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Nonlinear Parabolic Equation on Manifolds

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Abstract

In this work we investigate the existence and the uniqueness of solution for a nonlinear differential equation of parabolic type on the lateral boundary Σ of a cylinder Q, cf. (1). An important part of our study is to transform this initial value problem into another one whose differential operator equation is of the type

$$u_t + a\left(\int_\Gamma u dx\right)\mathcal{A}u - \Delta_\Gamma u + u^{2k+1} = f \text{ on } \Sigma,$$

cf. (9), where k is a positive integer. The operator $\mathcal{A}$ acts in Sobolev spaces on Γ, boundary of Ω. The initial value problem (9) will be studied in Section 4. Thus, we obtain the existence and the uniqueness of weak solution for (9).

Keywords: parabolic equation, manifolds, Wentzell boundary condition

1. Introduction

We consider Ω a bounded open set of $\mathbb{R}^n$ ($n \geq 2$) with C^∞ boundary Γ. By ν we denote the outward normal unit vector field defined on Γ. For each $T > 0$, $Q = \Omega \times]0, T[$ denotes a cylindrical domain whose lateral boundary will be represented by $\Sigma = \Gamma \times]0, T[$.

Our main objective is to investigate existence and uniqueness of solution for the following problem:

$$\left|\begin{array}{l} \Delta w = 0 \text{ in } Q \\ w_t + a\left(\int_\Gamma w d\Gamma\right)\dfrac{\partial w}{\partial \nu} - \Delta_\Gamma w + w^{2k+1} = f \text{ on } \Sigma \\ w(x,0) = w_0(x) \text{ on } \Gamma, \end{array}\right. \tag{1}$$

where k is a positive integer, the derivatives are in the sense of the theory of distributions, $\dfrac{\partial w}{\partial \nu}$ is the normal derivative of w, by Δ_Γ we denote the Laplace Beltrami operator on Γ, the Laplace operator Δ acts only on space variables and $w = w(x,t)$, $x \in \Omega$, $0 < t < T$. This work was motivated by J. L. Lions who has considered, in 1969, the existence and uniqueness of solution for nonlinear problems on manifolds whose unknown function satisfies the Laplace equation in Ω and a nonlinear evolution equation on its lateral boundary Σ.

The nonlinearity of the type $a\left(\int_\Gamma w d\Gamma\right)$ was motivated by the study of problems of diffusion of population cf. Chipot (2000, Chapters 1 and 12) and also Menezes (2006).

Similar problems on manifolds, also motivated by Lions (1969), can be seen in Antunes, Araruna, and Medeiros (2002), Antunes, Lopez, Silva, and Araújo (2013) and Cavalcanti and Domingos Cavalcanti (2004). See also similar questions in Coclite, G. R. Goldstein, and J. A. Goldstein (2008), Vázquez and Vitillaro (2009) and Wentzell(1939) which, we think, was the initial motivation for this type of questions.

Our paper is organized as follows: in section 2, we establish the appropriate notation and the functional setting to the treatment of our problem. In section 3 we develop a classical formalism in order to investigate (1) as a differential operator equation whose operator $\mathcal{A}$ acts on Sobolev spaces defined on the manifold Γ. In this way the Equation $(1)_2$ is formulated as a differential operator equation of the type

$$u_t + a\left(\int_\Gamma u dx\right)\mathcal{A}u - \Delta_\Gamma u + u^{2k+1} = f,$$

for which we can apply a known methodology for the initial value problem (9). In section 4, we investigate the initial value problem (9) by approximate method and we succeed to prove existence and uniqueness for weak solutions.

2. Preliminaries

We represent by $H^s(\Omega)$ and $H^s(\Gamma)$, the Sobolev spaces of order $s \in \mathbb{R}$ on Ω and Γ, see (Frota, Medeiros, & Vicente, 2011; Hebey, 1999; Lions, 1969). When $s = 0$, $H^0(\Omega)$ and $H^0(\Gamma)$ are denoted by $L^2(\Omega)$ and $L^2(\Gamma)$, the Lebesgue spaces of square integrable functions on Ω and Γ. We denote by $|\cdot|_p$ and $|\cdot|_{p,\Gamma}$, $1 \le p < \infty$, the norms of $L^p(\Omega)$ and $L^p(\Gamma)$, the usual Lebesgue L^p spaces. We also denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ the scalar product in $L^2(\Omega)$ and $L^2(\Gamma)$. Set also $((\cdot,\cdot))_{H^s(\Gamma)}$ and $\|u\|_{H^s(\Gamma)}$ the inner product and norm in the Hilbert spaces $H^s(\Gamma)$.

By ∇_Γ we denote the gradient tangent on the manifold Γ and by Δ_Γ the Laplace Beltrami operator, defined on a real function u on Γ as the divergence of the $\nabla_\Gamma u$. For details, see (Frota, Medeiros, & Vicente, 2011; Hebey, 1999; Lions & Magenes, 1968; Vázquez & Vitillaro, 2009; Vicente, 2010).

For $u, v \in C^\infty(\Gamma)$, the space of C^∞ real function defined on Γ, we have the following integral relation between Δ_Γ and ∇_Γ:

$$-\int_\Gamma (\Delta_\Gamma u)\, v\, d\Gamma = \int_\Gamma \nabla_\Gamma u . \nabla_\Gamma v d\Gamma. \tag{2}$$

Let us consider $C^\infty(\Gamma)$ with the scalar products:

$$((u,v))_{H^1(\Gamma)} = \int_\Gamma uv\, d\Gamma + \int_\Gamma \nabla_\Gamma u . \nabla_\Gamma v\, d\Gamma, \tag{3}$$

with induced norm $\|u\|^2_{H^1(\Gamma)} = ((u,u))_{H^1(\Gamma)}$ and

$$((u,v))_{H^2(\Gamma)} = \int_\Gamma uv\, d\Gamma + \int_\Gamma \Delta_\Gamma u \Delta_\Gamma v\, d\Gamma, \tag{4}$$

with induced norm $\|u\|^2_{H^2(\Gamma)} = ((u,u))_{H^2(\Gamma)}$.

Observe that $C^\infty(\Gamma)$ with scalar products (3) and (4) is a pre-Hilbert space. The completions of $C^\infty(\Gamma)$ with respect to the norms induced by (3) and (4) are represented by $H^1(\Gamma)$ and $H^2(\Gamma)$, respectively. Then, the Equation (2) can be extended to the case where $u \in H^2(\Gamma)$ and $v \in H^1(\Gamma)$.

We have $H^2(\Gamma) \hookrightarrow H^1(\Gamma)$, consequence of (2) and the embedding is compact, see (Hebey, 1999). Observe that $\hookrightarrow$ means continuous embedding.

In the present paper we need the embedding of $H^s(\Gamma)$ into $L^{4k+2}(\Gamma)$, for $s \ge 2$ and k a positive integer. In fact, by Sobolev embedding theorem, cf. Lions (2003) or Lions and Magenes (1968), if $s \ge 2$ such that $s > \dfrac{k(n-1)}{(2k+1)}$, we have

$$H^s(\Gamma) \hookrightarrow L^{4k+2}(\Gamma) \hookrightarrow L^2(\Gamma). \tag{5}$$

Finally, we suppose $a(s)$, $s \in \mathbb{R}$, real continuous function, with bounded derivative and

$$a(s) \ge a_0 > 0, \text{ for all } s \in \mathbb{R}. \tag{6}$$

3. Formulation of the Problem (1) on Σ

In (Antunes, Lopez, Silva, & Araújo 2013), we defined an operator $\mathcal{A} \in \mathcal{L}\left(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)\right)$ which is a composition of the traces γ_0, γ_1, these are, roughly speaking, respectively, $\dfrac{\partial w}{\partial \nu}$ and w restricted to Γ. To avoid duality pairing in the process of approximation we define, in the present argument, an operator $\mathcal{A}$: $H^1(\Omega) \longrightarrow L^2(\Gamma)$ and we obtain scalar product instead of duality.

We have the Dirichlet problem:

$$\left|\begin{array}{l} -\Delta w = 0, \text{ in } \Omega \\ w = u \text{ on } \Gamma. \end{array}\right. \tag{7}$$

If $u \in H^1(\Gamma)$ it has a unique solution $w \in H^{3/2}(\Omega)$. We have γ_0: $H^{3/2}(\Omega) \longrightarrow H^1(\Gamma)$ and γ_1: $H^{3/2}(\Omega) \longrightarrow L^2(\Gamma)$, the mapping γ_0, γ_1 are continuous. The composition $\gamma_1 \circ \gamma_0^{-1}$ is a bounded linear mapping from $H^1(\Gamma)$ to $L^2(\Gamma)$. We define $\mathcal{A}$: $H^1(\Gamma) \longrightarrow L^2(\Gamma)$ by $\mathcal{A} = \gamma_1 \circ \gamma_0^{-1}$. This bounded operator will be the "substitute" of the normal derivative in (1).

Moreover, we have, from (7)

$$(\mathcal{A}u, u) = (\gamma_1 w, \gamma_0 w)_\Gamma = \int_\Omega |\nabla w|^2\, dx \geq 0. \tag{8}$$

We formulate now the problem (1) on Σ. In fact, we define

$$w(t)|_\Gamma = u(t) \qquad \text{and} \qquad \left.\frac{\partial w}{\partial \nu}(t)\right|_\Gamma = \mathcal{A}u(t).$$

Thus, the problem (1) can be rewritten as follows:

$$\left|\begin{array}{l} u_t + a\left(\int_\Gamma u d\Gamma\right)\mathcal{A}u - \Delta_\Gamma u + u^{2k+1} = f \text{ on } \Sigma \\ u(x,0) = u_0(x) \text{ on } \Gamma. \end{array}\right. \tag{9}$$

From now on, our objective will be to prove existence and uniqueness of solutions for the problem (9).

4. Main Results

In this section we formulate and prove existence and uniqueness of weak solutions for the mixed problem (9) on Σ.

Theorem 1 *Let us suppose that* $u_0 \in H^1(\Gamma) \cap L^{2k+2}(\Gamma)$, $f \in L^2\left(0,T;L^2(\Gamma)\right)$ *and* $\mathcal{A} \in \mathcal{L}\left(H^1(\Gamma), L^2(\Gamma)\right)$ *as defined above. Then, there exists u:* $\Sigma \longrightarrow \mathbb{R}$, *in the class:*

$$\begin{array}{l} u \in L^2\left(0,T;H^2(\Gamma)\right) \cap L^\infty\left(0,T;H^1(\Gamma)\cap L^{2k+2}(\Gamma)\right), \\ u' \in L^2\left(0,T;L^2(\Gamma)\right), \end{array}$$

which is the unique weak solution of the initial value problem (9).

Proof. We recall that $\mathcal{A} \in \mathcal{L}\left(H^1(\Gamma), L^2(\Gamma)\right)$ is defined by $\mathcal{A} = \gamma_1 \circ \gamma_0^{-1}$ with γ_0, γ_1 the traces operators of order zero and one.

About the operator Δ_Γ, we obtain its spectral resolution and we realize it as an operator from $H^2(\Gamma)$ in $L^2(\Gamma)$. For this argument we call attention to the reader to see (Lions & Magenes, 1968, p. 42). In fact, we deduce that the domain of $-\Delta_\Gamma$ is $H^2(\Gamma)$ and its range is $L^2(\Gamma)$. We have the spectral resolution:

$$-\Delta_\Gamma w_j = \lambda_j w_j, \quad j = 1,2,...$$

where the eigenvectors w_j are normalized in $L^2(\Gamma)$ and complete in $H^2(\Gamma)$.

We consider in $H^2(\Gamma)$ the complete orthonormal basis $\left\{w_j\right\}_{j\in\mathbb{N}}$ of eigenvectors of $-\Delta_\Gamma$ and we define $V_m = [w_1,...,w_m] \subset H^2(\Gamma)$, the subspace generated by the m first eigenvectors of $-\Delta_\Gamma$.

Furthermore, $V_m \subset H^s(\Gamma)$, for all $s > 0$ (see Lions & Magenes, 1968, Chapter 1, Remark 7.5).

For each $m \in \mathbb{N}$, we look for a function $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j$ in V_m, such that $u_m(t)$ is solution of the approximate problem:

$$\left|\begin{array}{l} (u_m'(t), v)_\Gamma + a\left(\int_\Gamma u_m(t)\, d\Gamma\right)(\mathcal{A}u_m(t), v)_\Gamma - (\Delta_\Gamma u_m(t), v)_\Gamma + \\ \left(u_m^{2k+1}(t), v\right)_\Gamma = (f(t), v)_\Gamma, \text{ for all } v \in V_m, \\ u_m(0) = u_{0m} \longrightarrow u_0 \text{ in } H^1(\Gamma) \cap L^{2k+2}(\Gamma). \end{array}\right. \tag{10}$$

Observe that by section 2, we obtained $H^s(\Gamma) \hookrightarrow L^{4k+2}(\Gamma)$, consequentely, since $u_m(t) \in H^2(\Gamma)$, it follows that $u^{2k+1}(t) \in L^2(\Gamma)$, making sense $\left(u_m^{2k+1}(t), v\right)_\Gamma$ for $v \in V_m$. □

Observe that (10) is a nonlinear system of first order ordinary differential equation in $g_{jm}(t)$, the coordinates of the approximations $u_m(t)$. It has local solution defined in $0 \le t \le t_m < T$. By mean of estimates we extend these local solutions to the interval $[0, T]$.

Estimate 1 Setting $v = 2u_m(t)$ in (10), we have:

$$\begin{aligned} &\frac{d}{dt}|u_m(t)|^2_{2,\Gamma} + 2a\left(\int_\Gamma u_m(t)\,d\Gamma\right)(\mathcal{A}u_m(t), u_m(t))_\Gamma - 2(\Delta_\Gamma u_m(t), u_m(t))_\Gamma + 2\left(u_m^{2k+1}(t), u_m(t)\right)_\Gamma \\ = \quad & 2(f(t), u_m(t))_\Gamma \end{aligned} \tag{11}$$

We observe that:

- From (8), we have $(\mathcal{A}u_m(t), u_m(t))_\Gamma \ge 0$.
- $-(\Delta_\Gamma u_m(t), u_m(t))_\Gamma = (\nabla_\Gamma u_m(t), \nabla_\Gamma u_m(t))_\Gamma = |\nabla_\Gamma u_m(t)|^2_{2,\Gamma}$, because $u_m(t) \in H^2(\Gamma)$ is approximated by $C^\infty(\Gamma)$, see (2).
- $\left(u_m^{2k+1}(t), u_m(t)\right)_\Gamma = \|u_m(t)\|^{2(k+1)}_{2(k+1),\Gamma}$.

Going back to (11), employing the results above and the results (8), about $\mathcal{A}$, and (6), about $a(s)$, we obtain:

$$\frac{d}{dt}|u_m(t)|^2_{2,\Gamma} + 2|\nabla_\Gamma u_m(t)|^2_{2,\Gamma} + 2|u_m(t)|^{2(k+1)}_{2(k+1),\Gamma} \le |f(t)|^2_{2,\Gamma} + |u_m(t)|^2_{2,\Gamma}. \tag{12}$$

From (12), applying Gronwall lemma we conclude the existence of a positive constant C_1 depending only on $|f|_{L^2(0,T;L^2(\Gamma))}$, $|u_0|_{L^2(\Gamma)}$ and T, such that:

$$|u_m(t)|_{2,\Gamma} \le C_1. \tag{13}$$

Then, by the extension theorem for ordinary differential equation, the local solution $u_m(t)$ has an extension to the whole interval $[0, T]$. We represent the extension by the same notation $u_m(t)$. Thus, for the extension $u_m(t)$ we have (12) true for all t in $[0, T]$. Consequentely, it makes sense to integrate (12) on $[0, t) \subset [0, T]$.

Integrating (12) on $(0, t) \subset [0, T]$, by the hypothesis of f, the convergence in $(10)_2$ and the estimate in (13), we obtain:

$$\int_0^t |\nabla_\Gamma u_m(s)|^2_{2,\Gamma}\,ds + \int_0^t |u_m(s)|^{2(k+1)}_{2(k+1),\Gamma}\,ds \le C_2. \tag{14}$$

Estimate 2 Setting $v = 2u'_m(t)$ in (10), we obtain, after some calculus, that

$$\left|u'_m(t)\right|^2_{2,\Gamma} + \frac{d}{dt}|\nabla_\Gamma u_m(t)|^2_{2,\Gamma} + \frac{1}{k+1}\frac{d}{dt}|u_m(t)|^{2k+2}_{2k+2,(\Gamma)} \le |f(t)|^2_{2,\Gamma} + 2\left|a\left(\int_\Gamma u_m(t)\,d\Gamma\right)\right| |\mathcal{A}u_m(t)|_{2,\Gamma}\left|u'_m(t)\right|_{2,\Gamma}. \tag{15}$$

Now we observe that

$$\left|\int_\Gamma u_m(t)\,d\Gamma\right| \le \int_\Gamma |u_m(t)|\,d\Gamma \le C_4|u_m(t)|_{2,\Gamma}, \tag{16}$$

where C_4 is a positive constant depending on the measure of Γ. By (16) and (13) we obtain

$$\left|\int_\Gamma u_m(t)\,d\Gamma\right| \le C_5. \tag{17}$$

From (17) and the continuity of the function $a(s)$ it follows that

$$\left|a\left(\int_\Gamma u_m(t)\,d\Gamma\right)\right| \le C_6. \tag{18}$$

Returning with (18) in (15) we get

$$\left|u'_m(t)\right|^2_{2,\Gamma} + \frac{d}{dt}|\nabla_\Gamma u_m(t)|^2_{2,\Gamma} + \frac{1}{k+1}\frac{d}{dt}|u_m(t)|^{2k+2}_{2k+2,(\Gamma)} \le |f(t)|^2_{2,\Gamma} + 2C_6{}^2|\mathcal{A}u_m(t)|^2_{2,\Gamma} + \frac{\left|u'_m(t)\right|^2_{2,\Gamma}}{2}.$$

As the operator $\mathcal{A} \in \mathcal{L}\left(H^1(\Gamma), L^2(\Gamma)\right)$ then the last inequality is transformed into

$$\left|u_m'(t)\right|_{2,\Gamma}^2 + 2\frac{d}{dt}\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \frac{2}{k+1}\frac{d}{dt}\left|u_m(t)\right|_{2k+2,(\Gamma)}^{2k+2} \leq 2\left|f(t)\right|_{2,\Gamma}^2 + C_7\left\|u_m(t)\right\|_{H^1(\Gamma)}^2. \tag{19}$$

Integrating (19) from 0 to t,

$$\begin{aligned}&\int_0^t \left|u_m'(s)\right|_{2,\Gamma}^2 ds + 2\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \frac{2}{k+1}\left|u_m(t)\right|_{2k+2,(\Gamma)}^{2k+2} \\ \leq\ & 2\int_0^T \left|f(t)\right|_{2,\Gamma}^2 dt + C_7\int_0^T \left\|u_m(t)\right\|_{H^1(\Gamma)}^2 dt + 2\left|\nabla_\Gamma u_m(0)\right|_{2,\Gamma}^2 + \frac{2}{k+1}\left|u_m(0)\right|_{2k+2,(\Gamma)}^{2k+2}.\end{aligned}$$

By the hypothesis of f, the convergence in $(10)_2$ and the definition of the norm in $H^s(\Gamma)$, observing (13) and (14), we obtain:

$$\frac{1}{2}\int_0^t \left|u_m'(s)\right|_{2,\Gamma}^2 ds + \left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \frac{1}{k+1}\left|u_m(t)\right|_{2k+2,(\Gamma)}^{2k+2} \leq C_8. \tag{20}$$

Thus,

$$(u_m)_{m\in\mathbb{N}} \text{ is bounded in } L^\infty\left(0,T; H^1(\Gamma) \cap L^{2k+2}(\Gamma)\right). \tag{21}$$

Estimate 3 Set $v = -2\Delta_\Gamma u_m(t)$ in (10), that makes sense because $\Delta_\Gamma V_m \subset V_m$, we have:

$$\begin{aligned}&\frac{d}{dt}\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + 2\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2 - 2\left(u_m^{2k+1}(t), \Delta_\Gamma u_m(t)\right)_\Gamma \\ =\ & 2a\left(\int_\Gamma u_m(t)\,d\Gamma\right)\left(\mathcal{A}u_m(t), \Delta_\Gamma u_m(t)\right)_\Gamma - 2\left(f(t), \Delta_\Gamma u_m(t)\right)_\Gamma\end{aligned}$$

and therefore

$$\begin{aligned}&\frac{d}{dt}\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + 2\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2 - 2\left(u_m^{2k+1}(t), \Delta_\Gamma u_m(t)\right)_\Gamma \\ \leq\ & 2\left|a\left(\int_\Gamma u_m(t)\,d\Gamma\right)\right|\left|\mathcal{A}u_m(t)\right|_{2,\Gamma}\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma} + 2\left|f(t)\right|_{2,\Gamma}\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}.\end{aligned} \tag{22}$$

Remark 1 For all $v \in C^\infty(\Gamma)$, we have $-2\left(v^{2k+1}, \Delta_\Gamma v\right) \geq 0$. Since $C^\infty(\Gamma)$ is dense in $H^2(\Gamma) \cap L^{4k+2}(\Gamma)$ and $u_m \in H^2(\Gamma) \cap L^{4k+2}(\Gamma)$, we obtain that $-2\left(u_m^{2k+1}, \Delta_\Gamma u_m\right)$ is non-negative. Note also the elementary inequality for positive real numbers: $2\alpha\beta \leq \alpha^2 + \beta^2$.

From (22), employing (18) and *Remark* 1, we conclude that

$$\frac{d}{dt}\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + 2\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2 \leq 2C_6^2\left|\mathcal{A}u_m(t)\right|_{2,\Gamma}^2 + \frac{1}{2}\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2 + 2\left|f(t)\right|_{2,\Gamma}^2 + \frac{1}{2}\left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2,$$

that is,

$$\frac{d}{dt}\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \left|\Delta_\Gamma u_m(t)\right|_{2,\Gamma}^2 \leq 2C_6^2\left|\mathcal{A}u_m(t)\right|_{2,\Gamma}^2 + 2\left|f(t)\right|_{2,\Gamma}^2. \tag{23}$$

Integrating (23) from 0 to $t \leq T$, we get

$$\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \int_0^t \left|\Delta_\Gamma u_m(s)\right|_{2,\Gamma}^2 ds \leq 2C_6^2\int_0^T \left|\mathcal{A}u_m(t)\right|_{2,\Gamma}^2 dt + 2\int_0^T \left|f(t)\right|^2 dt + \left|\nabla_\Gamma u_m(0)\right|_{2,\Gamma}^2. \tag{24}$$

Considering the convergence in $(10)_2$, the hypothesis on f, the fact that the operator $\mathcal{A} \in \mathcal{L}\left(H^1(\Gamma), L^2(\Gamma)\right)$ and the limitation obtained in (21), from (24) we obtain the third estimate:

$$\left|\nabla_\Gamma u_m(t)\right|_{2,\Gamma}^2 + \int_0^t \left|\Delta_\Gamma u_m(s)\right|_{2,\Gamma}^2 ds \leq C_9. \tag{25}$$

Finally we observe that from (21), (25) and the norm defined in (4) we get

$$(u_m)_{m\in\mathbb{N}} \text{ is bounded in } L^2\left(0,T; H^2(\Gamma)\right). \tag{26}$$

Passage to the limit

Thus we proved that the sequence of approximatios $(u_m)_{m\in\mathbb{N}}$ is bounded in the spaces: $L^\infty\left(0,T;H^1(\Gamma)\right)$, by (21); $L^2\left(0,T;H^2(\Gamma)\right)$, by (26); $L^\infty\left(0,T;L^{2k+2}(\Gamma)\right)$ by (21) and $\left(u_m'\right)_{m\in\mathbb{N}}$ bounded in $L^2\left(0,T;L^2(\Gamma)\right)$ by (20).

From the above estimates we deduce that there exists $\left(u_\mu\right)_{\mu\in\mathbb{N}}$, subsequence of $(u_m)_{m\in\mathbb{N}}$, and a function u, such that

$$\begin{aligned}
&u_\mu \rightharpoonup u \text{ weak-star in } L^\infty\left(0,T;H^1(\Gamma)\right),\\
&u_\mu \rightharpoonup u \text{ weak-star in } L^\infty\left(0,T;L^{2k+2}(\Gamma)\right),\\
&u_\mu \rightharpoonup u \text{ weak in } L^2\left(0,T;H^2(\Gamma)\right),\\
&u_\mu' \rightharpoonup u' \text{ weak in } L^2\left(0,T;L^2(\Gamma)\right).
\end{aligned} \tag{27}$$

On the other hand, from (18) and as the operator $\mathcal{A}\in\mathcal{L}\left(H^1(\Gamma),L^2(\Gamma)\right)$, we conclude that

$$\left|a\left(\int_\Gamma u_\mu(t)\,d\Gamma\right)\mathcal{A}u_\mu(t)\right|_{2,\Gamma} \le C_{10} \text{ a.e. on } (0,T)$$

and therefore

$$\left|a\left(\int_\Gamma u_\mu(t)\,d\Gamma\right)\mathcal{A}u_\mu(t)\right|_{L^2\left(0,T;L^2(\Gamma)\right)} \le C_{11}. \tag{28}$$

From now on, we will consider some subsequeces of $\left(u_\mu\right)_{\mu\in\mathbb{N}}$ that will be still denoted by $\left(u_\mu\right)_{\mu\in\mathbb{N}}$.

We have $H^2(\Gamma)\hookrightarrow H^1(\Gamma)\hookrightarrow L^2(\Gamma)$, see Section 2 and $H^2(\Gamma)\hookrightarrow H^1(\Gamma)$ also compact, cf. Hebey (1999). Thus, by compactness argument, see (Lions, 1969, p. 12) or (Aubin, 1963), we obtain a subsequence, such that

$$u_\mu \to u \text{ strongly in } L^2\left(0,T;H^1(\Gamma)\right) \tag{29}$$

and therefore

$$\mathcal{A}u_\mu \to \mathcal{A}u \text{ in } L^2\left(0,T;L^2(\Gamma)\right),$$

tthen,

$$\mathcal{A}u_\mu \to \mathcal{A}u \text{ a.e. on } \Gamma\times(0,T). \tag{30}$$

From (29), it follows that

$$u_\mu \to u \text{ strongly in } L^2\left(0,T;L^2(\Gamma)\right) \tag{31}$$

and therefore

$$\int_\Gamma u_\mu(t)\,d\Gamma \to \int_\Gamma u(t)\,d\Gamma \text{ a.e. on } (0,T). \tag{32}$$

As a is continuous, from (32) we get

$$a\left(\int_\Gamma u_\mu(t)\,d\Gamma\right) \to a\left(\int_\Gamma u(t)\,d\Gamma\right) \text{ a.e. on } (0,T). \tag{33}$$

From (30) and (33), we conclude that

$$a\left(\int_\Gamma u_\mu(t)\,d\Gamma\right)\mathcal{A}u_\mu \to a\left(\int_\Gamma u(t)\,d\Gamma\right)\mathcal{A}u \text{ a.e on } \Gamma\times(0,T). \tag{34}$$

From (28) and (34), applying the result contained in Lions (1969, pp. 12-13), we obtain:

$$a\left(\int_\Gamma u_\mu(t)\,d\Gamma\right)\mathcal{A}u_\mu \rightharpoonup a\left(\int_\Gamma u(t)\,d\Gamma\right)\mathcal{A}u \text{ weakly in } L^2\left(0,T;L^2(\Gamma)\right). \tag{35}$$

Now note that from the first estimate we have

$$\left(u_\mu^{2k+1}\right)_{\mu\in\mathbb{N}} \text{ is bounded in } L^q\left(0,T;L^q(\Gamma)\right), \tag{36}$$

where $q = \dfrac{2k+2}{2k+1} > 1$, and from (31) it follows that

$$u_\mu^{2k+1} \to u^{2k+1} \text{ a.e on } \Gamma \times (0,T). \tag{37}$$

Therefore, from (36) and (37), by a similar argument as that one employed to obtain (35), we get

$$u_\mu^{2k+1} \rightharpoonup u^{2k+1} \text{ weakly in } L^q(0,T;L^q(\Gamma)) \tag{38}$$

with $q = \dfrac{2k+2}{2k+1}$. Taking into account the convergences in $(27)_3$, $(27)_4$, (35) and (38), we can pass to the limit in the approximate equation to obtain

$$u_t + a\left(\int_\Gamma u d\Gamma\right)\mathcal{A}u - \Delta_\Gamma u + u^{2k+1} = f \text{ in } L^2\left(0,T;L^2(\Gamma)\right). \tag{39}$$

We observe that, from the regularity obtained for u and u', we have that $u(0)$ makes sense and, in fact, we can prove that $u(0) = u_0$.

Uniqueness

Let us consider u_1 and u_2 solutions of (9). From Theorem 4.1 we know that

$$\begin{aligned} &u_1, u_2 \in L^2\left(0,T;H^2(\Gamma)\right) \cap L^\infty\left(0,T;H^1(\Gamma)\cap L^{2k+2}(\Gamma)\right), \\ &u_1', u_2' \in L^2\left(0,T;L^2(\Gamma)\right). \end{aligned}$$

From (39), we have, after some calculations, that $z = u_1 - u_2$ satisfies

$$\left(z'(t), v\right)_\Gamma + \left(a\left(\int_\Gamma u_1 d\Gamma\right)\mathcal{A}u_1(t) - a\left(\int_\Gamma u_2 d\Gamma\right)\mathcal{A}u_2(t), v\right)_\Gamma - (\Delta_\Gamma z(t), v)_\Gamma + \left(u_1^{2k+1}(t) - u_2^{2k+1}(t), v\right)_\Gamma = 0,$$

for all $v \in L^2(\Gamma)$.

Taking $v = z(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|z(t)|_{2,\Gamma}^2 + |\nabla_\Gamma z(t)|_{2,\Gamma}^2 + \left(u_1^{2k+1}(t) - u_2^{2k+1}(t), z(t)\right)_\Gamma + \\ &\left(a\left(\int_\Gamma u_1 d\Gamma\right)\mathcal{A}u_1(t) - a\left(\int_\Gamma u_2 d\Gamma\right)\mathcal{A}u_2(t), z(t)\right)_\Gamma = 0. \end{aligned} \tag{40}$$

By the mean value theorem, we obtain

$$\left(u_1^{2k+1}(t) - u_2^{2k+1}(t), z(t)\right)_\Gamma \geq 0. \tag{41}$$

Applying (41) to (40) we get

$$\frac{1}{2}\frac{d}{dt}|z(t)|_{2,\Gamma}^2 + |\nabla_\Gamma z(t)|_{2,\Gamma}^2 \leq \left|\left(a\left(\int_\Gamma u_2 d\Gamma\right)\mathcal{A}u_2(t) - a\left(\int_\Gamma u_1 d\Gamma\right)\mathcal{A}u_1(t), z(t)\right)\right|. \tag{42}$$

Let us do some estimates with the right-side term of the above inequality, in fact employing the hypothesis about the function a, we obtain

$$\begin{aligned} &\left|\left(a\left(\int_\Gamma u_2 d\Gamma\right)\mathcal{A}u_2(t) - a\left(\int_\Gamma u_1 d\Gamma\right)\mathcal{A}u_1(t), z(t)\right)\right| \\ = \; &\left|\left(\left(a\left(\int_\Gamma u_2 d\Gamma\right) - a\left(\int_\Gamma u_1 d\Gamma\right)\right)\mathcal{A}u_2(t) + \; a\left(\int_\Gamma u_1 d\Gamma\right)(\mathcal{A}u_2(t) - \mathcal{A}u_1(t)), z(t)\right)\right| \\ \leq \; &C_{12}\left|\int_\Gamma (u_2 - u_1)\, d\Gamma\right| |(\mathcal{A}u_2(t), z(t))| + C_{13}\,|(\mathcal{A}z(t), z(t))|. \end{aligned} \tag{43}$$

By Cauchy–Schwartz's inequality, Young's inequality, by the regularity of the solution and recalling that $\mathcal{A} \in \mathcal{L}\left(H^1(\Gamma), L^2(\Gamma)\right)$, we obtain

$$C_{12}\left|\int_\Gamma (u_2 - u_1)\, d\Gamma\right| |(\mathcal{A}u_2(t), z(t))| + C_{13}\,|(\mathcal{A}z(t), z(t))| \leq C_{13}\,|z(t)|^2_{2,\Gamma} + \frac{1}{2}\,|\nabla_\Gamma z(t)|^2_{2,\Gamma}. \tag{44}$$

Combining (43) and (44) then returning in (42), we conclude

$$\frac{1}{2}\frac{d}{dt}\,|z(t)|^2_{2,\Gamma} + \frac{1}{2}\,|\nabla_\Gamma z(t)|^2_{2,\Gamma} \leq C_{13}\,|z(t)|^2_{2,\Gamma}. \tag{45}$$

As $z(0) = 0$, uniqueness follows from Gronwall's inequality by applying it to (45).

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