: $M \to R_n$ a morphism of $A-$ modules.

Considering S and R two chains complexes of $A-$ modules andα a chain of S into R such:

$$\begin{tikzcd}
S : \ldots \arrow[r] \arrow[d, "\alpha"'] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\alpha_{n+1}"'] & S_n \arrow[r, "u_n"] \arrow[d, "\alpha_n"'] & S_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "v_{n+1}"] & R_n \arrow[r, "v_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where for all $n \in \mathbb{Z}$. $S_n = M$ and $u_n = Id_M$ with $R_n = N$ and $v_n = Id_N$ α_n is a monomorphism of $A-$ modules, for all $n \in \mathbb{Z}$.

Let ϕ a chain map of S into R such:

$$\begin{tikzcd}
S : \ldots \arrow[r] \arrow[d, "\phi"'] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\phi_{n+1}"'] & S_n \arrow[r, "u_n"] \arrow[d, "\phi_n"'] & S_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where for any morphismes ϕ_k: $M \to R_k$ of A-modules.

Given that R is quasi-injective then it exists ψ a chain map:

$$\begin{tikzcd}
R : \ldots \arrow[r] \arrow[d, "\psi"'] & S_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\psi_{n+1}"'] & S_n \arrow[r, "v_n"] \arrow[d, "\psi_n"'] & S_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such: $\psi \circ \alpha = \phi$ hence $\psi_n \circ f = \phi_n$. Then it exists ψ_n: $R_n \to R_n$ such $\psi_n \circ f = \phi_n$ then R_n is quasi-injective.

Reciprocally suppose that for all $n \in \mathbb{Z}$, R_n is an A-quasi-injective module and let us prove that R quasi-injective chain complex.

Let be γ a monomorphism of chains complexes:

$$\begin{tikzcd}
S : \ldots \arrow[r] \arrow[d, "\gamma"'] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & S_n \arrow[r, "u_n"] \arrow[d, "\gamma_n"'] & S_{n-1} \arrow[r] \arrow[d, "\gamma_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Considering β a chain map of complexes such:

$$\begin{tikzcd}
R : \ldots \arrow[r] \arrow[d, "\beta"'] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\beta_{n+1}"'] & S_n \arrow[r, "u_n"] \arrow[d, "\beta_n"'] & S_{n-1} \arrow[r] \arrow[d, "\beta_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then given for all $n \in \mathbb{Z}$, R_n is quasi- injective and γ_n is a monomorphism of A-modules hence it exists λ_n: $R_n \to R_n$ verifying $\beta_n = \lambda_n \circ \gamma_n$.

Let be λ such:

$$\begin{tikzcd}
R : \ldots \arrow[r] \arrow[d, "\lambda"'] & R_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\lambda_{n+1}"'] & R_n \arrow[r, "v_n"] \arrow[d, "\lambda_n"'] & R_{n-1} \arrow[r] \arrow[d, "\lambda_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Let us demonstrate λ is a chain map:

$$\begin{tikzcd}
S \ldots \arrow[r] \arrow[d, "\gamma"] \arrow[dd, bend right=60, "\beta"'] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\gamma_{n+1}"] & S_n \arrow[r, "u_n"] \arrow[d, "\gamma_n"] & S_{n-1} \arrow[r, "u_{n-1}"] \arrow[d, "\gamma_{n-1}"] & \cdots \\
R \ldots \arrow[r] \arrow[d, "\lambda"] & R_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\lambda_{n+1}"] & R_n \arrow[r, "v_n"] \arrow[d, "\lambda_n"] & R_{n-1} \arrow[r, "v_{n-1}"] \arrow[d, "\lambda_{n-1}"] & \cdots \\
R \ldots \arrow[r] & R_{n+1} \arrow[r, "w_{n+1}"] & R_n \arrow[r, "w_n"] & R_{n-1} \arrow[r, "w_{n-1}"] & \cdots
\end{tikzcd}$$

It is enough that: $w_{n+1} \circ \lambda_{n+1} = \lambda_n \circ v_{n+1}$.

We have: $w_{n+1} \circ \beta_{n+1} = \beta_n u_{n+1}$ but β is chain map, because $\beta_n = \lambda_n \circ \gamma_n$. Then $w_{n+1} \circ (\lambda_{n+1} \circ \gamma_{n+1}) = (\lambda_n \circ \gamma_n) \circ u_{n+1}$. Therefore $(w_{n+1} \circ \lambda_{n+1}) \circ \gamma_{n+1} = \lambda_n \circ (\gamma_n \circ u_{n+1})$. So $(w_{n+1}) \circ \lambda_{n+1}) \circ \gamma_{n+1} = \lambda_n \circ (v_{n+1} \circ \gamma_{n+1})$. Finally $(w_{n+1} \circ \lambda_{n+1}) \circ \gamma_{n+1} = (\lambda_n \circ v_{n+1}) \circ \gamma_{n+1}$. But γ_{n+1} is a monomorphism of A-modules then $w_{n+1} \circ \lambda_{n+1} = \lambda_n \circ v_{n+1}$, for all $n \in \mathbb{Z}$. so λ is achain map.

Let us verify that: $\lambda \circ \gamma = \beta$. We know that for all $n \in \mathbb{Z}$, $\beta_n = \lambda_n \circ \gamma_n$ with $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ and $\lambda = (\lambda_n)$ so $\lambda \circ \gamma = \beta$. That prove R is a quasi-injective complex. □

Theorem 4 *Given C a chain complex of A-modules. If C is quasi-injective and strongly hopfian, then C is strongly cohopfian.*

Proof. Suppose that C is *quasi-injective* and *strongly hopfian*, then C_n is quasi-injective using Theorem 3 and (Imf_n^k) stabilizing this implies that C_n is quasi-injective and $(kerf^k)$ stabilizes, so C is strongly cohopfian. □

3.3 Quasi Projective Chain Complex

Definition 10 An A-module P is said to be quasi-projective if for any A-module N and any epimorphism π: $P \to N$ and any homomorphism ϕ: $P \to N$, here exists an endomorphism ψ: $P \to P$ such $\pi \circ \psi = \phi$ illustrated by the following commutative diagramm:

$$\begin{tikzcd}
 & P \arrow[dl, "\psi"'] \arrow[d, "\phi"] & . \\
P \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

Definition 11 Given C and E two chains complexes of A-modules. C is said to be quasi-projective chain complex if for any epimorphism: $C \to E$ and for any morphism f: $C \to E$, there exisrsa chain map h: $C \to C$ verifying: $f = g \circ h$, illustrated by the following commutative diagramm:

$$\begin{tikzcd}
 & C \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\
C \arrow[r, "g"] & E \arrow[r] & 0
\end{tikzcd}$$

Theorem 5 *Let C: $\ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \to \ldots$ a chain complex of A-modules. C is quasi-projective if and only if for all $n \in \mathbb{Z}$, C_n is a quasi-projective module.*

Proof. Suppose that E is quasi-projective.

Considering f: $N \to M$ an epimorphism of A-modules and ϕ_n: $E_n \to M$ a morphisme of $A-$ modules.

Let S and E two chains complexes and α a chain map of E into S such:

$$\begin{tikzcd}
E : \ldots \arrow[r] \arrow[d, "\alpha"'] & E_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\alpha_{n+1}"'] & E_n \arrow[r, "v_n"] \arrow[d, "\alpha_n"'] & E_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"'] & \cdots \\
S : \ldots \arrow[r] & S_{n+1} \arrow[r, "u_{n+1}"] & S_n \arrow[r, "u_n"] & S_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where $E_n = M$ et $v_n = Id_M$. $S_n = N$ and $u_n = Id_N$.

Given α_n an epimorphism of $A-$ modules. Let ϕ a chain of E into S verifying:

$$\begin{tikzcd}
E : \ldots \arrow[r] \arrow[d, "\phi"'] & E_{n+1} \arrow[r, "w_{n+1}"] \arrow[d, "\phi_{n+1}"'] & E_n \arrow[r, "w_n"] \arrow[d, "\phi_n"'] & E_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \\
S : \ldots \arrow[r] & S_{n+1} \arrow[r, "u_{n+1}"] & S_n \arrow[r, "u_n"] & S_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

For any morphisms of A-modules ϕ_k: $E_k \to M$.

Given E is quasi-projective then it exists a chain map ψ such:

$$\begin{tikzcd}
E : \ldots \arrow[r] \arrow[d, "\psi"'] & E_{n+1} \arrow[r, "w_{n+1}"] \arrow[d, "\psi_{n+1}"'] & E_n \arrow[r, "w_n"] \arrow[d, "\psi_n"'] & E_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}"'] & \cdots \\
R : \ldots \arrow[r] & R_{n+1} \arrow[r, "v_{n+1}"] & R_n \arrow[r, "v_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where: $\alpha \circ \psi = \phi$ so for all $n \in \mathbb{Z}$, $\alpha_n \circ \psi_n = \phi_n$. Then it exists $\psi_n : E_n \to R_n$ such $f \circ \psi_n = \phi_n$. So E_n is quasi-projective.

Reciprocally suppose that for all $n \in \mathbb{Z}$, E_n is quasi-projective $A-$ and let us demonstrate that E is quasi-projective chain complex.

Considering γ an epimorphism such:

$$\begin{tikzcd}
E : \ldots \arrow[r] \arrow[d, "\gamma"'] & E_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & E_n \arrow[r, "v_n"] \arrow[d, "\gamma_n"'] & E_{n-1} \arrow[r] \arrow[d, "\gamma_{n-1}"'] & \cdots \\
S : \ldots \arrow[r] & S_{n+1} \arrow[r, "u_{n+1}"] & S_n \arrow[r, "u_n"] & S_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Let β a chain map complex verifying:

$$\begin{tikzcd}
E:\ldots \arrow[r] \arrow[d, "\beta"'] & E_{n+1} \arrow[r, "w_{n+1}"] \arrow[d, "\beta_{n+1}"'] & E_n \arrow[r, "w_n"] \arrow[d, "\beta_n"'] & E_{n-1} \arrow[r] \arrow[d, "\beta_{n-1}"'] & \cdots \\
S:\ldots \arrow[r] & S_{n+1} \arrow[r, "u_{n+1}"] & S_n \arrow[r, "u_n"] & S_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then given for all $n \in \mathbb{Z}$, E_n is projective and γ_n is an epimorphism of A-modules so it exists λ_n: $E_n \to R_n$ verifying $\gamma_n \circ \lambda_n$.

Considering λ such:

$$\begin{tikzcd}
E:\ldots \arrow[r] \arrow[d, "\lambda"'] & E_{n+1} \arrow[r, "w_{n+1}"] \arrow[d, "\lambda_{n+1}"'] & E_n \arrow[r, "w_n"] \arrow[d, "\lambda_n"'] & E_{n-1} \arrow[r] \arrow[d, "\lambda_{n-1}"'] & \cdots \\
R:\ldots \arrow[r] & R_{n+1} \arrow[r, "v_{n+1}"] & v_n \arrow[r, "v_n"] & R_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Let us demonstrate that λ is a chain map

$$\begin{tikzcd}
E\ldots \arrow[r] \arrow[d, "\gamma"] \arrow[dd, bend right=60, "\lambda"'] & E_{n+1} \arrow[r, "v_{n+1}"] \arrow[d, "\gamma_{n+1}"] & E_n \arrow[r, "v_n"] \arrow[d, "\gamma_n"] & E_{n-1} \arrow[r, "v_{n-1}"] \arrow[d, "\gamma_{n-1}"] & \cdots \\
S\ldots \arrow[r] \arrow[d, "\beta"] & S_{n+1} \arrow[r, "u_{n+1}"] \arrow[d, "\beta_{n+1}"] & S_n \arrow[r, "u_n"] \arrow[d, "\beta_n"] & S_{n-1} \arrow[r, "u_{n-1}"] \arrow[d, "\beta_{n-1}"] & \cdots \\
E\ldots \arrow[r] & E_{n+1} \arrow[r, "w_{n+1}"] & E_n \arrow[r, "w_n"] & E_{n-1} \arrow[r, "w_{n-1}"] & \cdots
\end{tikzcd}$$

It is enough that: $\lambda_{n+1} \circ w_{n+1} = v_{n+1} \circ \lambda_n$. But $\beta_{n+1} \circ w_{n+1} = u_{n+1} \circ \beta_n$, because β is a chain map.

Given $\gamma_n \circ \lambda_n = \beta_n$, then: $\gamma_{n+1} \circ \lambda_{n+1} \circ w_{n+1} = u_{n+1} \circ (\gamma_n \circ \lambda_n)$. So $(\gamma_{n+1} \circ \lambda_{n+1}) \circ w_{n+1} = u_{n+1} \circ (\gamma_n \circ \lambda_n)$. Then $\gamma_{n+1} \circ (\lambda_{n+1} \circ w_{n+1}) = \gamma_{n+1} \circ (v_{n+1} \circ \lambda_n)$. But γ_{n+1} is an epimorphism hence $\lambda_{n+1} \circ w_{n+1} = v_{n+1} \circ \lambda_n$ What justifies that λ is a chain complex.

Let us verify that: $\gamma \circ \lambda = \beta$. We know for all $n \in \mathbb{Z}, \beta_n = \gamma_n \circ \lambda_n$ so $\beta = \gamma \circ \lambda$. What justifies E is a quasi-projective chain complex. □

Theorem 6 *Given C a chain complex of A-modules. If C is quasi-projective and strongly cohopfian then C is strongly hopfian.*

Proof. Suppose that C is quasi-projective and *strongly cohopfian*, then C_n is quasi-projective using Theorem 4 and $(kerf_n^k)$ stabilizing this implies that C_n is quasi-projective and $(kerf^k)$ stabilizes so C is strongly hopfian. □

3.4 Fitting Chains Complexes

Definition 12 An A-module M is said to be $FITTING$ module if for any endomorphism f of M, there exists a positive integer $n \geq 1$ such: $M = Kerf^n \oplus Imf^n$.

Definition 13 A chain complex of A-modules is said to be $FITTING$ chain complex if for any endomorphism f of C, it exists $n \geq 1$ such $C = Kerf^n \oplus Imf^n$.

Theorem 7 *Considering C a chain complex of A-modules such C: $\ldots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} \ldots$. C is a $FITTING$ chain complex if and only if for all $n \in \mathbb{Z}$, C_n is A-$FITTING$ module.*

Proof. Suppose that C is a $FITTING$ chain complex of A-modules then it exists an positive integer k such $C = Kerf^k \oplus Imf^k$ donc $Kerf^k \sqcap Imf^k = 0$ and $C = Kerf^k + Imf^k$ hence for all $n \in \mathbb{Z}$, $C_n = Kerf_n^k + Imf_n^k$ so C_n is strongly hopfian and strongly cohpfian then C_n is a $FITTING$ A-module. Reciprocally suppose that C_n is an A-$FITTING$ module then $(Kerf_n^k)$ and (Imf_n^k) stabilizes so $(Kerf^k)$ stabilizes and (Imf^k) also. Which prove that C is a $FITTING$ chain complex. □

Theorem 8 *Any quasi-projective and strongly-hopfian or quasi-injective and strongly-cohofian chain complex of A-modules is a Fitting chain complex.*

Proof. Suppose that C is *quasi-projective* and *cohopfian* then using the previous theorem we can say C is *hopfian*, so C is *cohopfian* and *hopfian*, then is a $FITTING$ chain complex.

Suppose that C is *quasi-injective* and *hopfian* then using the previous theorem we can say C is *cohopfian*, so C is *cohopfian* and *hopfian*, then is a *FITTING* chain complex. □

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