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Further Promotion of Cauchy Inequality

Fengqing Sun (Corresponding author)
Department of Computer Science, Dezhou Vocational and Technical College
Dezhou 253034, Shandong, China
E-mail: sunfengqing1963@126.com

Abstract

Cauchy inequality is a very important inequality in mathematics and physics, especially has wild application in resolving the proof of inequalities, which has significance for researching inequalities in mathematics. This paper promotes the original Cauchy inequality. The original n-dimension and 2-order Cauchy inequality is promoted to a n-dimension and m-order Cauchy inequality and the rigorous proof is also given. Simultaneously, this paper summarizes several corollaries for this n-dimension and m-order Cauchy inequality, and gives the corresponding proof.

Keywords: Cauchy inequality(n-dimension and 2-order Cauchy inequality), Promotion of Cauchy inequality(n-dimension and m-order Cauchy inequality), Mean inequality, Important corollary

1. Cauchy inequality(n-dimension and 2-order Cauchy inequality)

Suppose $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ are two real series, then

$$(\sum_{i=1}^{n} a_i b_i)^2 \le (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2),$$

and equal sign holds if and only if $a_i = 0$ or $b_i = 0$ or there exists constant k satisfying $a_i = kb_i (i = 1, 2, \cdots, n)$.

2. Promotion of Cauchy inequality(n-dimension and m-order Cauchy inequality)

Suppose $a_j = (a_{j1}, a_{j2}, \cdots, a_{jn})(j = 1, 2, \cdots, m)$ are m real series(when m is odd, $a_{ji} \ge 0(j = 1, 2, \cdots, m; i = 1, 2, \cdots, n)), m \ge 2$, then

$$(\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji})^m \le \prod_{j=1}^{m} (\sum_{i=1}^{n} a_{ji}^m),$$

and equal sign holds if and only if $a_{1i} = 0$ or $a_{2i} = 0 \cdots$ or $a_{mi} = 0$ or existing m constants $k_1, k_2, \cdots, k_m$ satisfying $k_1 a_{1i} = k_2 a_{2i} = \cdots = k_m a_{mi} (i = 1, 2, \cdots, n)$.

Proof. It is divided into even and odd situations:

(1) When m is even:

(1.1) When $a_{1i} = 0$ or $a_{2i} = 0 \cdots$ or $a_{mi} = 0, (i = 1, 2, \cdots, n)$, the inequality holds obviously and the equal sign holds.

(1.2) When $\sum\limits_{i=1}^{n} a_{ji}^m \ne 0, (j = 1, 2, \cdots, m)$, denoted $\sum\limits_{i=1}^{n} a_{ji}^m = A_j (j = 1, 2, \cdots, m)$, then the original inequality turns into $(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji})^m \le \prod\limits_{j=1}^{m} A_j$, i.e. $|\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji}| \le \sqrt[m]{\prod\limits_{j=1}^{m} A_j}$. So we just prove this inequality: $\frac{|\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji}|}{\sqrt[m]{\prod_{j=1}^{m} A_j}} \le 1$.

$$\frac{|\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji}|}{\sqrt[m]{\prod\limits_{j=1}^{m} A_j}} \le \frac{\sum\limits_{i=1}^{n} |\prod\limits_{j=1}^{m} a_{ji}|}{\sqrt[m]{\prod\limits_{j=1}^{m} A_j}} = \sum_{i=1}^{n} \frac{|\prod\limits_{j=1}^{m} a_{ji}|}{\sqrt[m]{\prod\limits_{j=1}^{m} A_j}} = \sum_{i=1}^{n} \sqrt[m]{\prod_{j=1}^{m} \frac{a_{ji}^m}{A_j}} \le \sum_{i=1}^{n} \frac{\sum\limits_{j=1}^{m} \frac{a_{ji}^m}{A_j}}{m} = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{a_{ji}^m}{A_j}$$

$$= \frac{1}{m} \sum_{j=1}^{m} \frac{\sum\limits_{i=1}^{n} a_{ji}^m}{A_j} = \frac{1}{m} \sum_{j=1}^{m} \frac{A_j}{A_j} = 1.$$

So $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji})^m \leq \prod\limits_{j=1}^{m}(\sum\limits_{i=1}^{n}a_{ji}^m)$.

From the necessary and sufficient condition for the equal sign of absolute value inequality and mean inequality, we can see the equal sign holds if and only if $|\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji}| = \sum\limits_{i=1}^{n}|\prod\limits_{j=1}^{m}a_{ji}|$ and $\frac{a_{1i}^m}{A_1} = \frac{a_{2i}^m}{A_2} = \cdots = \frac{a_{mi}^m}{A_m}$, i.e. $k_1a_{1i} = k_2a_{2i} = \cdots = k_ma_{mi}(i = 1,2,\cdots,n)$, where $k_j = \pm\sqrt[m]{\frac{1}{A_j}}(j = 1,2,\cdots,m)$.

(2) When m is odd:

(2.1) When $a_{1i} = 0$ or $a_{2i} = 0\cdots$ or $a_{mi} = 0, (i = 1,2,\cdots,n)$, the inequality holds obviously and the equal sign holds.

(2.2) When $\sum\limits_{i=1}^{n}a_{ji}^m \neq 0$ (i.e. $\sum\limits_{i=1}^{n}a_{ji}^m > 0), (j = 1,2,\cdots,m)$, denoted $\sum\limits_{i=1}^{n}a_{ji}^m = A_j, (j = 1,2,\cdots,m)$, then the original inequality turns into $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji})^m \leq \prod\limits_{j=1}^{m}A_j$, i.e. $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji} \leq \sqrt[m]{\prod\limits_{j=1}^{m}A_j}$. So we just prove this inequality: $\frac{\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji}}{\sqrt[m]{\prod\limits_{j=1}^{m}A_j}} \leq 1$.

$$\frac{\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji}}{\sqrt[m]{\prod\limits_{j=1}^{m}A_j}} = \sum_{i=1}^{n}\frac{\prod\limits_{j=1}^{m}a_{ji}}{\sqrt[m]{\prod\limits_{j=1}^{m}A_j}} = \sum_{i=1}^{n}\sqrt[m]{\prod_{j=1}^{m}\frac{a_{ji}^m}{A_j}} \leq \sum_{i=1}^{n}\frac{\sum\limits_{j=1}^{m}\frac{a_{ji}^m}{A_j}}{m} = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{a_{ji}^m}{A_j} = \frac{1}{m}\sum_{j=1}^{m}\frac{\sum\limits_{i=1}^{n}a_{ji}^m}{A_j} = \frac{1}{m}\sum_{j=1}^{m}\frac{A_j}{A_j} = 1.$$

So $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji})^m \leq \prod\limits_{j=1}^{m}(\sum\limits_{i=1}^{n}a_{ji}^m)$.

From the necessary and sufficient condition for the equal sign of mean inequality, we can see the equal sign holds if and only if $\frac{a_{1i}^m}{A_1} = \frac{a_{2i}^m}{A_2} = \cdots = \frac{a_{mi}^m}{A_m}$, i.e. $k_1a_{1i} = k_2a_{2i} = \cdots = k_ma_{mi}, (i = 1,2,\cdots,n)$, where $k_j = \sqrt[m]{\frac{1}{A_j}}, (j = 1,2,\cdots,m)$.

From the above (1) and (2), the inequality $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji})^m \leq \prod\limits_{j=1}^{m}(\sum\limits_{i=1}^{n}a_{ji}^m)$ holds.

3. Several important corollaries for n-dimension and m-order Cauchy inequality(Only simple proof is given here)

Corollary 1. Suppose $a_1, a_2, \cdots, a_n$ are real numbers(when m is odd, $a_i \geq 0(i = 1,2,\cdots,n)$), then

$$(\sum_{i=1}^{n}a_i)^m \leq n^{m-1}(\sum_{i=1}^{n}a_i^m),$$

and equal sign holds if and only if $a_1 = a_2 = \cdots = a_n$, where $m \geq 2$ is a positive integer.

Proof: For the n-dimension and m-order Cauchy inequality $(\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}a_{ji})^m \leq \prod\limits_{j=1}^{m}(\sum\limits_{i=1}^{n}a_{ji}^m)$, let $a_{1i} = a_{2i} = \cdots = a_{m-1,i} = 1, a_{mi} = a_i, (i = 1,2,\cdots,n)$, we obtain $(\sum\limits_{i=1}^{n}a_i)^m \leq n^{m-1}(\sum\limits_{i=1}^{n}a_i^m)$.

And from the sufficient and necessary condition for the equal sign of the above n-dimension and m-order Cauchy inequality, we have the equal sign of this inequality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Corollary 2. Suppose $a_j = (a_{j1}, a_{j2}, \cdots, a_{jn})(j = 1,2,\cdots,m)$ are m non-negative real series($m \geq 2$), then

$$\prod_{j=1}^{m}(\sum_{i=1}^{n}a_{ji}) \geq (\sum_{i=1}^{n}\sqrt[m]{\prod_{j=1}^{m}a_{ji}})^m,$$

and equal sign holds if and only if $a_{1i} = 0$ or $a_{2i} = 0\cdots$ or $a_{mi} = 0$ or there exist positive constants $k_1, k_2, \cdots, k_m$ satisfying $k_1a_{1i} = k_2a_{2i} = \cdots = k_ma_{mi}(i = 1,2,\cdots,n)$.

Proof: From the above n-dimension and m-order Cauchy inequality , we have

$$(\sum_{i=1}^{n}\sqrt[m]{\prod_{j=1}^{m}a_{ji}})^m = (\sum_{i=1}^{n}\prod_{j=1}^{m}\sqrt[m]{a_{ji}})^m \leq \prod_{j=1}^{m}[\sum_{i=1}^{n}(\sqrt[m]{a_{ji}})^m] = \prod_{j=1}^{m}(\sum_{i=1}^{n}a_{ji}).$$

i.e.

$$\prod_{j=1}^{m}(\sum_{i=1}^{n}a_{ji}) \geq (\sum_{i=1}^{n}\sqrt[m]{\prod_{j=1}^{m}a_{ji}})^m,$$

and equal sign holds if and only if $a_{1i} = 0$ or $a_{2i} = 0 \cdots$ or $a_{mi} = 0$ or there exist positive constants $k_1, k_2, \cdots, k_m$ satisfying $k_1 a_{1i} = k_2 a_{2i} = \cdots = k_m a_{mi} (i = 1, 2, \cdots, n)$.

Corollary 3. Suppose $a_1, a_2, \cdots, a_n$ are real numbers(when m is odd, $a_i \geq 0 (i = 1, 2, \cdots, n)$) and $p_1, p_2, \cdots p_n$ are n positive numbers, then

$$\sum_{i=1}^{n} p_i a_i^m \geq \frac{(\sum\limits_{i=1}^{n} a_i)^m}{(\sum\limits_{i=1}^{n} \frac{1}{\sqrt[m-1]{p_i}})^{m-1}}.$$

and equal sign holds if and only if $\sqrt[m-1]{p_1} a_1 = \sqrt[m-1]{p_2} a_2 = \cdots = \sqrt[m-1]{p_n} a_n$, where $m \geq 2$ is a positive integer.

Proof: For the n-dimension and m-order Cauchy inequality $(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji})^m \leq \prod\limits_{j=1}^{m} (\sum\limits_{i=1}^{n} a_{ji}^m)$, let $a_{1i} = \sqrt[m]{p_i} a_i, a_{2i} = a_{3i} = \cdots = a_{mi} = \frac{1}{\sqrt[m(m-1)]{p_i}}, (i = 1, 2, \cdots, n)$, we obtain

$$(\sum_{i=1}^{n} a_i)^m \leq \sum_{i=1}^{n} (\sqrt[m]{p_i} a_i)^m (\sum_{i=1}^{n} \frac{1}{\sqrt[m-1]{p_i}})^{m-1} = \sum_{i=1}^{n} p_i a_i^m (\sum_{i=1}^{n} \frac{1}{\sqrt[m-1]{p_i}})^{m-1},$$

i.e.

$$\sum_{i=1}^{n} p_i a_i^m \geq \frac{(\sum\limits_{i=1}^{n} a_i)^m}{(\sum\limits_{i=1}^{n} \frac{1}{\sqrt[m-1]{p_i}})^{m-1}}.$$

And from the sufficient and necessary condition for the equal sign of the above n-dimension and m-order Cauchy inequality, we have the equal sign of this inequality holds if and only if $\sqrt[m-1]{p_1} a_1 = \sqrt[m-1]{p_2} a_2 = \cdots = \sqrt[m-1]{p_n} a_n$.

Corollary 4. Suppose $a_1, a_2, \cdots, a_n$ are non-negative real numbers and $p_1, p_2, \cdots p_n$ are n nonzero real numbers(when m is odd, $p_i > 0 (i = 1, 2, \cdots, n)$), then

$$|\sum_{i=1}^{n} p_i \sqrt[m]{a_i}| \leq (\sum_{i=1}^{n} \sqrt[m-1]{p_i^m})^{\frac{m-1}{m}} \sqrt[m]{\sum_{i=1}^{n} a_i},$$

and equal sign holds if and only if $\frac{a_1}{\sqrt[m-1]{p_1}} = \frac{a_2}{\sqrt[m-1]{p_2}} = \cdots = \frac{a_n}{\sqrt[m-1]{p_n}}$, where $m \geq 2$ is a positive integer.

Proof: For the n-dimension and m-order Cauchy inequality $(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji})^m \leq \prod\limits_{j=1}^{m} (\sum\limits_{i=1}^{n} a_{ji}^m)$, let $a_{1i} = a_{2i} = \cdots = a_{m-1,i} = \sqrt[m-1]{p_i}, a_{mi} = \sqrt[m]{a_i} (i = 1, 2, \cdots, n)$, we obtain

$$(\sum_{i=1}^{n} p_i \sqrt[m]{a_i})^m \leq (\sum_{i=1}^{n} \sqrt[m-1]{p_i^m})^{m-1} [\sum_{i=1}^{n} (\sqrt[m]{a_i})^m] = (\sum_{i=1}^{n} \sqrt[m-1]{p_i^m})^{m-1} (\sum_{i=1}^{n} a_i).$$

So

$$|\sum_{i=1}^{n} p_i \sqrt[m]{a_i}| \leq (\sum_{i=1}^{n} \sqrt[m-1]{p_i^m})^{\frac{m-1}{m}} \sqrt[m]{\sum_{i=1}^{n} a_i}.$$

And from the sufficient and necessary condition for the equal sign of the above n-dimension and m-order Cauchy inequality, we have the equal sign of this inequality holds if and only if $\frac{a_1}{\sqrt[m-1]{p_1}} = \frac{a_2}{\sqrt[m-1]{p_2}} = \cdots = \frac{a_n}{\sqrt[m-1]{p_n}}$.

Corollary 5. Suppose $a_j = (a_{j1}, a_{j2}, \cdots, a_{jn}) (j = 1, 2, \cdots, m)$ are m real series($m \geq 2$) and $\prod\limits_{j=1}^{m} a_{ji} > 0, a_{1i} a_{ji} > 0 (j = 2, 3, \cdots, m), (i = 1, 2, \cdots, n)$, then

$$\prod_{j=2}^{m} \sum_{i=1}^{n} \frac{a_{1i}}{a_{ji}} \geq \frac{(\sum\limits_{i=1}^{n} a_{1i})^m}{\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji}},$$

and equal sign holds if and only if $a_{21} = a_{22} = \cdots = a_{2n}$, $a_{31} = a_{32} = \cdots = a_{3n}, \cdots$, and $a_{m1} = a_{m2} = \cdots = a_{mn}$.

Proof: Replace a_{1i} by $\sqrt[m]{\prod\limits_{j=1}^{m} a_{ji}}$ and a_{ji} by $\sqrt[m]{\frac{a_{1i}}{a_{ji}}}, (j = 2, 3, \cdots, m), (i = 1, 2, \cdots, n)$ in the above n-dimension and m-order Cauchy inequality $(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ji})^m \leq \prod\limits_{j=1}^{m} (\sum\limits_{i=1}^{n} a_{ji}^m)$, we obtain

$$(\sum_{i=1}^{n} \sqrt[m]{\prod_{j=1}^{m} a_{ji}} \cdot \prod_{j=2}^{m} \sqrt[m]{\frac{a_{1i}}{a_{ji}}})^m \leq \sum_{i=1}^{n} (\sqrt[m]{\prod_{j=1}^{m} a_{ji}})^m \cdot \prod_{j=2}^{m} [\sum_{i=1}^{n} (\sqrt[m]{\frac{a_{1i}}{a_{ji}}})^m],$$

i.e. $(\sum_{i=1}^{n} a_{1i})^m \le \sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji} \cdot \prod_{j=2}^{m}(\sum_{i=1}^{n} \frac{a_{1i}}{a_{ji}})$.

So

$$\prod_{j=2}^{m}\sum_{i=1}^{n}\frac{a_{1i}}{a_{ji}} \ge \frac{(\sum_{i=1}^{n} a_{1i})^m}{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}},.$$

And from the sufficient and necessary condition for the equal sign of the above n-dimension and m-order Cauchy inequality, we have the equal sign of this inequality holds if and only if $a_{21} = a_{22} = \cdots = a_{2n}$, $a_{31} = a_{32} = \cdots = a_{3n}, \cdots$, and $a_{m1} = a_{m2} = \cdots = a_{mn}$.

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