Strongly Multiplicative Labeling in the Context
of Arbitrary Supersubdivision

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Abstract
We investigate some new results for strongly multiplicative labeling of graph. We prove that the graph obtained by
arbitrary supersubdivision of tree $T$, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and one-point union of
$m$ cycle of length $n$ are strongly multiplicative.

Keywords: Strongly multiplicative labeling, Strongly multiplicative graphs, Arbitrary supersubdivision

1. Introduction
We begin with simple, finite, undirected and connected graph $G = (V, E)$. In the present work $T$, $P_n \times P_m$ and $K_{m,n}$
denote the tree, grid graph, and complete bipartite graph respectively. $C_n \odot P_m$ is the graph obtained by identifying an
end point of $P_m$ with every vertex of cycle $C_n$. One point union of $m$ cycles of length $n$ denoted as $C_n^{(m)}$ is the graph
obtained by identifying one vertex of each cycles. If $V_1$ and $V_2$ are two partitions correspond to complete bipartite graph
$K_{m,n}$ then $V_1$ is called $m$-vertices part and $V_2$ is called $n$-vertices part of $K_{m,n}$. In the graph $G$ eccentricity of a vertex $u$ is
$max_{v \in V(G)}d(u, v)$. For all other terminology and notations we refer to (Harary, F., 1972). We will give brief summary of
definitions and other information which are useful for the present investigations.

Definition 1.1 Let $G$ be a graph with $q$ edges. A graph $H$ is called a supersubdivision of $G$ if $H$ is obtained from $G$ by
replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i$, $1 \leq i \leq q$ in such a way that the end vertices
of each $e_i$ are merged with the two vertices of 2-vertices part of $K_{2,m_i}$ after removing the edge $e_i$ from graph $G$.

A supersubdivision $H$ of $G$ is said to be an arbitrary supersubdivision of $G$ if every edge of $G$ is replaced by an arbitrary
$K_{2,m}$ ($m$ may vary for each edge arbitrarily). Arbitrary supersubdivision of $G$ is denoted by $SS(G)$.

Definition 1.2 If the vertices of the graph are assigned values subject to certain conditions then it is known as graph
labeling.
Most interesting graph labeling problems have following three important characteristics.

1. a set of numbers from which the labels are chosen;
2. a rule that assigns a value to each edges;
Definition 1.3 A graph $G = (V,E)$ with $p$ vertices is said to be multiplicative if the vertices of $G$ can be labeled with $p$ distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced in (Beineke, L., 2001, p. 63-75) where it is shown that every graph admits multiplicative labeling and strongly multiplicative labeling is defined as follows.

Definition 1.4 A graph $G = (V,E)$ with $p$ vertices is said to be strongly multiplicative if the vertices of $G$ can be labeled with $p$ distinct integers $1, 2, ... p$ such that label induced on the edges by the product of labels of end vertices are all distinct.

In the present investigations we prove that the graphs obtained by arbitrary supersubdivision of tree $T$, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and $C_n$ are strongly multiplicative for all $n$ and $m$.

2. Main Results

Theorem 2.1: Arbitrary supersubdivisions of tree $T$ are strongly multiplicative.

Proof: Let $T$ be the tree with $n$ vertices. Arbitrary supersubdivision $SS(T)$ of tree $T$ obtained by replacing every edge of tree with $K_{2,m}$ and we denote such graph by $G$. Let $K = \sum m_i (1 \leq i \leq n - 1)$. Let $v_j (1 \leq j \leq K + n)$ be the vertices of $G$. Denote the vertex with minimum eccentricity as $v_1$. Then $v_2$ will be the vertex which is at 1-distance apart from $v_1$. If there are more than one such vertices then throughout the work we will follow one of the direction (clockwise or anticlockwise) and denote them as $v_3, v_4, \ldots$. Next consider the vertices which are at 2-distance apart from $v_1, 3$-distance apart from $v_1$ and so on. (e.g. if there are seven vertices and two vertices are at distance 1-apart, one vertex is at distance 2-apart and three vertices are at distance 3-apart respectively form $v_1$. In this situation the vertices which are at 1-distance apart from $v_1$ will be identified as $v_2$ and $v_3$, the vertex which is at distance 2-apart will be identified as $v_4$ and the vertices which are at distance 3-apart will be identified as $v_5, v_6$ and $v_7$.) We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K + n\}$ as follows.

For any $1 \leq i \leq n + K$ define

$$f(v_i) = i$$

Then the graph $G$ under consideration admits strongly multiplicative labeling.

Illustration 2.2: In Fig. 2 strongly multiplicative labeling of $SS(T)$ corresponding to tree $T$ of Fig. 1 is shown where $n = 13$ and $K = 26$.

Theorem 2.3: Arbitrary supersubdivisions of complete bipartite graph $K_{m,n}$ are strongly multiplicative.

Proof: Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices of $m$-vertices part and $v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{m+n}$ be the vertices of $n$-vertices part of $K_{m,n}$. Arbitrary supersubdivision $SS(K_{m,n})$ of $K_{m,n}$ obtained by replacing every edge of $K_{m,n}$ with $K_{2,m}$ and we denote such graph by $G$. Let $K = \sum m_i (1 \leq i \leq mn)$. Let $u_j$ be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq K$. We denote vertices by $u_j$ which are used for supersubdivision of edges $v_1v_{m+1}, v_1v_{m+2}, \ldots, v_1v_{m+n}, v_2v_{m+1}, \ldots v_2v_{m+n}$. Let $p_o$ be the highest prime less than $K + m + n$. We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K + m + n\}$ as follows.

If $p_o \leq K + m$

$$f(v_i) = \begin{cases} i, & \text{if } 1 \leq i \leq m, \\ k + i, & \text{if } m + 2 \leq i \leq m + n \end{cases}$$

$$f(v_{m+1}) = p_o;$$

$$f(u_j) = \begin{cases} m + j, & \text{if } 1 \leq j < p_o, \\ m + j + 1, & \text{if } p_o \leq j \leq K \end{cases}$$

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If \( p_o > K + m \)

\[
    f(v_i) = \begin{cases} 
    i; & \text{if } 1 \leq i \leq m, \\
    k + i - 1; & \text{if } m + 2 \leq i < p_o, \\
    k + i; & \text{if } p_o \leq i \leq m + n 
    \end{cases}
\]

\[
    f(v_{m+1}) = p_o;
\]

\[
    f(u_j) = m + j, \quad \text{where } 1 \leq j \leq K
\]

Then in each possibilities described above the graph \( G \) under consideration admits strongly multiplicative labeling.

**Illustration 2.4:** Consider \( SS(K_{2,3}) \). Here \( m = 2, n = 3 \) and \( K = 14 \). The strongly multiplicative labeling is as shown in Fig.3.

**Theorem 2.5:** Arbitrary supersubdivisions of grid graph \( P_n \times P_m \) are strongly multiplicative.

**Proof:** Arbitrary supersubdivision \( SS(P_n \times P_m) \) of \( P_n \times P_m \) obtained by replacing every edge of grid graph with \( K_{2,m} \) and we denote such graph by \( G \). Let \( K = \sum m_i \) \( (1 \leq i \leq mn) \). Let \( v_1 \) \( (1 \leq i \leq mn + K) \) be the vertices of \( G \). Denote the vertex of left upper corner with \( v_1 \). Here we designate vertices by \( v_i \) \( (2 \leq i \leq mn + K) \) according to the procedure described in Theorem 2.1. We define vertex labeling \( f : V(G) \rightarrow \{1, 2, \ldots, mn + K\} \)

\[
    f(v_i) = i; \quad \text{where } 1 \leq i \leq mn + K
\]

Then the graph \( G \) under consideration admits strongly multiplicative labeling.

**Illustration 2.6:** Consider \( SS(P_4 \times P_3) \). Here \( n = 4, m = 3 \) and \( K = 41 \). The corresponding strongly multiplicative labeling is shown in Fig.4.

**Theorem 2.7:** Arbitrary supersubdivisions of \( C_n \odot P_m \) are strongly multiplicative.

**Proof:** Arbitrary supersubdivision \( SS(C_n \odot P_m) \) of \( C_n \odot P_m \) obtained by replacing every edge of \( C_n \odot P_m \) with \( K_{2,m} \) and we denote such graph by \( G \). Let \( K = \sum m_i \) \( (1 \leq i \leq mn) \). Let \( v_1 \) \( (1 \leq i \leq mn + K) \) be the vertices of \( G \). Designate arbitrary vertex of \( C_n \) as \( v_1 \) and employing the scheme used in Theorem 2.1 the remaining vertices will receive labels \( v_2, v_3, \ldots, v_{mn+K} \). We define vertex labeling \( f : V(G) \rightarrow \{1, 2, \ldots, mn + K\} \) as follows.

\[
    f(v_i) = i; \quad \text{where } 1 \leq i \leq mn + K
\]

Then the graph \( G \) under consideration admits strongly multiplicative labeling.

**Illustration 2.8:** Consider \( SS(C_5 \odot P_3) \). Here \( n = 5, m = 3 \) and \( K = 37 \). The corresponding strongly multiplicative labeling is as shown in Fig.5.

**Theorem 2.9:** Arbitrary supersubdivisions of \( C_n^{(m)} \) are strongly multiplicative.

**Proof:** Arbitrary supersubdivision of \( C_n^{(m)} \) is obtained by replacing every edge of \( C_n^{(m)} \) with \( K_{2,m} \) and we denote this graph by \( G \). Let \( K = \sum m_i \). Let \( v_1(1 \leq i \leq m(n-1) + K + 1) \) be the vertices of \( G \). Denote the common vertex of cycles by \( v_1 \). According to the procedure followed in previous results the remaining vertices will be designated as \( v_1 \) \( (2 \leq i \leq m(n-1) + K + 1) \). We define vertex labeling \( f : V(G) \rightarrow \{1, 2, \ldots, m(n-1) + K + 1\} \) as follows.

For any \( 1 \leq i \leq m(n-1) + K + 1 \) we define

\[
    f(v_i) = i;
\]

Then the graph \( G \) under consideration admits strongly multiplicative labeling.

**Illustration 2.10:** Consider \( SS(C_4^{(3)}) \). Here \( n = 4, m = 3 \) and \( K = 26 \). The strongly multiplicative labeling is as shown in Fig.6.

3. Concluding Remarks And Open Problem

Labeled graph is the topic of current interest for many researchers as it has diversified applications. It is also very interesting to investigate graph or families of graph which admits particular type of labeling. In (Sethuraman, G., 2001 p.1059-1064) and (Kathiresan, K., 2004 p.81-84) graceful labeling in the context of arbitrary supersubdivision is discussed while we discuss here strongly multiplicative labeling in the context of arbitrary supersubdivision. We consider five different graph families and investigate their strongly multiplicative labeling. This work is a nice combination of combinatorial number theory and graph theory which will provide enough motivation to any researcher.

**Open Problems:**
• Similar investigations are possible for other graph families.
• Parallel results can be investigated corresponding to other graph labeling techniques.

References
Harary, F. (1972). *Graph theory*. Addison-Wesley, Reading, Massachusetts.
Figure 3. Strongly multiplicative labeling of $SS(K_{2,3})$

Figure 4. Strongly multiplicative labeling of $SS(P_4 \times P_3)$

Figure 5. Strongly multiplicative labeling of $SS(C_5 \odot P_3)$
Figure 6. Strongly multiplicative labeling of $SS(C_4^{(3)})$