Strong Law of Large Numbers for Weighted Sum of Exchangeable Random Variables

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Abstract
In this paper, the Marcinkiewicz type theorem is extended to the case of exchangeable random variables. As a generalization, we also obtain two strong laws of large numbers on the weighted sum of exchangeable random variables.

Keywords: Exchangeability, Law of large numbers

1. Introduction
If the replacement the joint distribution of $X_1, \ldots, X_n$ is unchanged, that is, for each replacement $\pi$ of 1, 2, $\ldots$, $n$, the joint distribution of $X_1, \ldots, X_n$ is the same with that of $X_{\pi(1)}, \ldots, X_{\pi(n)}$; then the random variable finite series $X_1, \ldots, X_n$ is known as the exchangeable. Obviously, the independent identical distribution random variables is the simplest exchangeable random variables. The concept of exchangeable random variables is the first proposed by De Finetti 1930. The most famous property of exchangeable random variables is its basic structured theorem, called De Finetti theorem; that is, the infinite series of exchangeable random variables is independent identical distribution, if its tail is $\sigma$ algebra.

The aim of this paper is generalize the independent identical distribution variables Bai, 2000, P.105-112 and Sung, 2001, P.413-419 to the exchangeable random variables. As the selection method for truncated random variables is different when deal with random variables, so the prove method is more simple than that of Bai, 2000, P.105-112 and Sung, 2001, P.413-419.

Definition Wu, 2006, P.132-133. The positive valued function $l(x)$ defined on $[1, \infty)$ is called slowly changed, if for any $c > 0$, we have $\lim_{x \to \infty} \frac{l(cx)}{l(x)}$. Suppose $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is real positive series that satisfy $A_{ni} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}$, and $A_{\alpha} = \lim_{n \to \infty} \sup A_{ni}, \quad n < \infty$. (1)

Lemma Wu, 2006, P.132-133. Suppose $\{X_n, n \geq 1\}$ are exchangeable random variables, that satisfy $\text{Cov}(f_1(X_1), f_2(X_2)) \leq 0$.

Let $A_1, \ldots, A_m$ be the disjoint non-empty subset of $\{1, 2, \ldots, n\}$, with $m \geq 2$. Suppose $f_i, i = 1, 2, \ldots, m$ is a non-increase (non-decrease) function, then (1) If $f_i \geq 0, i = 1, \ldots, m$ then

$E \prod_{i=1}^{n} f(X_i, j \in A_i) \leq \prod_{i=1}^{n} Ef(X_i, j \in A_i)$.

(2). Particularly, for any $x_i \in R, i = 1, \ldots, m$, we have

$P(X_i < x_1, \ldots, X_m < x_m) \leq \prod_{i=1}^{n} P(X_i < x_i)$.

Subsequently, we will outline several lemmas, which will be used in the proof of the main theorems. If necessary, we will also give the proof.
Lemma 2. Suppose $X_1, \ldots, X_n$ are exchangeable random variables, that satisfy

$$Cov(f_1(X_1), f_2(X_2)) \leq 0$$

and $EX_k = 0, \sigma_k^2 = EX_k^2 < \infty, k = 1, 2, \ldots, n$. Suppose there exists a positive constant $H$ such that

$$|EX_k^m| \leq \frac{m!}{2} \sigma_k^2 H^{m-2}, \quad k = 1, \ldots, n,$$

then we have

$$P\left(\sum_{i=1}^{n} X_i \geq x\right) \leq \exp(-x^2/4) \sum_{i=1}^{n} \sigma_i^2 0 \leq x \leq \sum_{i=1}^{n} \sigma_i^2 \frac{1}{H},$$

$$P\left(\sum_{i=1}^{n} X_i \leq -x\right) \leq \exp(-x^2/4) \sum_{i=1}^{n} \sigma_i^2 0 \leq x \leq \sum_{i=1}^{n} \sigma_i^2 \frac{1}{H}.$$

Proof. Based on Theorem 2.5 in Taylor, 2002, P.643-656 and Lemma 1 in Sung, 2001, P.413-419, this Lemma is easy to prove.

Lemma 3. Suppose $[X_n, n \geq 1]$ are the exchangeable random variables and there exist $h > 0, r > 0$ such that

$$E[exp(hx^r)] < \infty.$$ (2)

Let $[X_{ni}, 1 \leq i \leq n, n \geq 1]$ are the exchangeable random variables that satisfy

$$Cov(f_1(X_{ni}), f_2(X_{n2})) \leq 0$$

with $EX_{ni} = 0, 1 \leq i \leq n, n \geq 1$; and $[a_{ni}], 1 \leq i \leq n, n \geq 1$ are real constant array that satisfy

(i). There exist $\beta$ with $0 < \beta \leq r$ and $[u_{ni}, n \geq 1]$ with $\lim_{n \to \infty} u_{ni} = 0$ such that

$$|a_{ni}X_{ni}| \leq \frac{u_{ni}|X_{ni}|^\beta}{\log n}, a.s.$$

(ii). There exists $\delta > 0$ and array $[v_{ni}]$ that satisfy $\lim_{n \to \infty} v_{ni} = 0$ such that

$$X_{ni}^2 \sum_{i=1}^{n} a_{ni}^2 \leq \frac{v_{ni}|X_{ni}|^\beta}{\log n}, a.s.$$

then

$$\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$$

a.s. $n \to \infty$.

Proof. Based on Theorem 2.5 in Taylor, 2002, P.643-656 and Theorem 18 in Petrov, 1991,83-84, the lemma is easy to prove.

Lemma 3. Suppose $[X_n, n \geq 1]$ are the exchangeable random variables and there exist $h > 0, r > 0$ such that

$$E[exp(hx^r)] < \infty,$$

$[X_{ni}, 1 \leq i \leq n, n \geq 1]$ are the exchangeable random variables that satisfy

$$Cov(f_1(X_{ni}), f_2(X_{n2})) \leq 0$$

with $EX_{ni} = 0, 1 \leq i \leq n, n \geq 1$; and $[a_{ni}], 1 \leq i \leq n, n \geq 1$ are real constant array that satisfy

(1). $E[exp(hx^r)] < \infty$.

(2). There exist $\beta$ with $0 < \beta \leq r$ and constant $c > 0$ such that

$$|a_{ni}X_{ni}| \leq \frac{c|X_{ni}|^\beta}{\log n}, a.s.$$

(3). There exists $\delta > 0$ and array $[v_{ni}]$ that satisfy $\lim_{n \to \infty} v_{ni} = 0$ such that

$$X_{ni}^2 \sum_{i=1}^{n} a_{ni}^2 \leq \frac{v_{ni}|X_{ni}|^\beta}{\log n}, a.s.$$

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then
\[ \sum_{i=1}^{n} a_{ii}X_{ii} \to 0 \]
a.s. \( n \to \infty. \)

2. The main results and proof

**Theorem 1.** Suppose \( \{X, X_n, n \geq 1\} \) are the exchangeable random variables that satisfy

\[ \text{Cov}(f_1(X_1), f_2(X_2)) \leq 0. \]

Suppose \( f_i, i = 1, 2 \) are functions satisfy the above rule and non-decrease with \( X_1 \) and \( X_2, EX_1, \alpha p > 1, p < 2, I(x) > 0 \) is monotonous non-decrease function when \( x \to +\infty \), \( [a_{ii}], 1 \leq i \leq n, n \geq 1 \) are real constant array with \( A_{n,1}^{-1} = n^{-1} \sum_{i=1}^{n} |a_{ii}|^2 \). Further, suppose

\[ A_{ii} = \lim_{d \to \infty} A_{ii}, \quad n \leq \infty; \]

and

\[ E|X|^p \leq \infty; \]

and

\[ EX = 0; \]

with \( 1 < \alpha, \beta < \infty, 1 < p < 2 \) and \( \frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta} \). Then we have

\[ \frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ii}X_i \to c.o.a.s. n \to \infty. \tag{3} \]

**Proof.** Without loss of generality, for any \( 1 \leq i \leq n, n \geq 1 \), suppose \( a_{ii} > 0 \). As \( \{X, X_n, n \geq 1\} \) are the exchangeable random variables, and \( a_{ii}X_1, \ldots, a_{ii}X_n \) also satisfy

\[ \text{Cov}(f_1(a_{ii}X_1), f_2(a_{ii}X_n)) \leq 0 \]

and \( 1 < \alpha, \beta < \infty, 1 < p < 2 \), and \( \frac{1}{p} = \frac{1}{\alpha} + \frac{1}{\beta} \). Then \( p < \alpha \cap \beta \cap 2 \). From Yang, 2000, P.218-223 and (1) we have

\[ E[n^{-1/p} \sum_{i=1}^{n} a_{ii}X_{i}^{-\alpha/\beta + 2}] \leq Cr^{\alpha/\beta + 2/p} \sum_{i=1}^{n} |a_{ii}|^{-\alpha/\beta + 2} X^{-\alpha/\beta + 2} \leq Cr^{\alpha/\beta + 2/p + 1} A_{n,1}^{-\alpha/\beta + 2} \to 0, n \to \infty. \]

Therefore, \( n^{-1/p} \sum_{i=1}^{n} a_{ii}X_i \to 0, n \to \infty \). From the symmetrized inequality proved in Lemma 14 in Petrov, 1991, P.83-84, we know that, in order to prove \( n^{-1/p} \sum_{i=1}^{n} a_{ii}X_i \to 0, n \to \infty \), we just need to prove

\[ n^{-1/p} a_{ii}X_i^S \to 0, \quad a.s. \quad n \to \infty, \]

where \( X_i^S \) is the symmetrized form of \( X_i \). From Lemma 3 in Chi, 1997, P.199-203. we have the symmetrized series of

\[ \text{Cov}(f_1(X_1), f_2(X_2)) \leq 0 \]

also satisfy the inequality, i.e.

\[ \text{Cov}(f_1(X_1^S), f_2(X_2^S)) \leq 0. \]

Without loss of generality, we assume that \( \{X_n, n \geq 1\} \) are the symmetrized exchangeable random variables that satisfy \( \text{Cov}(f_1(X_1), f_2(X_2)) \leq 0 \) for all \( 1 \leq i \leq n, n \geq 1 \). Letting

\[ X_i^f = X_i I[X_i] \leq n^{1/\beta} + n^{1/\beta} I[X_i > \beta] - n^{1/\beta} I[X_i < -n^{1/\beta}], \]

\[ X_i^f = X_i I[X_i] > n^{1/\beta} - n^{1/\beta} I[X_i > \beta] + n^{1/\beta} I[X_i < -n^{1/\beta}], \]

\[ X_i^S = X_i I[X_i] \leq n^{1/\beta}, \]

\[ a_{ii} = a_{ii} I[|a_{ii}| \leq n^{1/\alpha}], \]

\[ a_{ii} = a_{ii} I[|a_{ii}| > n^{1/\alpha}]. \]

Then we have

\[ \sum_{i=1}^{n} a_{ii}X_i = \sum_{i=1}^{n} a_{ii}X_i^f = \sum_{i=1}^{n} a_{ii}X_i^S = \sum_{i=1}^{n} a_{ii}X_i^S. \tag{4} \]
As \( \frac{1}{p} = \frac{1}{a} + \frac{1}{b} \), \( \beta = \frac{\alpha}{n-1} \langle 1 + \beta (1 - \frac{1}{2}) \rangle \), \( |X_i| < |X_i| \frac{d\alpha}{n} n^{-(1 - \frac{1}{2})} \), and \( E|X|^\beta < \infty \), which is equivalent to \( \sum_{i=1}^{\infty} P(|X_i|^\beta > n) < \infty \). From Borel-Cantelli Lemma, we have \( P(|X|^\beta > n, i.o.) < \infty = 0 \), hence

\[
\frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i|^\beta \rightarrow 0, \text{ a.s.}(n \rightarrow \infty).
\]

From Hölder inequality, \( |X_i| \leq |\tilde{X}_i| \) and \( |\tilde{X}_i| \leq |X_i|^{\alpha(a-1)/\alpha} n^{-(1-\frac{1}{2})} \) and

\[
\frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i|^\beta \rightarrow 0, \text{ a.s.}(n \rightarrow \infty)
\]

then we have

\[
n^{-1/p} \sum_{i=1}^{n} a_m|\tilde{X}_i| \leq n^{-1/p} \sum_{i=1}^{n} |a_m||\tilde{X}_i| \leq n^{-1} \sum_{i=1}^{n} |a_m||X_i|^{\alpha(a-1)/\alpha} \leq A_{\alpha,\beta}(\frac{1}{n} \sum_{i=1}^{n} |X_i|^{\alpha(a-1)/\alpha}, \text{ a.s.} n \rightarrow \infty).
\]

Therefore, we have

\[
n^{-1/p} \sum_{i=1}^{n} a_m\tilde{X}_i \leq n^{-1/p} \rightarrow 0, \text{ a.s. } n \rightarrow \infty.
\]

As \( 1 < p < 2 \), and \( \frac{1}{p} = \frac{1}{a} + \frac{1}{b} \), and \( \alpha \vee \beta < 2 \), so

\[
1 + \frac{(2 - \alpha)}{\alpha} + \frac{(2 - \beta)}{\beta} = \left\{ \begin{array}{ll}
\frac{2}{\alpha \beta}, & \text{if } \alpha \vee \beta < 2, \\
1, & \text{if } \alpha \vee \beta \geq 2.
\end{array} \right.
\]

Therefore, we have

\[
\sum_{i=1}^{n} E(a_m a_i \tilde{X}_i^2) \leq C_a A_{a,\beta}^{a/2} n^{\frac{\alpha(a-1)/\alpha}{\alpha} + \frac{\alpha}{\beta}} |X_i|^{\alpha(a-1)/\alpha} = O(\max[n^{2/a}, n^{2/\beta}]).
\]

Moreover, for any \( 1 \leq i \leq n \), \( n \geq 1 \), we have \( |n^{-1/p} a_m X_i| \leq n^{1/a} n^{1/p} n^{-1/p} \), and \( \max\{n^{2/a}, n^{2/\beta} \} = O(n^{2/p} \log^2 n) \). From Lemma 2, for sufficient small \( \epsilon \) and sufficient large \( n \), we have

\[
P(n^{-1/p} \sum_{i=1}^{n} a_m \tilde{X}_i > \epsilon) \leq \exp(-\epsilon^2 / 4n^{2/p}O(\max[n^{2/a}, n^{2/\beta}])) \leq \exp(-\epsilon^2 (\log n)^2).
\]

By the same procedures, we can also prove that

\[
P(n^{-1/p} \sum_{i=1}^{n} a_m \tilde{X}_i < -\epsilon) \leq \exp(-\epsilon^2 (\log n)^2).
\]

Therefore,

\[
\sum_{i=1}^{n} P(n^{-1/p} \sum_{i=1}^{n} a_m \tilde{X}_i > \epsilon) < \infty.
\]

From the above inequality, we have

\[
n^{-1/p} \sum_{i=1}^{n} a_m \tilde{X}_i \rightarrow 0, \text{ a.s. } n \rightarrow \infty.
\]

Also, based on \( \alpha > 1 \) and \( \frac{1}{p} = \frac{1}{a} + \frac{1}{b} \), we have

\[
n^{-1/p} \sum_{i=1}^{n} a_m X_i \leq n^{-1/p} n^{1/b} \sum_{i=1}^{n} a_m I(a_m > n^{1/a}) \leq n^{-1/p+1/b} n^{(1-alpha)/a} \alpha \sum_{i=1}^{n} |a_m|^{\alpha} = A_{\alpha,\beta}^{a/\alpha}.
\]

Then from (4),(5),(6),(7), we have

\[
\lim \sup_{n \rightarrow \infty} n^{-1/p} \sum_{i=1}^{n} a_m X_i \leq A_{\alpha,\beta}^{a} \text{ a.s. } n \rightarrow \infty.
\]

By replacing \( X_i \) with \( t X_i \), we have

\[
\lim \sup_{n \rightarrow \infty} n^{-1/p} \sum_{i=1}^{n} a_m X_i \leq \frac{A_{\alpha,\beta}^{a}}{t} \text{ a.s. } n \rightarrow \infty.
\]
Let $t \to \infty$, we have $n^{-1/p} \sum_{i=1}^{n} a_{ni} X_i \to \infty$ a.s. $n \to \infty$, i.e. the inequality (3) is true.

**References**


