

On Positive Integer Powers of Toeplitz Matrices

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Abstract

In this paper, we restate a necessary and sufficient condition for all positive integer powers of a Toeplitz matrix to be still Toeplitz matrices which was ever studied by T. N. E. Greville (1983) and Tamir Shalom (1987), and our new proof is clear and self-contained. We also derive some identities involving the entries of the Toeplitz matrix and those of its positive integer powers in case that the powers are still Toeplitz matrices.

Keywords: Toeplitz matrix, positive integer power, main diagonal, upper- r th diagonal, lower- r th diagonal

AMS Subject Classifications: 15A23, 05B20, 65F99

1. Introduction

A Toeplitz matrix is an $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} = a_{j-i}$, i.e., a matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_1 \\ a_{-n+1} & a_{-n+2} & \cdots & a_{-1} & a_0 \end{pmatrix},$$

where $a_{-n+1}, a_{-n+2}, \dots, a_{-1}, a_0, a_1, a_2, \dots, a_{n-1} \in \mathbb{C}$. The entries of A are constant down the diagonals parallel to the main diagonal. Toeplitz matrix arises in scientific computing and engineering, for example, image processing, numerical differential equations and integral equations, time series analysis and control theory (see Chan & Ng, 1996; Gray, 2006; Horn & Johnson, 1985; Kailath & Sayed, 1995). A great deal is known about the behavior of such matrices—the most common and complete references being Grenander and Szegö (1958). A more recent text devoted to the subject is Böttcher and Silbermann (1999).

For an $n \times n$ matrix $A = (a_{ij})$, we call the diagonal parallel to the main diagonal which has the entries $a_{i,i+r}$ ($i = 1, 2, \dots, n-r$) the upper- r th diagonal for $r = 1, 2, \dots, n-1$, and call the diagonal parallel to the main diagonal which has the entries $a_{i,i-r}$ ($i = r+1, r+2, \dots, n$) the lower- r th diagonal for $r = 1, 2, \dots, n-1$. Given an $n \times n$ Toeplitz matrix $A = (a_{ij}) = (a_{j-i})$, the upper- r th diagonal has the constant entry a_r , and the lower- r th diagonal has the constant entry a_{-r} for $r = 1, 2, \dots, n-1$.

Notice that the positive integer powers of Toeplitz matrices are not necessarily Toeplitz matrices. However, we can still find a great deal of Toeplitz matrices whose positive integer powers are Toeplitz matrices, for example, circulant matrices. So what additional conditions do those kinds of Toeplitz matrices have? In section 2, we restate a necessary and sufficient condition for Toeplitz matrices to have Toeplitz positive integer powers which was ever studied before. Moreover, we also get some additional results about this problem. In section 3, we give some examples to interpret our result.

2. Main Results

T. N. E. Greville obtained a criterion for a nonsingular Toeplitz matrix to have a Toeplitz inverse in 1983 (see Greville, 1983). In 1987, Tamir Shalom derived an equivalent necessary and sufficient condition for a nonsingular Toeplitz matrix to have a Toeplitz inverse, and proved that the statement “All positive powers of a nonsingular Toeplitz matrix are Toeplitz matrices” is equivalent to the statement “The inverse of a nonsingular Toeplitz matrix is a Toeplitz matrix” (see Shalom, 1987). We restate the necessary and sufficient condition for Toeplitz matrices to have Toeplitz positive integer powers and give a new proof.

Obviously, any power of a Toeplitz matrix of order $n = 1$ is naturally a Toeplitz matrix, and any power of a scalar matrix which is a diagonal matrix whose diagonal elements are the same scalar is also a scalar matrix that is naturally a Toeplitz matrix. So we only need to consider Toeplitz matrices of order $n \geq 2$ which are not scalar matrices in the following. Now we derive an important result for the product of two Toeplitz matrices which are not scalar matrices to be still a Toeplitz matrix, and obtain some connections between the entries of the two Toeplitz matrices and those of their product.

Lemma 1 Let $A = (a_{j-i})_{i,j=1}^n$, $B = (b_{j-i})_{i,j=1}^n$ be Toeplitz matrices which are not scalar matrices, $n \geq 2$. If $a_i b_{j-n} = a_{i-n} b_j$ for $i, j = 1, \dots, n-1$, then the product AB is also a Toeplitz matrix. Furthermore, if we let $AB = C = (c_{j-i})_{i,j=1}^n$, then $c_i a_{j-n} = c_{i-n} a_j$ and $c_i b_{j-n} = c_{i-n} b_j$ for $i, j = 1, \dots, n-1$.

Proof. Suppose $A = (a_{ij})_{i,j=1}^n = (a_{j-i})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n = (b_{j-i})_{i,j=1}^n$ are Toeplitz matrices which are not scalar matrices, and $AB = C = (c_{ij})_{i,j=1}^n$. Then $c_{ij} = \sum_{p=1}^n a_{i,p} b_{p,j} = \sum_{p=1}^n a_{p-i} b_{j-p}$ for $1 \leq i, j \leq n$. Thus, for $r = 1, 2, \dots, n-1$, the entries of C down the upper- r th diagonal are

$$c_{i,i+r} = \sum_{p=1}^n a_{p-i} b_{i+r-p} = \sum_{q=1-i}^{n-i} a_q b_{r-q}, \quad i = 1, \dots, n-r;$$

the entries of C down the lower- $(n-r)$ th diagonal are

$$c_{i,i+r-n} = \sum_{p=1}^n a_{p-i} b_{i+r-n-p} = \sum_{q=1-i}^{n-i} a_q b_{r-n-q}, \quad i = n-r+1, \dots, n.$$

And the entries of C on the main diagonal are

$$c_{i,i} = \sum_{p=1}^n a_{p-i} b_{i-p} = \sum_{q=1-i}^{n-i} a_q b_{-q}, \quad i = 1, 2, \dots, n.$$

Now we assume $a_i b_{j-n} = a_{i-n} b_j$ for $i, j = 1, \dots, n-1$.

Case (I) $a_i a_{i-n} = 0$ for any $i = 1, \dots, n-1$.

Then for any $j = 1, \dots, n-1$, we have $0 = a_i b_{j-n} a_{i-n} b_j = a_i^2 b_{j-n}^2 = a_{i-n}^2 b_j^2$. That is, $a_i b_{j-n} = a_{i-n} b_j = 0$ for any $i, j = 1, \dots, n-1$.

Subcase (I) There exists an i_0 ($1 \leq i_0 \leq n-1$) such that $a_{i_0} = 0$ and $a_{i_0-n} \neq 0$.

Since $a_{i_0-n} b_j = a_{i_0} b_{j-n} = 0$, we have $b_j = 0$ ($j = 1, \dots, n-1$), i.e., B is a lower triangular Toeplitz matrix. Considering that B is not a scalar matrix, so there exists an j_0 ($1 \leq j_0 \leq n-1$) such that $b_{j_0-n} \neq 0$. Then $a_i = 0$ ($i = 1, \dots, n-1$) since $a_i b_{j_0-n} = a_{i-n} b_{j_0} = 0$, i.e., A is also a lower triangular Toeplitz matrix. Thus we only need to show that the lemma holds for the case in which A and B are both lower triangular Toeplitz matrices. Note that, for $r = 1, 2, \dots, n-1$, the entries of C down the upper- r th diagonal are

$$c_{i,i+r} = \sum_{q=1-i}^{n-i} a_q b_{r-q} = 0, \quad i = 1, \dots, n-r;$$

the entries of C down the lower- $(n-r)$ th diagonal are

$$c_{i,i+r-n} = \sum_{q=1-i}^{n-i} a_q b_{r-n-q} = \sum_{q=r-n}^0 a_q b_{r-n-q},$$

for all $i = n - r + 1, \dots, n$. And the entries of C on the main diagonal are

$$c_{i,i} = \sum_{q=1-i}^{n-i} a_q b_{-q} = a_0 b_0, \quad i = 1, 2, \dots, n.$$

Thus, $C = AB$ is also a lower triangular Toeplitz matrix. Furthermore, if we let $AB = C = (c_{j-i})_{i,j=1}^n$, then $c_i a_{j-n} = 0 = c_{i-n} a_j$ and $c_i b_{j-n} = 0 = c_{i-n} b_j$ for $i, j = 1, \dots, n-1$.

Subcase (II) There exists an i_0 ($1 \leq i_0 \leq n-1$) such that $a_{i_0} \neq 0$ and $a_{i_0-n} = 0$.

This subcase can be proved similarly to Subcase (I). In fact, A and B are both upper triangular Toeplitz matrices in this subcase.

Case (II) $b_i b_{i-n} = 0$ for any $i = 1, \dots, n-1$.

This case can be proved similarly to Case (I).

Case (III) There exist p, q ($1 \leq p, q \leq n-1$) such that $a_p a_{p-n} \neq 0$ and $b_q b_{q-n} \neq 0$.

Considering that $b_i a_{p-n} b_{j-n} a_p = b_{i-n} a_p b_j a_{p-n}$ and $a_i b_{q-n} a_{j-n} b_q = a_{i-n} b_q a_j b_{q-n}$ for $i, j = 1, \dots, n-1$, we have $b_i b_{j-n} = b_{i-n} b_j$ and $a_i a_{j-n} = a_{i-n} a_j$ for $i, j = 1, \dots, n-1$.

Since $a_i b_{j-n} = a_{i-n} b_j$ for $i, j = 1, \dots, n-1$, we get equations for $r = 0, 1, 2, \dots, n-2$:

$$\begin{aligned} a_{n-1} b_{r-n+1} &= a_{-1} b_{r+1}, \\ a_{n-2} b_{r-n+2} &= a_{-2} b_{r+2}, \\ &\dots \\ a_{r+1} b_{-1} &= a_{r-n+1} b_{n-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{q=0}^{n-2} a_q b_{r-q} + a_{n-1} b_{r-n+1} &= a_{-1} b_{r+1} + \sum_{q=0}^{n-2} a_q b_{r-q}, \\ \sum_{q=-1}^{n-3} a_q b_{r-q} + a_{n-2} b_{r-n+2} &= a_{-2} b_{r+2} + \sum_{q=-1}^{n-3} a_q b_{r-q}, \\ &\dots \quad \dots \quad \dots \\ \sum_{q=r-n+2}^r a_q b_{r-q} + a_{r+1} b_{-1} &= a_{r-n+1} b_{n-1} + \sum_{q=r-n+2}^r a_q b_{r-q}. \end{aligned}$$

That is,

$$\begin{aligned} c_{1,1+r} &= \sum_{q=0}^{n-1} a_q b_{r-q} = \sum_{q=-1}^{n-2} a_q b_{r-q} = c_{2,2+r}, \\ c_{2,2+r} &= \sum_{q=-1}^{n-2} a_q b_{r-q} = \sum_{q=-2}^{n-3} a_q b_{r-q} = c_{3,3+r}, \\ &\dots \quad \dots \quad \dots \\ c_{n-r-1,n-1} &= \sum_{q=r-n+2}^{r+1} a_q b_{r-q} = \sum_{q=r-n+1}^r a_q b_{r-q} = c_{n-r,n}. \end{aligned}$$

Therefore, the entries down the upper- r th diagonal of C are equal for $r = 1, 2, \dots, n-1$ and the entries on the main diagonal of C are also equal.

Similarly, we get equations for $r = 2, 3, \dots, n-1$ (suppose $n \geq 3$ now without loss of generality since it is not necessary to prove the case $n = 2$ here):

$$\begin{aligned} a_{r-1} b_{1-n} &= a_{r-n-1} b_1, \\ a_{r-2} b_{2-n} &= a_{r-n-2} b_2, \\ &\dots \\ a_1 b_{r-n-1} &= a_{1-n} b_{r-1}. \end{aligned}$$

So

$$\begin{aligned}
 \sum_{q=r-n}^{r-2} a_q b_{r-n-q} + a_{r-1} b_{1-n} &= a_{r-n-1} b_1 + \sum_{q=r-n}^{r-2} a_q b_{r-n-q}, \\
 \sum_{q=r-n-1}^{r-3} a_q b_{r-n-q} + a_{r-2} b_{2-n} &= a_{r-n-2} b_2 + \sum_{q=r-n-1}^{r-3} a_q b_{r-n-q}, \\
 &\dots \quad \dots \quad \dots \\
 \sum_{q=2-n}^0 a_q b_{r-n-q} + a_1 b_{r-n-1} &= a_{1-n} b_{r-1} + \sum_{q=2-n}^0 a_q b_{r-n-q}.
 \end{aligned}$$

That is,

$$\begin{aligned}
 c_{n-r+1,1} &= \sum_{q=r-n}^{r-1} a_q b_{r-n-q} = \sum_{q=r-n-1}^{r-2} a_q b_{r-n-q} = c_{n-r+2,2}, \\
 c_{n-r+2,2} &= \sum_{q=r-n-1}^{r-2} a_q b_{r-n-q} = \sum_{q=r-n-2}^{r-3} a_q b_{r-n-q} = c_{n-r+3,3}, \\
 &\dots \quad \dots \quad \dots \\
 c_{n-1,r-1} &= \sum_{q=2-n}^1 a_q b_{r-n-q} = \sum_{q=1-n}^0 a_q b_{r-n-q} = c_{n,r}.
 \end{aligned}$$

Thus, the entries down the lower- $(n-r)$ th diagonal of C are equal for $r = 1, 2, \dots, n-1$.

Therefore, AB is also a Toeplitz matrix. Furthermore, if we let $AB = C = (c_{j-i})_{i,j=1}^n$, then

$$c_i = c_{n-i,n} = \sum_{q=i-n+1}^i a_q b_{i-q} = a_i b_0 + a_0 b_i + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-q}$$

and

$$c_{i-n} = c_{n-i+1,1} = \sum_{q=i-n}^{i-1} a_q b_{i-n-q} = a_{i-n} b_0 + a_0 b_{i-n} + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-n-q}.$$

Hence,

$$\begin{aligned}
 c_i a_{j-n} &= b_0 a_i a_{j-n} + a_0 b_i a_{j-n} + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-q} a_{j-n} \\
 &= b_0 a_{i-n} a_j + a_0 b_{i-n} a_j + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-q-n} a_j = c_{i-n} a_j
 \end{aligned}$$

and

$$\begin{aligned}
 c_i b_{j-n} &= b_0 a_i b_{j-n} + a_0 b_i b_{j-n} + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-q} b_{j-n} \\
 &= b_0 a_{i-n} b_j + a_0 b_{i-n} b_j + \sum_{\substack{q=i-n+1 \\ q \neq 0}}^{i-1} a_q b_{i-q-n} b_j = c_{i-n} b_j
 \end{aligned}$$

for any $i, j = 1, 2, \dots, n-1$. □

Next we obtain a sufficient condition for a Toeplitz matrix to have Toeplitz positive integer powers by the lemma above, and clarify the connections between the entries of the Toeplitz matrix and those of its positive integer powers.

Lemma 2 Let $A = (a_{j-i})_{i,j=1}^n$ be a Toeplitz matrix, $n \geq 2$. If $a_i a_{j-n} = a_{i-n} a_j$ for $i, j = 1, 2, \dots, n-1$, then A^k is also a Toeplitz matrix for any positive integer k . Furthermore, if we let $A^k = (a_{j-i}^{(k)})$, then $a_i a_{j-n}^{(k)} = a_{i-n} a_j^{(k)}$ for $i, j = 1, 2, \dots, n-1$.

Proof. If A is a scalar matrix, then the lemma holds naturally. Otherwise, We prove by induction on k . The case $k = 1$ holds naturally and the case $k = 2$ is proved by choosing $B = A$ in Lemma 1.

Now we suppose that the lemma holds for $k-1$. Since $A^k = AA^{k-1}$, the lemma holds by taking $B = A^{k-1}$ in Lemma 1. $\square$

Now we derive a necessary condition for Toeplitz matrices to have Toeplitz squares.

Lemma 3 Let $A = (a_{j-i})_{i,j=1}^n$, and A^2 be Toeplitz matrices, $n \geq 2$. Then $a_i a_{j-n} = a_{i-n} a_j$ for $i, j = 1, 2, \dots, n-1$.

Proof. Suppose that $A = (a_{j-i})_{i,j=1}^n$, $A^2 = (c_{j-i})_{i,j=1}^n$. Then for $r = 1, 2, \dots, n-2$, the entries of A^2 down the upper- r th diagonal are equal:

$$\sum_{q=0}^{n-1} a_q a_{r-q} = \sum_{q=-1}^{n-2} a_q a_{r-q} = \dots = \sum_{q=1-n+r}^r a_q a_{r-q};$$

the entries of A^2 down the lower- $(n-r)$ th diagonal are equal:

$$\sum_{q=r-n}^{r-1} a_q a_{r-n-q} = \sum_{q=r-n-1}^{r-2} a_q a_{r-n-q} = \dots = \sum_{q=1-n}^0 a_q a_{r-n-q}.$$

And the entries of A^2 on the main diagonal are also equal:

$$c_0 = \sum_{q=0}^{n-1} a_q a_{-q} = \sum_{q=-1}^{n-2} a_q a_{-q} = \dots = \sum_{q=1-n}^0 a_q a_{-q}.$$

Compare both sides of each “=”, we derive $(n-1)^2$ identities: $a_i a_{j-n} = a_{i-n} a_j$ for $i, j = 1, 2, \dots, n-1$. $\square$

Finally, our main theorem of this paper can be easily obtained.

Theorem 4 Let $A = (a_{j-i})_{i,j=1}^n$ be a Toeplitz matrix, $n \geq 2$. Then any positive integer power of A is also a Toeplitz matrix if and only if $a_i a_{j-n} = a_{i-n} a_j$ for $i, j = 1, 2, \dots, n-1$.

Proof. The sufficiency is proved by Lemma 2, and the necessity is proved by Lemma 3. $\square$

The condition in Theorem 4 can be examined by computer easily, so it is a valuable condition for us to study the powers of Toeplitz matrices. We note that circulant matrices are well-known examples of such Toeplitz matrices satisfying the necessary and sufficient condition above.

3. Examples

Now we give some matrices, and compare the powers of them.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}, & A^5 &= \begin{pmatrix} 25056 & 25072 & 24928 & 24944 \\ 24944 & 25056 & 25072 & 24928 \\ 24928 & 24944 & 25056 & 25072 \\ 25072 & 24928 & 24944 & 25056 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 2 & 4 & 8 \\ 4 & 1 & 2 & 4 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 4 & 1 \end{pmatrix}, & B^5 &= \begin{pmatrix} 28361 & 34266 & 40476 & 47576 \\ 23788 & 28361 & 34266 & 40476 \\ 20238 & 23788 & 28361 & 34266 \\ 17133 & 20238 & 23788 & 28361 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 2 & 4 & 0 \\ 4 & 1 & 2 & 4 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 4 & 1 \end{pmatrix}, & C^5 &= \begin{pmatrix} 10681 & 11674 & 12988 & 9520 \\ 14668 & 16841 & 18394 & 12988 \\ 13214 & 14668 & 16841 & 11674 \\ 11997 & 13214 & 14668 & 10681 \end{pmatrix} \end{aligned}$$

The matrix A is a circulant matrix, hence any positive integer power of A , A^5 for example, is still a Toeplitz matrix (a circulant matrix actually) by Theorem 4. The matrices B and C are both Toeplitz matrices, however, B^5 is a Toeplitz matrix while C^5 is no longer a Toeplitz matrix since $C = (c_{j-i})_{i,j=1}^4$ does not satisfy the conditions in Theorem 4: $c_3c_{2-4} = 0 \times 2 \neq c_{3-4}c_2 = 4 \times 4$.

Toeplitz matrices have nice properties and great applications in science and industry, there remain many problems for us to study.

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